Extending the pebbling threshold spectrum

David Moews

October 30, 2017

Abstract. Given a distribution of pebbles on the vertices of a graph, say that we can *pebble* a vertex if a pebble is left on it after some sequence of moves, each of which takes two pebbles from some vertex and places one on an adjacent vertex. A distribution is *solvable* if all vertices are pebblable; the *pebbling threshold* is, roughly speaking, the total number of pebbles for which random distributions change from being almost never solvable to almost always solvable. We show that any sequence of connected graphs with strictly increasing orders always has some pebbling threshold which is $\Omega(\sqrt{n})$ and $O(2^{\sqrt{2\log_2 n}} n/\sqrt{\log_2 n})$, and that it is possible to construct a sequence of connected graphs with strictly increasing orders which has any desired pebbling threshold, as long as it is always between $\sqrt{n}$ and $2^{\sqrt{2\log_2 n}} n/\sqrt{\log_2 n}$. (Here, n is the order of a graph in the sequence.) It follows that the sequence of paths, which has pebbling threshold $\Theta(2^{\sqrt{\log_2 n}} n/\sqrt{\log_2 n})$, does not have the greatest possible pebbling threshold.

Introduction

In the mathematical game of *pebbling*, one starts with a *distribution* on a graph assigning a nonnegative integral number of pebbles to each vertex of the graph. A *pebbling move* consists of taking two pebbles away from a vertex with at least two pebbles and adding one pebble to any adjacent vertex. A vertex is *pebblable* for a given distribution if there is some sequence of pebbling moves starting at the distribution and finishing with at least one pebble on that vertex, and a distribution is *solvable* if each vertex is pebblable for that distribution. In [3], Czygrinow et al. introduce the *pebbling threshold* for a sequence of graphs, which, roughly speaking, is the number of pebbles at which a random distribution with that number of pebbles changes from being almost always unsolvable to almost always solvable.

In [6], we determined that the pebbling threshold of the sequence of paths was $\Theta(2^{\sqrt{\log_2 n}} n/\sqrt{\log_2 n})$, where n is the order of a path in the sequence. In this paper, we show that this is not the largest possible pebbling threshold. The reason is that most vertices in the path can be moved onto from both directions; the ends are harder to reach since they can only be reached from one direction, but there are only two ends. A graph which contains a bouquet of paths joined at a point will then be harder since it has more ends (for an appropriate choice of path lengths and number of paths.) Using this construction we find the highest possible pebbling threshold for a sequence of connected graphs with strictly increasing orders, which is, roughly speaking, $\Theta(2^{\sqrt{2\log_2 n}} n/\sqrt{\log_2 n})$, and show that all intermediate thresholds between this and the lowest possible (roughly speaking, $\Theta(\sqrt{n})$), exist. (Here, n is the order of a graph in the sequence.) This resolves the question [5, RP17].

This paper will use definitions and notations from [6]; in particular, we let $(W_i)_{i\in\mathbb{Z}_{>0}}$ be an independent family of standard exponential random variables, and let

$$Y_\infty := \sum_{i\geq 0} \frac{W_{i+1}}{2^i}.$$

In addition, $d(x, y)$ will mean the distance between vertices x and y in a given graph; $\mathcal{Z}(x)$ will mean the number of pebbles on vertex x, in a given distribution on a given graph.

1 The bouquet of paths

Lemma 1. *Let H be a graph which contains $L \geq 1$ vertices, $v_1, \ldots, v_L$, such that $d(v_1, v_i) = i - 1$ for all $i = 2, \ldots, L$. There exists some absolute constant $G_- \geq 3$ such that, if $g \geq G_-$ and an independent, geometrically distributed number of pebbles with parameter $p := (1 + (2^{\sqrt{2\log_2 g}} e/(2(1 + (\log_2 g)^{-1/4})\sqrt{\log_2 g})))^{-1}$ is placed on each of $v_1, \ldots, v_L$, then, with probability at least $2/g$, v_1 is unpebblable, provided that, for the set of vertices V of H apart from $v_1, \ldots, v_L$,*

$$\sum_{x\in V} \mathcal{Z}(x) 2^{-d(x,v_1)} \leq \frac{2e}{\sqrt{\log_2 g}}. \tag{1}$$

