An exact pebbling threshold for the path

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Abstract. Given a distribution of pebbles on the vertices of a graph, say that we can *pebble* a vertex if a pebble is left on it after some sequence of moves, each of which takes two pebbles from some vertex and places one on an adjacent vertex. A distribution is *solvable* if all vertices are pebblable. Using new methods, we improve earlier estimates by showing that the *pebbling threshold* of the sequence of paths, i.e., the total number of pebbles for which random distributions change from being almost never solvable to almost always solvable, is $\Theta(2^{\sqrt{\log_2 n}} n/\sqrt{\log_2 n})$, where n is the number of vertices of a path in the sequence.

Introduction

In the mathematical game of *pebbling*, one starts with a *distribution* on a graph assigning a nonnegative integral number of pebbles to each vertex of the graph. A *pebbling move* consists of taking two pebbles away from a vertex with at least two pebbles and adding one pebble to any adjacent vertex. A vertex is *pebblable* for a given distribution if there is some sequence of pebbling moves starting at the distribution and finishing with at least one pebble on that vertex, and a distribution is *solvable* if each vertex is pebblable for that distribution. In [4], Czygrinow et al. introduce the *pebbling threshold* for a sequence of graphs, which, roughly speaking, is the number of pebbles at which a random distribution with that number of pebbles changes from being almost always unsolvable to almost always solvable. In [7, RP15], Hurlbert asks for the pebbling threshold of the sequence of paths. In this paper, we determine that it is $\Theta(2^{\sqrt{\log_2 n}} n/\sqrt{\log_2 n})$, where n is the number of vertices of a path in the sequence. This makes more precise the estimates of [1], [4], [5], and [9]. We also prove some subsidiary results that may be of interest. To do this, we first (§1) define uniform and geometric probability distributions over multisets and the corresponding thresholds of sequences of families of multisets. In §2, we improve some estimates used in [1], and in §3, we use this to relate the uniform and geometric thresholds. In §4, we begin to compute the pebbling threshold of the sequence of paths, relating it to a certain hypoexponential distribution. In §5, we estimate asymptotically some probabilities of this distribution and finally complete the computation of the pebbling threshold of the sequence of paths in §6.

1 Definitions and notation

We use $\mathbb{Z}$, $\mathbb{Z}_{>0}$, ω, $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{R}_{>0}$, $\mathbb{C}$, $\mathbb{P}$, $\mathbb{E}$, and Var to denote the integers, the positive integers, the nonnegative integers, the reals, the nonnegative reals, the positive reals, the complex numbers, probability, expectation, and variance. ι is the imaginary unit. For $x \in \mathbb{R}$, $\lfloor x \rfloor$ will be the largest integer no larger than x, $\lceil x \rceil$ the smallest integer no smaller than x, and $\{x\}$ the fractional part of x, $\{x\} := x - \lfloor x \rfloor$; for $x \in \mathbb{R}_{>0}$, $\log x$ will be the natural logarithm of x, and $\log_2 x$ will be the logarithm of x to the base 2, $\log_2 x := \log x/(\log 2)$. The cardinality of a set S is written $\#S$. For nonnegative integers $k \leq n$, $\binom{n}{k}$ denotes the binomial coefficient $n!/(k!\,(n-k)!)$. A^B will be the set of functions from B to A. If f, $g \in A^B$ and A is ordered, we define $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in B$; similarly, if A has an addition operation, we define $f + g \in A^B$ by $(f+g)(x) = f(x) + g(x)$ for all $x \in B$. We call the elements of ω^B *multisets* and write 0 for the empty multiset, i.e., the element of ω^B whose value is always 0.

For any $b \in B$, we take $e_b \in \omega^B$ to have $e_b(c) = 1$ if $b = c$, $e_b(c) = 0$ if $b \neq c$. For S a subset of some ω^B, we let ∂S be $\{f \in \omega^B \mid f + e_b \in S \text{ for some } b \in B\}$, and if B is finite and $T \in \omega$, we take $[S]_T$ to be $\{f \in S \mid \sum_{x \in B} f(x) = T\}$.

The geometric distribution on ω with parameter $0 < p \leq 1$ is the probability measure χ with $\chi(\{n\}) = p(1-p)^n$, where we take $0^0 = 1$. For B finite and nonempty, if $T \in \omega$, we let μ_T be the probability measure on ω^B that is uniform on $[\omega^B]_T$ and zero elsewhere, and if $T \in \mathbb{R}_{\geq 0}$, we let ν_T be the probability measure on ω^B which is the product of $\#B$ copies of the geometric distribution with parameter $(1 + (T/\#B))^{-1}$.

If $(M_i)_{i \in \omega}$ is a sequence such that $\emptyset \neq M_i \subseteq \omega^{B_i}$ for each i, where each B_i is a nonempty finite set, and each M_i is an *upper set* ($x \in M_i$ and $x \leq y$ implies that $y \in M_i$), then we define the *uniform threshold* of $(M_i)_{i \in \omega}$ to be the sequence $(T_i)_{i \in \omega}$, where each $T_i \in \omega$ is minimal such that $\mu_{T_i}(M_i) \geq \frac{1}{2}$. If in addition $0 \notin M_i$ for each i, we define the *geometric threshold* of $(M_i)_{i \in \omega}$ to be the sequence $(T_i)_{i \in \omega}$, where each $T_i \in \mathbb{R}_{>0}$ is the unique T_i satisfying $\nu_{T_i}(M_i) = \frac{1}{2}$. (We will see in §2 and §3 that these definitions are sensible.)

2 Improved thresholds for multisets

The main result of this section is the following, which can be used to improve estimates like those used in [1, Theorem 1.5].