Proof. This is similar to half of the proof of [6, Theorem 15]. Let Z_i be the number of pebbles on v_i, $i = 1, \ldots, L$. The quantity

$$Q := \sum_{x \text{ a vertex of } H} \mathcal{Z}(x) 2^{-d(x,v_1)}$$

is at least 1 if there is a pebble on v_1, and it cannot be increased by pebbling moves. It follows that v_1 will be unpebblable provided that $Q < 1$, which, by (1), will certainly be true if $\sum_{1\leq i\leq L} Z_i 2^{-(i-1)} < 1 - (2e/\sqrt{\log_2 g})$; so, if we let Z_{L+1}, $Z_{L+2}, \ldots$ be additional independent geometric random variables with parameter p, v_1 will be be unpebblable if $\sum_{i\geq 1} Z_i 2^{-(i-1)} < 1 - (2e/\sqrt{\log_2 g})$. We can now set $Z_i := \lfloor W_i/\lambda \rfloor$, where $\lambda := -\log(1 - p)$, so it suffices for unpebblability that

$$Y_\infty < \lambda(1 - \frac{2e}{\sqrt{\log_2 g}}).$$

We can compute the probability q of this event using [6, Theorem 13], setting

$$c'' := \sqrt{2\log_2 g}, \quad p' := (p^{-1}-1)^{-1} = \frac{2\sqrt{\log_2 g}}{e2^{\sqrt{2\log_2 g}}}(1+(\log_2 g)^{-1/4}),$$

$$y := \frac{\sqrt{2}}{e}\Delta, \quad \Delta := \frac{\lambda}{p}\frac{1}{p'+1}(1+(\log_2 g)^{-1/4})(1-\frac{2e}{\sqrt{\log_2 g}}).$$

For large g, Δ will be close to 1, so after choosing G_- appropriately, y will be between $\frac{1}{2}$ and 1. According then to the theorem, if we choose G_- so as to make c'' sufficiently large, there is some positive constant C such that

$$q \geq C(ey)^{c''}2^{-c''(c''+1)/2}/\sqrt{c''},$$

or such that $q \geq C\Delta^{c''}/(g\sqrt{c''})$. Now Δ is a function only of g and, for large g, $\log\Delta = 2^{1/4}c''^{-1/2} + O(c''^{-1})$, so choose G_- large enough to ensure that $\Delta^{c''}/\sqrt{c''} \geq 2/C$. □

Lemma 2. *Let H be a graph which contains a path with $L \geq 1$ vertices, v_1, …, v_L. Then there exists some absolute constant $G_+ \geq 3$ such that, if $g \geq G_+$ and an independent, geometrically distributed number of pebbles with parameter $p := (1+(2^{\sqrt{2\log_2 g}}e/(2(1-(\log_2 g)^{-1/4})\sqrt{\log_2 g})))^{-1}$ is placed on each of v_1, …, v_L, then (A) if $L \geq 1.1\sqrt{2\log_2 g}$, with probability at least $1-1/(4g)$, v_1 is pebblable, and (B) if*

$$2.2\sqrt{2\log_2 g} \leq L \leq \exp(2\log_2 g)^{1/4},$$

with probability at least $1-1/(4g)$, all of v_1, …, v_L are pebblable.

Proof. This is similar to the other half of the proof of [6, Theorem 15]. We start with (A). Let Z_j be the number of pebbles on v_j, $j = 1, \ldots, L$. Set $M := \lfloor(\log_2 g)^{1/16}\rfloor$. Since $L \geq 1.1\sqrt{2\log_2 g}$, we can choose G_+ large enough to ensure that $L \geq M+1$. For v_1 to be unpebblable, we must have $Z_j < 2^M$, $j = 1, \ldots, M+1$, and $\sum_{0\leq j\leq L-M-2} Z_{M+2+j}2^{-j} < 2^{M+1}$; since the Z_js are independent and geometrically distributed with parameter p, this will have probability no more than $(2^M p)^{M+1}X$, where

$$X := \mathbb{P}(Z_1 + \cdots + \frac{Z_{L-M-1}}{2^{L-M-2}} < 2^{M+1}),$$

and as in the proof of [6, Theorem 15], $X \leq q + X'$, where

$$q := \mathbb{P}(Y_\infty < (2^{M+1}+3)\lambda), \quad X' := \mathbb{P}(Y_\infty > 2^{L-M-1}\lambda),$$

$$\lambda := -\log(1-p).$$

By [6, Theorem 14],

$$\begin{aligned} X' &\leq 4\exp -2^{L-M-1}\lambda \\ &\leq 4\exp -2^{0.09\sqrt{2\log_2 g}}, \quad \text{for an appropriate choice of } G_+. \qquad (2) \end{aligned}$$

By choosing G_+ large enough, we can force the right-hand side of (2) to be less than $\frac{1}{8}$ when multiplied by $(2^M p)^{M+1} g$.

To estimate q, use [6, Theorem 13], setting

$$c'' := \sqrt{2\log_2 g}, \quad p' := (p^{-1} - 1)^{-1} = \frac{2\sqrt{\log_2 g}}{e 2^{\sqrt{2\log_2 g}}}(1 - (\log_2 g)^{-1/4}),$$

$$y := 2^{M+1}\frac{\sqrt{2}}{e}\Delta', \quad \Delta' := \frac{\lambda}{p}\frac{1}{p'+1}(1 - (\log_2 g)^{-1/4})(1 + \frac{3}{2^{M+1}}).$$

For large g, Δ' will be close to 1, so $2^M \leq y \leq 2^{M+1}$, and by the theorem, if we choose G_+ appropriately, we will have, for some constant $C > 0$,

$$q \leq \frac{C}{\sqrt{c''} g} 2^{(M+1)c''} 2^{-(M-1)M/2} \Delta'^{c''},$$

so

$$(2^M p)^{M+1} q g \leq \frac{C}{\sqrt{c''}} (\frac{\sqrt{2}}{e} c'')^{M+1} 2^{M(M+3)/2} \Delta''^{M+1} \Delta'^{c''}, \tag{3}$$

where

$$\Delta'' := \frac{1}{p'+1}(1 - (\log_2 g)^{-1/4}).$$

The logarithm of the right-hand side of (3) is

$$-2^{1/4} c''^{1/2} + \frac{M(M+3)\log 2}{2} + O(M \log\log g),$$

so we can choose G_+ so that the right-hand side of (3) is less than $\frac{1}{8}$.

For (B), it will suffice to show that for each $i = 1, \ldots, L$, v_i is unpebblable with probability no more than $1/(4gL)$. We fix some i and let $\delta := 1$ if $i \leq L/2$, $\delta := -1$ if $i > L/2$; we now try to move pebbles onto v_i from $v_{i+\delta}$, $v_{i+2\delta}$, $\ldots$, $v_{i+\lfloor L/2 \rfloor \delta}$. The proof is then similar to (A), except that $L - M - 1$ is replaced by $\lfloor L/2 \rfloor - M$; also, since we have assumed that $L \leq e^{\sqrt{c''}}$, we must bound $e^{\sqrt{c''}}(2^M p)^{M+1} X$ instead of $(2^M p)^{M+1} X$. □

For positive n and L and nonnegative g such that $g(L-1)+1 \leq n$, let the graph $\mathcal{B}_{n,g,L}$ be the graph which has n vertices and is made by taking g paths with L vertices each and a complete graph, choosing one vertex from the complete graph and one end-vertex from each of the paths, and identifying these $g+1$ vertices into a single vertex. Also, for a graph H with $n > 0$ vertices, we define the *geometric pebbling threshold* of H to be the unique positive real x for which, if an independent, geometrically distributed number of pebbles with parameter $(1 + x/n)^{-1}$ is placed on each of the vertices of H, the probability of the distribution being solvable is $\frac{1}{2}$. (See [6, Theorem 7] for a proof that this probability is strictly increasing with x and that this definition is sensible.)

Lemma 3. *For all $\delta > 0$ there is some $m_0 = m_0(\delta)$ such that, if $\alpha \in \mathbb{R}_{\geq 0}$, N is a sum of m independent geometric random variables with parameter $(1+\alpha)^{-1}$, and both m and αm exceed m_0, then*

$$\mathbb{P}(|N - \alpha m| \leq \delta \alpha m) \geq \frac{8}{9}.$$

Proof. A geometric random variable with parameter $(1+\alpha)^{-1}$ has mean α and variance $\alpha(1+\alpha)$, so N has mean $m\alpha$ and variance $m\alpha(1+\alpha)$. Then, use Chebyshev's inequality. □

Theorem 4. *There is some integer $G_0 \geq 3$ such that if $g \geq G_0$, $2gL \leq n$, and*

$$\sqrt{2\log_2 g} \leq L - \log_2 n \leq \exp(2\log_2 g)^{1/4},$$

then the geometric pebbling threshold of $\mathcal{B}_{n,g,L}$ is αn, where

$$\alpha := \beta(1+\eta)^{-1}, \quad \beta := \frac{2^{\sqrt{2\log_2 g}}e}{2\sqrt{\log_2 g}}, \qquad |\eta| \leq (\log_2 g)^{-1/4}.$$