Theorem 1. *If B is nonempty and finite, $T \in \omega$, $x \in \mathbb{R}_{\geq 0}$, $S \subseteq \omega^B$, and $\mu_{T+1}(S) \geq x/(T+1+x)$, then $\mu_T(\partial S) \geq x/(T+x)$. (Here we take $0/0 = 0$ in the case where $T = x = 0$.)*

Lemma 2. *Given $x \in \mathbb{R}_{\geq 0}$, $r \in \omega$ and positive integers t, n and $d_0 \geq d_1 \geq \cdots \geq d_{r-1}$ with $t \geq r$ and (if d_0 exists) $n > d_0$, let*

$$p := \sum_{0 \leq i < r} \binom{t-i-1+d_i}{t-i} \bigg/ \binom{t-1+n}{t},$$

$$q := \sum_{0 \le i < r} \binom{t-i-2+d_i}{t-i-1} \Big/ \binom{t-2+n}{t-1}.$$

Then $0 \le p < 1$ and, if $p \ge x/(t+x)$, also $q \ge x/(t-1+x)$ (where we take $0/0 = 0$ in the case $t = 1$ and $x = 0$.)

Proof. We first prove that $0 \le p < 1$. This is clear if $r = 0$; otherwise

$$\begin{aligned} \sum_{0 \le i < r} \binom{t-i-1+d_i}{t-i} &< \sum_{0 \le j \le t} \binom{j+d_0-1}{j} \\ &= \binom{t+d_0}{t} \\ &\le \binom{t-1+n}{t}, \end{aligned}$$

so $p < 1$.

We can give an equivalent condition for $p \ge x/(t+x)$ implying that $q \ge x/(t-1+x)$ by observing that it will do to prove this for the maximal x for which $p \ge x/(t+x)$, which is $x := pt/(1-p)$. In this case, $q \ge x/(t-1+x)$ reduces to $(t-1+p)q \ge pt$.

We now induce on t to prove this. If $t = 1$, we have two cases. If $r = 0$, we must have $p = 0$, making the result trivial; if $r = 1$, $q = 1$, making the result again trivial. Otherwise, let $t > 1$. If $r = 0$, we again have $p = 0$, making the result trivial. If $r > 0$, set

$$\begin{aligned} \alpha &:= \sum_{1 \le i < r} \binom{t-i-1+d_i}{t-i}, \\ \beta &:= \sum_{1 \le i < r} \binom{t-i-2+d_i}{t-i-1}, \\ p' &:= \alpha \Big/ \binom{t-1+d_0}{t-1}, \qquad q' := \beta \Big/ \binom{t-2+d_0}{t-2}. \end{aligned}$$

By the induction hypothesis, we can assume that $(t-2+p')q' \ge p'(t-1)$; since $0 \le p' < 1$, this implies that $p' \le q'$. We need to show that $(t-1+p)q \ge pt$, which, after clearing denominators, is equivalent to

$$\begin{aligned} &\left((t-1)\binom{t-1+n}{t} + p'\binom{t-1+d_0}{t-1} + \binom{t-1+d_0}{t}\right) \cdot \\ &\left(q'\binom{t-2+d_0}{t-2} + \binom{t-2+d_0}{t-1}\right) \\ \ge\ &\binom{t-2+n}{t-1}\left(p'\binom{t-1+d_0}{t-1} + \binom{t-1+d_0}{t}\right)t. \qquad (1) \end{aligned}$$

Taking a forward first difference of (1) with respect to n gives

$$(t-1)\binom{t-1+n}{t-1}\left(q'\binom{t-2+d_0}{t-2} + \binom{t-2+d_0}{t-1}\right)$$

$$\geq \quad \binom{t-2+n}{t-2}\left(p'\binom{t-1+d_0}{t-1}+\binom{t-1+d_0}{t}\right)t,$$

which, using $\binom{t-1+n}{t-1} = \frac{t+n-1}{t-1}\binom{t-2+n}{t-2}$ and removing the common factor $\binom{t-2+n}{t-2}$, can be rewritten as

$$(t+n-1)\left(q'\binom{t-2+d_0}{t-2}+\binom{t-2+d_0}{t-1}\right) \geq \left(p'\binom{t-1+d_0}{t-1}+\binom{t-1+d_0}{t}\right)t.$$

Since we assume $n > d_0$, it's enough to prove this when $n = d_0+1$. Using $\binom{t-1+d_0}{t} = \frac{t-1+d_0}{t}\binom{t-2+d_0}{t-1}$, this simplifies to

$$q'(t+d_0)\binom{t-2+d_0}{t-2}+\binom{t-2+d_0}{t-1} \geq p'\binom{t-1+d_0}{t-1}t,$$

and since $p' \leq q'$ and $p' \leq 1$, it's enough to show that

$$(t+d_0)\binom{t-2+d_0}{t-2}+\binom{t-2+d_0}{t-1} = \binom{t-1+d_0}{t-1}t;$$

this can be rewritten as

$$\begin{aligned} & \binom{t-2+d_0}{t-2}+(t+d_0-1)\binom{t-2+d_0}{t-2}+\binom{t-2+d_0}{t-1} \\ = \quad & \binom{t-1+d_0}{t-1}(t-1)+\binom{t-1+d_0}{t-1}, \end{aligned}$$

which follows from $\frac{t+d_0-1}{t-1}\binom{t-2+d_0}{t-2} = \binom{t-1+d_0}{t-1}$ and $\binom{t-2+d_0}{t-2}+\binom{t-2+d_0}{t-1} = \binom{t-1+d_0}{t-1}$.