Proof. Let $H := \mathcal{B}_{n,g,L}$ and let $\alpha := \beta(1+\eta)^{-1}$, where η is now arbitrary but satisfies $|\eta| \leq (\log_2 g)^{-1/4}$. We have $L \geq 2$ and by taking G_0 large enough we can ensure that $g \geq G_-$, $g \geq G_+$, $\beta \geq 12$, $\alpha \geq 1$, $\log_2 n \geq 1.2\sqrt{2\log_2 g}$, and $\exp(2\log_2 g)^{1/4} \geq \lceil 2.2\sqrt{2\log_2 g}\rceil$. Suppose that an independently geometrically distributed number of pebbles with parameter $(1+\alpha)^{-1}$ is placed on each vertex of H.

Set $\eta := (\log_2 g)^{-1/4}$, consider one of the paths $v_1, \ldots, v_L$ which was identified to make H, and let its unidentified end-vertex be v_1. Let the total number of pebbles on H be N, assume that $N \leq 2\alpha n$, and let V be the set of vertices of H apart from $v_1, \ldots, v_{L-1}$. Then

$$\sum_{x\in V} \mathcal{Z}(x)2^{-d(x,v_1)} \leq 2^{-(L-1)} \sum_{x \in V} \mathcal{Z}(x) \leq N2^{-(L-1)} \leq 2\alpha n 2^{-(L-1)}$$

and

$$2\alpha n 2^{-(L-1)} \leq 4\beta 2^{-\sqrt{2\log_2 g}} = \frac{2e}{\sqrt{\log_2 g}},$$

so we can apply Lemma 1 to this path (with L decreased by 1) to show that v_1 is unpebblable with probability at least $2/g$. After doing this to each of the paths in H in turn we can conclude that the probability that the distribution is solvable is no more than

$$\mathbb{P}(N > 2\alpha n) + (1 - \frac{2}{g})^g \leq \mathbb{P}(N > 2\alpha n) + e^{-2}. \tag{4}$$

If we apply Lemma 3 (with $m := n$), then, since $\alpha \geq 1$ and $n \geq 2gL$, we can choose G_0 so that n and αn are forced to be so large that $\mathbb{P}(|N - \alpha n| \leq \alpha n) \geq \frac{8}{9}$.

Since $\frac{1}{9} + e^{-2} < \frac{1}{2}$, (4) then implies that the geometric pebbling threshold of $\mathcal{B}_{n,g,L}$ is at least $\beta n(1 + (\log_2 g)^{-1/4})^{-1}$.

Set $\eta := -(\log_2 g)^{-1/4}$. Let N' be the total number of pebbles on the vertices of the complete graph which was identified to make H, and let there be m of these vertices. If $N' \geq \alpha m/2$, then, since $m = n - g(L-1) \geq n - gL \geq n/2$,

$$N' \geq \frac{\alpha m}{2} \geq \frac{\alpha n}{4} \geq \frac{\beta n}{4} \geq 3n.$$

By moving from vertices in the complete graph to any vertex w of the complete graph, we can then place at least $(3n - m)/2 \geq (3n - n)/2 = n$ pebbles on w. Now, again consider one of the paths $v_1, \ldots, v_L$ which was identified to make H, letting its unidentified end-vertex be v_1. By moving from vertices in the complete graph to v_L, we can place at least n pebbles on v_L; by moving along the path, we can then place at least one pebble on v_{L-j}, for any $j = 1, \ldots, \lceil \log_2 n \rceil - 1$. If $L - \lceil \log_2 n \rceil \geq 2.2\sqrt{2 \log_2 g}$, we can apply Lemma 2 to the path with L decreased by $\lceil \log_2 n \rceil$ and conclude that, with probability at least $1 - 1/(4g)$, each of $v_1, \ldots, v_{L-\lceil \log_2 n \rceil}$ are pebblable; otherwise, we can apply Lemma 2 to the path with L replaced by $\lceil 2.2\sqrt{2 \log_2 g} \rceil$ and conclude that, with probability at least $1 - 1/(4g)$, each of $v_1, \ldots, v_{\lceil 2.2\sqrt{2 \log_2 g} \rceil}$ are pebblable. Applying this reasoning to each path, then, the probability that H is solvable is at least

$$\mathbb{P}(N' \geq \frac{\alpha m}{2}) - g\frac{1}{4g} = \mathbb{P}(N' \geq \frac{\alpha m}{2}) - \frac{1}{4}.$$