It remains to prove (1) when $n = d_0+1$. In this case, looking at a portion of the left-hand side of (1),

$$\begin{aligned} & q'\binom{t-2+d_0}{t-2}\left((t-1)\binom{t+d_0}{t}+p'\binom{t-1+d_0}{t-1}\right) \\ = \quad & q'\binom{t-2+d_0}{t-2}\left((t-2+p')\binom{t-1+d_0}{t-1}+\binom{t+d_0}{t}+\right. \\ & \left.(t-2)\left(\binom{t+d_0}{t}-\binom{t-1+d_0}{t-1}\right)\right) \\ \geq \quad & p'(t-1)\binom{t-2+d_0}{t-2}\binom{t-1+d_0}{t-1}+ \\ & q'\binom{t-2+d_0}{t-2}\left(\binom{t+d_0}{t}+(t-2)\binom{t-1+d_0}{t}\right). \end{aligned}$$

It's therefore enough to prove (1) with the left-hand side replaced by

$$\begin{aligned} & p'(t-1)\binom{t-2+d_0}{t-2}\binom{t-1+d_0}{t-1}+ \\ & q'\binom{t-2+d_0}{t-2}\left(\binom{t+d_0}{t}+(t-2)\binom{t-1+d_0}{t}\right)+ \end{aligned}$$

$$q'\binom{t-2+d_0}{t-2}\binom{t-1+d_0}{t}+$$
$$\left((t-1)\binom{t+d_0}{t}+p'\binom{t-1+d_0}{t-1}+\binom{t-1+d_0}{t}\right)\binom{t-2+d_0}{t-1}.$$

Using $p' \leq q'$ and separating terms which involve and do not involve p', it will do to show that

$$\begin{aligned}
&(t-1)\binom{t-2+d_0}{t-2}\binom{t-1+d_0}{t-1}+\\
&\binom{t-2+d_0}{t-2}\left(\binom{t+d_0}{t}+(t-2)\binom{t-1+d_0}{t}\right)+\\
&\binom{t-2+d_0}{t-2}\binom{t-1+d_0}{t}+\binom{t-1+d_0}{t-1}\binom{t-2+d_0}{t-1}\\
=\;&\binom{t-1+d_0}{t-1}^2 t \qquad (2)
\end{aligned}$$

and

$$\left((t-1)\binom{t+d_0}{t}+\binom{t-1+d_0}{t}\right)\binom{t-2+d_0}{t-1}=\binom{t-1+d_0}{t-1}\binom{t-1+d_0}{t}t. \quad (3)$$

Using $\binom{t-1+d_0}{t-1}=\binom{t-2+d_0}{t-1}+\binom{t-2+d_0}{t-2}$, (3) will follow from

$$\begin{aligned}
&(t-1)\binom{t+d_0}{t}\binom{t-2+d_0}{t-1}\\
=\;&\binom{t-2+d_0}{t-1}\binom{t-1+d_0}{t}(t-1)+\binom{t-2+d_0}{t-2}\binom{t-1+d_0}{t}t
\end{aligned}$$

and then, using $\binom{t+d_0}{t}=\binom{t-1+d_0}{t}+\binom{t-1+d_0}{t-1}$, from

$$(t-1)\binom{t-1+d_0}{t-1}\binom{t-2+d_0}{t-1}=\binom{t-2+d_0}{t-2}\binom{t-1+d_0}{t}t,$$

which follows from $\binom{t-1+d_0}{t}=\frac{t-1+d_0}{t}\binom{t-2+d_0}{t-1}$ and $\binom{t-1+d_0}{t-1}=\frac{t-1+d_0}{t-1}\binom{t-2+d_0}{t-2}$.

For (2), expanding the left-hand side, using the identity $\binom{t-1+d_0}{t-1}+\binom{t-1+d_0}{t}=\binom{t+d_0}{t}$, and collecting terms gives

$$t\binom{t-2+d_0}{t-2}\binom{t+d_0}{t}+\binom{t-1+d_0}{t-1}\binom{t-2+d_0}{t-1}=\binom{t-1+d_0}{t-1}^2 t$$

which, after using $\binom{t+d_0}{t}=\frac{t+d_0}{t}\binom{t+d_0-1}{t-1}$ and $\binom{t-1+d_0}{t-1}=\frac{t-1+d_0}{t-1}\binom{t-2+d_0}{t-2}$, can be rearranged to

$$\left(t\frac{t-1+d_0}{t-1}-(t+d_0)\right)\binom{t-1+d_0}{t-1}\binom{t-2+d_0}{t-2}=\binom{t-1+d_0}{t-1}\binom{t-2+d_0}{t-1},$$

or

$$\frac{d_0}{t-1}\binom{t-1+d_0}{t-1}\binom{t-2+d_0}{t-2} = \binom{t-1+d_0}{t-1}\binom{t-2+d_0}{t-1},$$

which follows from the definition of the binomial coefficient. □

We now prove Theorem 1.

Proof. Set $n := \#B$, and intersect S with $[\omega^B]_{T+1}$ if necessary so we can assume that $S \subseteq [\omega^B]_{T+1}$. If $S = [\omega^B]_{T+1}$, then $\partial S = [\omega^B]_T$, so $\mu_T(\partial S) = 1$ and the result is obvious; if S is empty, then we must have $x = 0$ so the result is again obvious. Otherwise, set $t := T + 1$. Since we have $\emptyset \subsetneq S \subsetneq [\omega^B]_t$, we must have $0 < \#S < \#[\omega^B]_t = \binom{t+n-1}{t}$. By the theorem in [3], there is then a representation

$$\#S = \sum_{0 \le i < r} \binom{t-i-1+d_i}{t-i}$$

with $t \ge r > 0$, $d_0 \ge d_1 \ge \cdots \ge d_{r-1} > 0$, and

$$\#(\partial S) \ge \sum_{0 \le i < r} \binom{t-i-2+d_i}{t-i-1}.$$

Since $\#S < \binom{t+n-1}{t}$ we must then have $n > d_0$, so we can apply Lemma 2. □