If we apply Lemma 3, since $m \geq n/2$, we can choose G_0 so that $\mathbb{P}(|N' - \alpha m| \leq \frac{1}{2}\alpha m) \geq \frac{8}{9}$. Since $\frac{8}{9} - \frac{1}{4} > \frac{1}{2}$, this means that the geometric pebbling threshold of $\mathcal{B}_{n,g,L}$ is no more than $\beta n(1 - (\log_2 g)^{-1/4})^{-1}$, completing the proof. □

Lemma 5. *If we define $\Phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ by*

$$\Phi(\alpha) := \frac{\alpha^2}{2\alpha + 1},$$

then Φ is a strictly increasing bijection, and if $\alpha_2 > \alpha_1 > 0$,

$$(\frac{\alpha_2}{\alpha_1})^2 \geq \frac{\Phi(\alpha_2)}{\Phi(\alpha_1)} \geq \frac{\alpha_2}{\alpha_1}.$$

Proof. Easy. □

The following theorem is similar to [4, Theorem 4].

Theorem 6. *For any $0 < \epsilon < 1$, there is some integer $L_0 = L_0(\epsilon) \geq 2$ such that if $g \geq 1$, $2gL \leq \epsilon n$, and $L_0 \leq L \leq (\log_2 n) - L_0$, then the geometric pebbling threshold of $\mathcal{B}_{n,g,L}$ is αn, where*

$$\alpha := \beta(1 + \eta), \quad \beta > 0 \text{ and } \Phi(\beta) = \frac{2^{L-1}}{n}, \qquad |\eta| \leq \epsilon.$$

Proof. Fix ϵ, let $H := \mathcal{B}_{n,g,L}$ and $\alpha := \beta(1+\eta)$, where η is arbitrary such that $|\eta| \leq \epsilon$, and suppose that an independently geometrically distributed number of pebbles with parameter $(1+\alpha)^{-1}$ is placed on each vertex of H.

Set $\eta := -\epsilon$ and let $v_1, \ldots, v_L$ be one of the paths which was identified to make H, with v_1 being the unidentified end vertex. If V is the set of vertices in H other than $v_1, \ldots, v_L$, then

$$\mathcal{Z}(v_1) + \frac{\mathcal{Z}(v_2)}{2} + \cdots + \frac{\mathcal{Z}(v_L)}{2^{L-1}} + 2^{-(L-1)} \sum_{x \in V} \lfloor \frac{\mathcal{Z}(x)}{2} \rfloor$$

is at least 1 if there is a pebble on v_1, and it cannot be increased by pebbling moves. Arguing as in Lemma 1, then, the probability that v_1 is unpebblable conditioned on the distribution on V is at least

$$\mathbb{P}(Y_\infty < \lambda(1 - 2^{-(L-1)}N)),$$

where

$$N := \sum_{x \in V} \lfloor \frac{\mathcal{Z}(x)}{2} \rfloor, \quad \lambda := \log(1 + \alpha^{-1}).$$

Now, for each x, $\lfloor \mathcal{Z}(x)/2 \rfloor$ is independently geometrically distributed with parameter $1 - (1 - (1+\alpha)^{-1})^2 = (1 + \Phi(\alpha))^{-1}$. Also, $n \geq 2gL \geq 2L_0$, and if m is the number of vertices in V, then

$$m \geq n - gL \geq n(1 - \frac{\epsilon}{2}) \geq L_0,$$

and, by Lemma 5,

$$\Phi(\alpha)m \leq \Phi(\beta)(1-\epsilon)m = \frac{2^{L-1}m}{n}(1-\epsilon) \leq 2^{L-1}(1-\epsilon)$$

and

$$\Phi(\alpha)m \geq \Phi(\beta)(1-\epsilon)^2 m = \frac{2^{L-1}m}{n}(1-\epsilon)^2 \geq 2^{L_0-1}(1-\epsilon)^2(1-\frac{\epsilon}{2}).$$