Theorem 3. *If B is finite and nonempty and $M \subseteq \omega^B$ is an upper set, $T \ge U \in \omega$, $x \in \mathbb{R}_{\ge 0}$, and $\mu_T(M) \le T/(T+x)$, then $\mu_U(M) \le U/(U+x)$. (Here we take $0/0 = 1$ if $U = x = 0$.)*

Proof. If $x = 0$, the result is obvious. Otherwise, since $\mu_T(M) \le T/(T+x)$, $\mu_T(\omega^B \setminus M) \ge x/(T+x)$, so by repeated application of Theorem 1, $\mu_U(\partial^{T-U}(\omega^B \setminus M)) \ge x/(U+x)$. However, if $v \in \partial^{T-U}(\omega^B \setminus M)$, we have $v \le w$ for some $w \in \omega^B \setminus M$, so we cannot have $v \in M$ since then, as M is an upper set, w would also be in M. Therefore, $\partial^{T-U}(\omega^B \setminus M)$ is disjoint from M so $\mu_U(M) \le \mu_U(\omega^B \setminus \partial^{T-U}(\omega^B \setminus M)) \le 1 - (x/(U+x)) = U/(U+x)$. □

Theorem 4. *If B is finite and nonempty and $M \subseteq \omega^B$ is an upper set, T and U are positive integers with $T \ge U$, $x \in \mathbb{R}_{\ge 0}$, and $\mu_U(M) \ge U/(U+x)$, then $\mu_T(M) \ge T/(T+x)$.*

Proof. Replace x by $x+\epsilon$, apply the contrapositive of Theorem 3, and let $\epsilon \to 0$ from above. □

Theorem 5. *If B is finite and nonempty, $M \subseteq \omega^B$ is an upper set, $U \in \omega$, and $\mu_U(M) > 0$, then $\mu_U(M) \le \mu_{U+1}(M)$ and $\lim_{i\to\infty} \mu_i(M) = 1$. Also, if $0 < \mu_U(M) < 1$, then $\mu_U(M) < \mu_{U+1}(M)$.*

Proof. If $U = 0$, then we must have $0 \in M$ so $M = \omega^B$ and $\mu_i(M) = 1$ for all i. Otherwise, we can apply Theorem 4 with some value of x to conclude that $\lim_{i\to\infty} \mu_i(M) = 1$. Applying it with the minimum possible value of x allows us to conclude that $\mu_U(M) \le \mu_{U+1}(M)$, or $\mu_U(M) < \mu_{U+1}(M)$ in the case where $x > 0$, i.e., $\mu_U(M) < 1$. □

Theorem 6. *If B is finite and nonempty and $M \subseteq \omega^B$ is an upper set, then the sequence $(\mu_0(M), \mu_1(M), \ldots)$ is either:*

1. $(0, 0, \ldots)$*, if M is empty.*
2. $(0, 0, \ldots, 0, r_0, \ldots, r_{N-1}, 1, 1, \ldots)$*, for some $N \in \omega$ and $0 < r_0 < \cdots < r_{N-1} < 1$.*
3. $(0, 0, \ldots, 0, r_0, r_1, \ldots)$*, for some strictly increasing sequence $(r_i)_{i \in \omega}$ of positive real numbers with $\lim_{i \to \infty} r_i = 1$.*

(The initial sequence of zeroes may be empty in cases 2 and 3.)

Proof. Apply Theorem 5 repeatedly. □

Theorem 6 shows that the definition of uniform threshold makes sense.

3 Uniform and geometric thresholds

In this section, we show that the definition of geometric threshold is sensible; also, if both uniform and geometric thresholds of a sequence of multiset families are defined, the geometric threshold approaches infinity, and the number of elements in the base sets of the multiset families approaches infinity, then the two thresholds have asymptotic ratio 1.

Theorem 7. *If B is finite and nonempty and $M \subseteq \omega^B$ is an upper set, then the function $x \mapsto \nu_x(M)$ on $\mathbb{R}_{\geq 0}$ is either:*

1. *Identically 0, if M is empty.*
2. *Identically 1, if $M = \omega^B$.*
3. *Strictly increasing and continuous with $\nu_0(M) = 0$ and $\lim_{x \to \infty} \nu_x(M) = 1$, otherwise.*

Proof. If M is empty or ω^B, this is obvious. Assume otherwise. Since $M \neq \omega^B$, $0 \notin M$ so $\nu_0(M) = 0$ and, since ν_x always assigns positive probability to 0, $\nu_x(M) < 1$ for all x.

If $(G_b)_{b \in B}$ is an i.i.d. family of geometric random variables, then, conditioned on $\sum_b G_b = T$, the function $b \mapsto G_b$ is uniform on $[\omega^B]_T$. Therefore, $\nu_x(M) = \mathbb{E}(\mu_{N_x}(M))$, where the random variable N_x is the sum of $\#B$ i.i.d. geometric random variables with parameter $(1 + (x/\#B))^{-1}$. Given a geometric random variable G_p with parameter p, we can realize G_p as the smallest i for which an i.i.d. sequence of random variables $U_0, U_1, \ldots$ uniform on $[0, 1]$ has $U_i < p$. This lets us realize G_p and G_q $(p < q)$ on the same probability space with $G_p = G_q + \Xi$, where Ξ is a nonnegative integral random variable which, conditioned on G_q, always assigns positive probability to each nonnegative integer; in fact, $\mathbb{P}(\Xi = 0 \mid G_q)$ is always p/q. Summing $\#B$ independent copies of this, we can realize N_x and N_y $(x < y)$ on the same probability space with $N_y = N_x + \Xi'$, where Ξ' is a nonnegative integral random variable which, conditioned on N_x, always assigns positive probability to each nonnegative integer; also, $\mathbb{P}(\Xi' = 0 \mid N_x)$ is always $((x + \#B)/(y + \#B))^{\#B}$. But then

$$\begin{aligned} \nu_y(M) &= \mathbb{E}(\mu_{N_y}(M)) \\ &= \mathbb{E}(\mathbb{E}(\mu_{N_y}(M) \mid N_x)) \\ &= \mathbb{E}(\mathbb{E}(\mu_{N_x+\Xi'}(M) \mid N_x)). \end{aligned}$$

By Theorem 6, $\mathbb{E}(\mu_{N_x+\Xi'}(M) \mid N_x) \geq \mu_{N_x}(M)$. Also, by Theorem 6 and the above property of Ξ', $\mathbb{E}(\mu_{N_x+\Xi'}(M) \mid N_x) > \mu_{N_x}(M)$ whenever $\mu_{N_x}(M) < 1$. Since $\nu_x(M) < 1$ we must have $\mathbb{P}(\mu_{N_x}(M) < 1) > 0$, so $\mathbb{E}(\mathbb{E}(\mu_{N_x+\Xi'}(M) \mid N_x)) > \mathbb{E}(\mu_{N_x}(M)) = \nu_x(M)$. This proves that $x \mapsto \nu_x(M)$ is strictly increasing. To show that it is continuous, observe that

$$\begin{aligned} \mathbb{E}(\mu_{N_x+\Xi'}(M) \mid N_x) &\leq \mathbb{P}(\Xi' = 0 \mid N_x)\mu_{N_x}(M) + 1 - \mathbb{P}(\Xi' = 0 \mid N_x) \\ &= (\frac{x + \#B}{y + \#B})^{\#B}\mu_{N_x}(M) + 1 - (\frac{x + \#B}{y + \#B})^{\#B} \\ &= \mu_{N_x}(M) + (1 - (\frac{x + \#B}{y + \#B})^{\#B})(1 - \mu_{N_x}(M)), \end{aligned}$$

and, taking expectations,

$$\nu_y(M) \leq \nu_x(M) + (1 - (\frac{x + \#B}{y + \#B})^{\#B})(1 - \nu_x(M))$$

so

$$0 < \nu_y(M) - \nu_x(M) \leq 1 - (\frac{x + \#B}{y + \#B})^{\#B},$$

which implies that $x \mapsto \nu_x(M)$ is continuous.

Using Theorem 6 again, to prove that $\lim_{x\to\infty} \nu_x(M) = 1$, it will do to prove that $\lim_{x\to\infty} \mathbb{P}(N_x \leq j) = 0$ for each fixed $j \in \omega$. This is so because

$$\begin{aligned} \mathbb{P}(N_x \leq j) &\leq \mathbb{P}(G_{(1+(x/\#B))^{-1}} \leq j) \\ &\leq (j + 1)(1 + (x/\#B))^{-1} \\ &\to 0 \text{ as } x \to \infty. \end{aligned}$$

□

Theorem 7 shows that the definition of geometric threshold makes sense.

Theorem 8. *If B is finite and nonempty, $M \subseteq \omega^B$ is an upper set, $T \geq U \in \omega$, and $\mu_U(M) \geq \frac{1}{2}$, then $\mu_T(M) \geq T/(T + U)$, where here we take $0/0 = 1$ if $T = U = 0$.*

Proof. If $U = 0$, M must contain 0, so $M = \omega^B$ and the result is obvious. Otherwise, set $x := U$ and use Theorem 4. □

Theorem 9. *If B is finite and nonempty, $\emptyset \neq M \subseteq \omega^B$ is an upper set, $U < T \in \omega$, and T is minimal such that $\mu_T(M) \geq \frac{1}{2}$, then $\mu_U(M) \leq U/(T+U-1)$, where here we take $0/0 = 0$ if $U = 0$ and $T = 1$.*

Proof. Since $T \neq 0$, $0 \notin M$, so $\mu_0(M) = 0$; this proves the result if $U = 0$. Otherwise, set $x := T - 1$ and use Theorem 3 with T decreased by 1. □

Theorem 10. *If B is finite and nonempty, $\emptyset \neq M \subseteq \omega^B$ is an upper set with $0 \notin M$, $T \in \omega$ is minimal such that $\mu_T(M) \geq \frac{1}{2}$, T' is the unique positive real number such that $\nu_{T'}(M) = \frac{1}{2}$, $S := \sqrt{T' + (T'^2/\#B)}$, and θ is a real number with $\sqrt{2} < \theta < T'/S$, then*

$$\lceil T' - \theta S\rceil(1 - \frac{2}{\theta^2}) \leq T \leq 1 + \lfloor T' + \theta S\rfloor(1 + \frac{2}{\theta^2 - 2}). \tag{4}$$

Proof. If G_p is a geometric random variable with parameter p, then $\mathbb{E}G_p = p^{-1} - 1$ and $\mathrm{Var}\ G_p = p^{-2} - p^{-1}$. Therefore, if $N_{T'}$ is the sum of $\#B$ i.i.d. geometric random variables with parameter $(1 + (T'/\#B))^{-1}$, then $\mathbb{E}N_{T'} = T'$ and $\mathrm{Var}\ N_{T'} = T' + (T'^2/\#B) = S^2$. If $V := \lceil T' - \theta S\rceil$ and $W := \lfloor T' + \theta S\rfloor$, it follows from Chebyshev's inequality that

$$\mathbb{P}(V \leq N_{T'} \leq W) \geq 1 - \frac{1}{\theta^2},$$

so since $\mu_i(M)$ is nondecreasing with i,

$$\frac{1}{2} = \nu_{T'}(M) = \mathbb{E}(\mu_{N_{T'}}(M)) \geq (1 - \frac{1}{\theta^2})\mu_V(M) \tag{5}$$

and

$$\frac{1}{2} = \nu_{T'}(M) = \mathbb{E}(\mu_{N_{T'}}(M)) \leq (1 - \frac{1}{\theta^2})\mu_W(M) + \frac{1}{\theta^2}. \tag{6}$$