Using Lemma 3, then, we can choose L_0 large enough so that N is no more than $2^{L-1}(1 - (\epsilon/2))$ with probability at least $\frac{8}{9}$, so the probability that v_1 is unpebblable is at least

$$\frac{8}{9}\mathbb{P}(Y_\infty < \frac{\lambda\epsilon}{2}). \tag{5}$$

Since $L \leq (\log_2 n) - L_0$, we have $\Phi(\beta) \leq 2^{-L_0-1}$, so we may choose L_0 large enough so that β is forced to be small enough, and $\lambda\epsilon/2$ large enough, so that (5) is greater than $\frac{1}{2}$. This proves that, for an appropriate choice of L_0, the geometric pebbling threshold of $\mathcal{B}_{n,g,L}$ is at least $\beta n(1-\epsilon)$.

Now, set $\eta := \epsilon$, let V' be the set of vertices on the complete graph which was identified to make H, let V' have size m', and let $N' := \sum_{x \in V'} \lfloor \mathcal{Z}(x)/2 \rfloor$. If $N' \geq 2^{L-1}$, we can place 2^{L-1} pebbles on the identified vertex in H, and from there place at least one pebble on any vertex in H. We need then to show that $\mathbb{P}(N' \geq 2^{L-1}) > \frac{1}{2}$. As before, for each x, $\lfloor \mathcal{Z}(x)/2 \rfloor$ is independently geometrically distributed with parameter $(1+\Phi(\alpha))^{-1}$, so since

$$m' \geq n - gL \geq n(1 - \frac{\epsilon}{2}) \geq L_0$$

and, by Lemma 5,

$$\begin{aligned} \Phi(\alpha)m' \geq \Phi(\beta)(1+\epsilon)m' \geq 2^{L-1}(1+\epsilon)(1-\frac{\epsilon}{2}) &= 2^{L-1}(1+\frac{\epsilon(1-\epsilon)}{2}) \\ &\geq 2^{L_0-1}, \end{aligned}$$

we can choose L_0 large enough so that, by Lemma 3, $\mathbb{P}(N' \geq 2^{L-1}) \geq \frac{8}{9}$. This proves that the geometric pebbling threshold of $\mathcal{B}_{n,g,L}$ is no more than $\beta n(1+\epsilon)$, completing the proof. □

2 The pebbling threshold spectrum

Lemma 7. *Let H be a connected graph with $n \geq 2$ vertices, let v be a vertex of H such that all vertices of H are within distance $d \in \mathbb{Z}$ of v, $d \geq 2$, and let each vertex of H have a number of pebbles which is independently geometrically distributed with parameter $(1+\alpha)^{-1}$, $\alpha \in \mathbb{R}_{>0}$. Then v is unpebblable with probability at most*

$$\left(\frac{e(2^{d-1} + \lceil n^{1/d} - 1 \rceil - 1)}{\lceil n^{1/d} - 1 \rceil (1 + \Phi(\alpha))}\right)^{\lceil n^{1/d} - 1 \rceil}.$$

Proof. For $i = 0, 1, \ldots$, let D_i be the number of vertices at distance i from v and let $D'_i := \sum_{0 \leq j \leq i} D_j$. Since $\log D'_0 = 0$ and $\log D'_d = \log n$, there must be some $0 \leq i \leq d-1$ for which $(\log D'_{i+1}) - (\log D'_i) \geq (\log n)/d$, and then $D_{i+1}/D'_i = (D'_{i+1}/D'_i) - 1 \geq n^{1/d} - 1$. Since $D'_i \geq D_i$, D_{i+1}/D_i is also at least $n^{1/d} - 1$. This means that there must be some vertex w at distance i from v which has at least $\lceil n^{1/d} - 1 \rceil$ neighbors at distance $i+1$. Letting a set of $\lceil n^{1/d} - 1 \rceil$ of these neighbors be V, v will be pebblable if

$$\sum_{x \in V} \lfloor \mathcal{Z}(x)/2 \rfloor \geq 2^{d-1}, \tag{6}$$

since if so we can move 2^{d-1} pebbles to w and then place a pebble on v. Each $\lfloor \mathcal{Z}(x)/2 \rfloor$ is independently geometrically distributed with parameter $p := (1 + \Phi(\alpha))^{-1}$, so the probability of (6) is the probability that, if we flip a coin with success probability p, it takes at least $2^{d-1} + \lceil n^{1/d} - 1 \rceil$ flips to get $\lceil n^{1/d} - 1 \rceil$ successes. Another way of saying this is that there are no more than $\lceil n^{1/d} - 1 \rceil - 1$ successes in $2^{d-1} + \lceil n^{1/d} - 1 \rceil - 1$ flips, so if (6) is false, there must be at least $\lceil n^{1/d} - 1 \rceil$ successes in this number of flips. Set