If $V \leq T$, then the left-hand inequality of (4) is satisfied. If $T < V$, then by Theorem 8 and (5),

$$\frac{\frac{1}{2}}{1 - \theta^{-2}} \geq \mu_V(M) \geq \frac{V}{T+V},$$

which, after rearrangement, gives the left-hand inequality of (4). If $T \leq W$, then the right-hand inequality of (4) is satisfied. If $W < T$, then by Theorem 9 and (6),

$$\frac{\frac{1}{2} - \theta^{-2}}{1 - \theta^{-2}} \leq \mu_W(M) \leq \frac{W}{T+W-1},$$

which, after rearrangement, gives the right-hand inequality of (4). □

Theorem 11. *If $(M_i)_{i\in\omega}$ is a sequence such that $\emptyset \neq M_i \subseteq \omega^{B_i}$ for each i, each B_i is a nonempty finite set, each M_i is an upper set, $0 \notin M_i$ for all i, $(T_i)_{i\in\omega}$ and $(T'_i)_{i\in\omega}$ are the uniform and geometric thresholds of $(M_i)_{i\in\omega}$, and $\#B_i$ and T'_i both approach infinity as $i \to \infty$, then $T_i/T'_i \to 1$ as $i \to \infty$.*

Proof. For sufficiently large i, use Theorem 10 on each $B := B_i$, $M := M_i$, $T := T_i$, and $T' := T'_i$, setting

$$\theta := (\frac{T'}{S})^{1/3} = (\frac{T'}{\sqrt{T' + (T'^2/\#B)}})^{1/3}$$

and observing that since $T'/S \to \infty$ as $i \to \infty$, θ and $T'/(\theta S)$ both approach ∞ as $i \to \infty$. □

4 The threshold of the sequence of paths, I

We now begin to compute the pebbling threshold of the sequence of n-paths, where the n-path has n vertices, 1, ..., n, and edges between vertices i and $i+1$ for all $i = 1, \ldots, n-1$. Because of Theorem 11 and [1], it will do to find the geometric threshold of the sequence of families of solvable distributions of the n-paths. Therefore, fix some positive n, and suppose that, for some parameter $0 < p < 1$, we have i.i.d. geometric random variables $(Z_i)_{i \in \mathbb{Z}_{>0}}$ with parameter p, and that, for $i = 1, \ldots, n$, we place Z_i pebbles on each vertex i. If $r := (2 \log n)/p$, then $\mathbb{P}(Z_i \geq r) = \mathbb{P}(Z_i \geq \lceil r \rceil) = (1-p)^{\lceil r \rceil} \leq e^{-pr} = n^{-2}$, so with probability at least $1 - n^{-1}$, $Z_i < r$ for all $i = 1, \ldots, n$. Let L be a positive integer such that $n \geq 2L+1$. Now, for each $L+1 \leq i \leq n-L$, i will be unpebblable iff $Z_i = 0$, $\sum_{1 \leq j < i} Z_j 2^{-(i-j)} < 1$, and $\sum_{i < j \leq n} Z_j 2^{-(j-i)} < 1$, so, given that $Z_j < r$ for all $j = 1, \ldots, n$, it is sufficient for unpebblability that $Z_i = 0$, $\sum_{1 \leq k \leq L} Z_{i-k} 2^{-k} < 1 - (r/2^L)$, and $\sum_{1 \leq k \leq L} Z_{i+k} 2^{-k} < 1 - (r/2^L)$. If we pick $i = L+1$, $L+1+(2L+1)$, ..., $L+1+(\lfloor \frac{n}{2L+1} \rfloor - 1)(2L+1)$, then, since the Z_i's are i.i.d., the probability that the distribution is unsolvable will be at least

$$-\frac{1}{n} + 1 - (1 - pq^2)^{\lfloor n/(2L+1) \rfloor}, \quad \text{where} \quad q := \mathbb{P}(\frac{Z_1}{2} + \cdots + \frac{Z_L}{2^L} < 1 - \frac{r}{2^L}),$$

so the probability that it is solvable will be no more than

$$\frac{1}{n} + (1 - pq^2)^{\lfloor n/(2L+1) \rfloor} \leq \frac{1}{n} + \exp -pq^2 \lfloor \frac{n}{2L+1} \rfloor.$$

It is easy to see that $\sum_{i>0} Z_i/2^i$ converges a.s., and then

$$q \geq q' := \mathbb{P}(\sum_{i>0} \frac{Z_i}{2^i} < 1 - \frac{r}{2^L}).$$

If we let $(W_i)_{i \in \mathbb{Z}_{>0}}$ be an i.i.d. family of standard exponential random variables, with $\mathbb{P}(W_i \geq x) = e^{-x}$ for each i, then we can realize Z_i by letting each Z_i be $\lfloor W_i/\lambda \rfloor$, where $\lambda := -\log(1-p)$. Since $\sum_{i>0} W_i/2^i$ also converges a.s., we have

$$q' \quad = \quad \mathbb{P}(\sum_{i>0} \frac{\lfloor W_i/\lambda \rfloor}{2^i} < 1 - \frac{r}{2^L})$$

$$\begin{aligned}\geq \quad q'' &:= \mathbb{P}(\sum_{i>0} \frac{W_i/\lambda}{2^i} < 1 - \frac{r}{2^L}) \\ &= \mathbb{P}(Y_\infty < 2\lambda(1 - \frac{r}{2^L})),\end{aligned}$$

where we have set

$$Y_\infty := W_1 + \frac{W_2}{2} + \frac{W_3}{4} + \cdots = \sum_{i\geq 0} \frac{W_{i+1}}{2^i}.$$

The probability that the distribution is solvable is then no more than

$$\frac{1}{n} + \exp -pq''^2 \lfloor \frac{n}{2L+1} \rfloor.$$

To estimate this, we must estimate the probability that Y_∞ is below a small threshold.

5 The asymptotics of Y_∞

Let $n \in \mathbb{Z}_{>0}$, $Y_n := W_1 + \cdots + (W_n/2^{n-1})$, and, for $i = 0, \ldots, n-1$, let

$$R_{i,n}(x) := \prod_{0\leq j\leq n-1, j\neq i} \frac{2^j - x}{2^j - 2^i}$$

be the degree $n-1$ polynomial which is 0 at 2^j, $j = 0, \ldots, n-1$, $j \neq i$, and 1 at 2^i. Then for all $x \in \mathbb{R}_{\geq 0}$ [6, §I.13, ex. 12],

$$\mathbb{P}(Y_n \leq x) = \sum_{0\leq i\leq n-1} (1 - e^{-2^i x}) R_{i,n}(0).$$

For $0 \leq k < n$, $\sum_{0\leq i\leq n-1} 2^{ik} R_{i,n}(x)$ is a polynomial of degree at most $n-1$ which is 2^{ik} at 2^i, $i = 0, \ldots, n-1$; it must therefore be x^k, so

$$\sum_{0\leq i\leq n-1} 2^{ik} R_{i,n}(0) = 0, \qquad 1 \leq k \leq n-1.$$

Therefore, if we write, for any $c \in \omega$,

$$e_c(x) := e^{-x} - \sum_{0\leq k\leq c} \frac{(-x)^k}{k!},$$

then

$$\mathbb{P}(Y_n \leq x) = - \sum_{0\leq i\leq n-1} e_c(2^i x) R_{i,n}(0), \qquad 0 \leq c \leq n-1.$$

Let

$$\mathcal{N} := \prod_{j\geq 1} \frac{2^j}{2^j - 1}, \qquad F_c(x) := \mathcal{N} \sum_{i\geq 0} \frac{(-1)^{i+1} e_c(2^i x)}{(2^1 - 1)\cdots(2^i - 1)}. \tag{7}$$

We have

$$\lim_{n\to\infty} \mathbb{P}(Y_n \le x) = \lim_{n\to\infty} \sum_{0\le i\le n-1} V_{c,i,n}(x),$$

where

$$\begin{aligned} V_{c,i,n}(x) &:= -e_c(2^i x) \prod_{0\le j\le n-1, j\ne i} \frac{2^j}{2^j - 2^i} \\ &= \left(\prod_{1\le j\le n-1-i} \frac{2^j}{2^j - 1} \right) (-1)^{i+1} e_c(2^i x) \prod_{1\le k\le i} \frac{1}{2^k - 1}. \end{aligned}$$

Then for all n,

$$|V_{c,i,n}(x)| \le U_{c,i}(x) := \left(\prod_{j\ge 1} \frac{2^j}{2^j - 1} \right) |e_c(2^i x)| \prod_{1\le k\le i} \frac{1}{2^k - 1},$$

and $\sum_{i\ge 0} U_{c,i}(x)$ converges, so by the dominated convergence theorem,

$$\begin{aligned} \lim_{n\to\infty} \mathbb{P}(Y_n \le x) &= \sum_{i\ge 0} \lim_{n\to\infty} V_{c,i,n}(x) \\ &= F_c(x). \end{aligned}$$

Since $Y_n \to Y_\infty$ a.s., Y_n also converges to Y_∞ in distribution, so $\mathbb{P}(Y_\infty \le x) = F_c(x)$ at every continuity point of $\mathbb{P}(Y_\infty \le x)$. Since $F_c(x)$ is continuous, it must equal $\mathbb{P}(Y_\infty \le x)$ everywhere.

Let c and x be positive. If, for complex z, we define $S_c(z)$ by

$$e^{cz} = \sum_{0\le k\le c} \frac{(cz)^k}{k!} + \frac{(cz)^c}{c!} S_c(z),$$

and z has negative real part, then [2, Lemmas 1 and 3]

$$|S_c(z) - \frac{z}{1-z}| \le \frac{224}{c}. \tag{8}$$

Substituting (8) and

$$e_c(2^i x) = \frac{(-2^i x)^c}{c!} S_c(-\frac{2^i x}{c})$$

into (7) gives

$$\begin{aligned} F_c(x) &= \mathcal{N}^2 \sum_{i\ge 0} (-1)^{i+1} 2^{-i(i+1)/2} e_c(2^i x) \prod_{j>i} (1 - 2^{-j}) \\ &= \mathcal{N}^2 \sum_{i\ge 0} (-1)^i \frac{(-2^i x)^c}{c!} 2^{-i(i+1)/2} (\frac{2^i x}{c + 2^i x} + O(\frac{1}{c}))(1 + O(2^{-i})). \end{aligned}$$

In this section, by $O(f(\cdots))$ for some function f, we mean any quantity for which there is an absolute constant $\mathcal{L}$ so that its absolute value is no larger than $\mathcal{L}f(\cdots)$ for all values of the parameters of f. Substituting $x := cy/2^c$, $i := j + c$, we get

$$\begin{aligned}
F_c(x) &= \mathcal{N}^2 \frac{(cy)^c}{c!} 2^{-c(c+1)/2} \sum_{j \geq -c} (-1)^j 2^{-j(j+1)/2} \left(\frac{2^j y}{2^j y + 1} + O(\frac{1}{c})\right)(1 + O(2^{-(j+c)})) \\
&= \mathcal{N}^2 \frac{(cy)^c}{c!} 2^{-c(c+1)/2} \left(O(\frac{1}{c}) + \sum_{j \geq -c} (-1)^j 2^{-j(j+1)/2} \frac{2^j y}{2^j y + 1}\right),
\end{aligned}$$

and since removing the lower limit at $-c$ changes the sum by only $O(2^{-c(c+1)/2})$, we have

$$\begin{aligned}
\mathbb{P}(Y_\infty \leq x) &= \mathcal{N}^2 \frac{(cy)^c}{c!} 2^{-c(c+1)/2} (\mathcal{P}(y) + O(\frac{1}{c})), \quad \text{where} \\
\mathcal{P}(z) &:= \sum_{j \in \mathbb{Z}} (-1)^j 2^{-j(j+1)/2} \frac{2^j z}{2^j z + 1}, \qquad z \in \mathbb{C}. \qquad (9)
\end{aligned}$$