$$p' := \lceil n^{1/d} - 1 \rceil / (2^{d-1} + \lceil n^{1/d} - 1 \rceil - 1).$$

If $\lceil n^{1/d} - 1 \rceil (1 + \Phi(\alpha)) \leq 2^{d-1} + \lceil n^{1/d} - 1 \rceil - 1$, then the claimed bound on the probability of unpebblability is 1 or greater and there is nothing to prove. We can assume then that $\lceil n^{1/d} - 1 \rceil (1 + \Phi(\alpha)) > 2^{d-1} + \lceil n^{1/d} - 1 \rceil - 1$; now $p < p' < 1$, so by [1, Theorem 1], the probability that (6) is false is no more than

$$\exp -(2^{d-1} + \lceil n^{1/d} - 1 \rceil - 1)\Omega, \quad \text{where}$$
$$\Omega := p' \log \frac{p'}{p} + (1 - p') \log \frac{1 - p'}{1 - p}. \tag{7}$$

Since $y \log y \geq y - 1$ for all $0 < y < 1$, $\Omega \geq p'(-1 + \log(p'/p))$. Together with (7), this gives the claimed bound. □

Theorem 8. *There is some $n_0 \geq 3$ such that if H is a connected graph with $n \geq n_0$ vertices, then the geometric pebbling threshold of H is no more than*

$$\frac{2^{\sqrt{2 \log_2 n}} e}{2\sqrt{\log_2 n}} (1 - (\log_2 n)^{-1/4})^{-1} n.$$

Proof. Choose n_0 such that $n_0 \geq G_+$. Let $p := (1 + (2^{\sqrt{2 \log_2 n}} e/(2(1 - (\log_2 n)^{-1/4})\sqrt{\log_2 n})))^{-1}$, and suppose that an independent, geometrically distributed number of pebbles with parameter p is placed on each vertex of H. Pick some vertex v of H. It will do to show that v is pebblable with probability at least $1 - 1/(4n)$. If there is some vertex x of H with $d(v, x) \geq 1.1\sqrt{2 \log_2 n}$, then this follows immediately from Lemma 2. Otherwise, apply Lemma 7 with $d := \lceil 1.1\sqrt{2 \log_2 n} \rceil$. For an appropriate choice of n_0, this will always show that v is unpebblable with probability at most $1/(4n)$. □

Theorem 9. *There is some $n_1 \geq 1$ such that, if H is a connected graph with $n \geq n_1$ vertices, then the geometric pebbling threshold of H is at least $\sqrt{n \log 2}$.*

Proof. A distribution on any graph with $n \geq 1$ vertices will not be solvable if no vertex has two or more pebbles and some vertex has no pebbles. If the number of pebbles on each vertex is independently geometrically distributed with parameter p, the probability of this event is $q := (1 - (1-p)^2)^n - (p(1-p))^n$. For large n, if $p := (1 + \sqrt{(\log 2)/n})^{-1} = 1 - \sqrt{(\log 2)/n} + ((\log 2)/n) + O(n^{-3/2})$, then $q = \frac{1}{2} + (\log 2)^{3/2} n^{-1/2} + O(n^{-1})$, which eventually exceeds $\frac{1}{2}$. □

Corollary 10. *If $(H_i)_{i \in \mathbb{Z}_{>0}}$ is any sequence of connected graphs such that the number of vertices in H_i is strictly increasing with i, then the sequence has some pebbling threshold $t(n)$ which is $\Omega(\sqrt{n})$ and $O(2^{\sqrt{2 \log_2 n}} n/\sqrt{\log_2 n})$, where n is the number of vertices in a graph in the sequence.*

Proof. By Theorems 8 and 9, for sufficiently large i, the geometric pebbling threshold T_i of H_i satisfies

$$\sqrt{n_i \log 2} \leq T_i \leq \frac{2^{\sqrt{2 \log_2 n_i}} e}{2\sqrt{\log_2 n_i}} (1 - (\log_2 n_i)^{-1/4})^{-1} n_i,$$

where n_i is the number of vertices in H_i. Define the function $t(n)$ by $t(n_i) := T_i$ for each $i \in \mathbb{Z}_{>0}$ and $t(n) := n$ if n is not equal to any n_i. Now apply [6, Theorem 11] and argue as in the proof of [2, Theorem 1.3] to prove that $t(n)$ is a pebbling threshold for $(H_i)_{i\in\mathbb{Z}_{>0}}$. □

Theorem 11. *There is some constant $K > 1$ such that, if $n \geq 2$ is an integer and*

$$\sqrt{n} \leq t \leq \frac{2^{\sqrt{2\log_2 n}}}{\sqrt{\log_2 n}} n,$$

then there is some connected graph H with n vertices whose geometric pebbling threshold is between t/K and Kt.

Proof. Set $L_0 := L_0(\frac{1}{2})$. We are free to choose an arbitrary connected graph for H for a finite number of values of n, at the cost of worsening K, so we can assume that $n \geq 2^{2L_0}$, $n/(4\log_2 n) \geq G_0$, and $\sqrt{2\log_2 n} + 1 \leq \log_2 n$. Then, we will always choose H to be some $\mathcal{B}_{n,g,L}$. Set $\beta := t/n$ and $\beta_c := 2^{\sqrt{2\log_2 G_0}} e/(2\sqrt{\log_2 G_0})$.

1. If $\beta < \beta_c$, g will always be 1. Let $\hat{L} := 1 + \log_2(\Phi(\beta)n)$. Then we let L be L_0 if $\hat{L} < L_0$, $\lfloor \hat{L} \rfloor$ if $L_0 \leq \hat{L} \leq (\log_2 n) - L_0$, and $\lfloor \log_2 n \rfloor - L_0$ if $\hat{L} > (\log_2 n) - L_0$.
2. If $\beta \geq \beta_c$, let g be the maximal integer in $G_0, G_0 + 1, \ldots, \lfloor n/(4\log_2 n) \rfloor$ with $2^{\sqrt{2\log_2 g}} e/(2\sqrt{\log_2 g}) \leq \beta$. Let L be $\lceil (\log_2 n) + \sqrt{2\log_2 g} \rceil$.

It is straightforward to verify that, regardless of t or n, the geometric pebbling threshold t' of H can then be computed with Theorem 4 or Theorem 6, and that there is some absolute constant $K > 1$ such that t'/t is always in $[1/K, K]$. □

Corollary 12. *If $t(n)$ is any function of integral $n \geq 1$ such that $\sqrt{n} \leq t(n) \leq 2^{\sqrt{2\log_2 n}} n/\sqrt{\log_2 n}$ for all $n \geq 2$, then there is some sequence of connected graphs $(H_n)_{n\in\mathbb{Z}_{>0}}$ with pebbling threshold $t(n)$ such that H_n has n vertices for each n.*

Proof. Let H_1 be the 1-vertex graph, and, for each $n \geq 2$, let H_n be the connected graph given by Theorem 11 which has n vertices and geometric pebbling threshold between $t(n)/K$ and $Kt(n)$. Then, apply [6, Theorem 11] and [2, Theorem 1.3] to prove that $t(n)$ is a pebbling threshold of $(H_n)_{n\in\mathbb{Z}_{>0}}$. □

References

[1] R. Arratia, L. Gordon, Tutorial on large deviations for the binomial distribution, *Bulletin of Mathematical Biology* **51** (1989), pp. 125–131.

[2] Airat Bekmetjev, Graham Brightwell, Andrzej Czygrinow, and Glenn Hurlbert, Thresholds for families of multisets, with an application to graph pebbling, *Discrete Mathematics* **269** (2003), pp. 21–34.

[3] Andrzej Czygrinow, Nancy Eaton, Glenn Hurlbert, and P. Mark Kayll, On pebbling threshold functions for graph sequences, *Discrete Mathematics* **247** (2002), pp. 93–105.

[4] Andrzej Czygrinow and Glenn H. Hurlbert, On the pebbling threshold of paths and the pebbling threshold spectrum, *Discrete Mathematics* **308** (2008), pp. 3297–3307.

[5] Glenn Hurlbert, Graph pebbling, in *Handbook of graph theory*, edited by Jonathan L. Gross, Jay Yellen and Ping Zhang, CRC Press, Boca Raton, 2nd ed., 2014.

[6] David Moews, An exact pebbling threshold for the path, ver. 0.2. `<http://djm.cc/dmoews/pebbling-path-threshold-0.2.pdf>`.