For small $\epsilon > 0$, the region $\mathcal{D}(\epsilon) := \{z \in \mathbb{C} \mid |z + 1| \leq \epsilon |z|\}$ is a small disk containing -1. On $\mathbb{C} \setminus \{0\}$ with $\mathcal{D}(\epsilon)$ and its scalings by 2^j $(j \in \mathbb{Z})$ removed, the sum defining $\mathcal{P}(z)$ converges uniformly, so it is analytic. Letting $\epsilon \to 0$, it follows that $\mathcal{P}(z)$ is analytic on $\mathbb{C} \setminus \{0, -2^j \mid j \in \mathbb{Z}\}$; similarly, it has simple poles at -2^j $(j \in \mathbb{Z}.)$ Where $\mathcal{P}$ is defined, we have

$$\begin{aligned}
\mathcal{P}(z) &= z \sum_{j \in \mathbb{Z}} (-1)^j 2^{-j(j+1)/2} \frac{2^j}{2^j z + 1} \\
&= z \sum_{j \in \mathbb{Z}} (-1)^j 2^{-(j-1)j/2} \frac{1}{2^j z + 1} \\
&= z \sum_{j \in \mathbb{Z}} (-1)^j 2^{-(j-1)j/2} \left(\frac{1}{2^j z + 1} - 1\right), \\
&\qquad \text{since } \textstyle\sum_{j \in \mathbb{Z}} (-1)^j 2^{-(j-1)j/2} = 0 \\
&= z \sum_{j \in \mathbb{Z}} (-1)^{j-1} 2^{-(j-1)j/2} \frac{2^{j-1} \cdot 2z}{2^{j-1} \cdot 2z + 1} \\
&= z\mathcal{P}(2z).
\end{aligned}$$

Now, set

$$\mathcal{Q}(z) := 2^{z(z-1)/2} \mathcal{P}(2^z); \qquad (10)$$

then $\mathcal{Q}(z)$ is analytic on $\mathbb{C} \setminus (\mathbb{Z} + (2\pi\iota/\log 2)\mathbb{Z} + \pi\iota/\log 2)$, has simple poles at $\mathbb{Z} + (2\pi\iota/\log 2)\mathbb{Z} + \pi\iota/\log 2$, and, where $\mathcal{Q}$ is defined,

$$\begin{aligned}
\mathcal{Q}(z + 1) &= 2^{(z+1)z/2} \mathcal{P}(2^{z+1}) = 2^{z(z-1)/2} \cdot 2^z \mathcal{P}(2 \cdot 2^z) = \mathcal{Q}(z) \quad \text{and} \\
\mathcal{Q}(z + \frac{2\pi\iota}{\log 2}) &= 2^{(z+(2\pi\iota/\log 2))(z+(2\pi\iota/\log 2)-1)/2} \mathcal{P}(2^z) = -e^{2\pi\iota z} e^{-2\pi^2/\log 2} \mathcal{Q}(z).
\end{aligned}$$

However, if $\theta_4(z,q)$ is the theta function [8, §21.1, §21.11, §21.12]

$$\theta_4(z,q) := 1 + 2\sum_{i\geq 1}(-1)^i q^{i^2}\cos(2iz), \qquad q = e^{\pi\iota\tau},\ |q|<1,\ z,q,\tau\in\mathbb{C},$$

then, for fixed q, $\theta_4(z,q)$ is analytic on all of $\mathbb{C}$ and has zeroes at $\pi\mathbb{Z}+\pi\tau\mathbb{Z}+\frac{1}{2}\pi\tau$, and

$$\theta_4(z+\pi,q) = \theta_4(z,q), \qquad \theta_4(z+\pi\tau,q) = -\frac{e^{-2\iota z}}{q}\theta_4(z,q).$$

If we set $\tau := 2\pi\iota/\log 2$, $q := \exp -2\pi^2/\log 2$, then, $\mathcal{Q}(z)\theta_4(\pi z,q)$ will be analytic on $\mathbb{C}$ and doubly periodic, so it is constant and

$$\mathcal{Q}(z) = \frac{\mathcal{K}}{\theta_4(\pi z,q)}$$

for some $\mathcal{K}\in\mathbb{C}$, which is real and positive since both $\mathcal{Q}(0)=\mathcal{P}(1)$ and $\theta_4(0,q)$ are real and positive. Therefore, $\mathcal{Q}(r)$ cannot be zero for $r\in\mathbb{R}$, and since $\mathcal{Q}(r)$ is then real, it must always be real and positive for $r\in\mathbb{R}$, so $\mathcal{P}(r)$ is also always real and positive for $r\in\mathbb{R}_{>0}$. Also, since $\theta_4(\pi z,q)$ is even, $\mathcal{Q}(z)$ is even.

Supposing now that $x = c'/2^{c'}$ for some real $c'\geq 1$, we may let $c := \lfloor c'\rfloor \geq 1$. Then

$$y = \frac{c'/c}{2^{c'-c}} = 2^{-\{c'\}}(1+\frac{\{c'\}}{c}) \tag{11}$$

so $\frac{1}{2} < y < 2$, and, from (9) and (10),

$$\begin{aligned}
\mathbb{P}(Y_\infty \leq x) &= \mathcal{N}^2\frac{(cy)^c}{c!}2^{-c(c+1)/2}(\mathcal{P}(y)+O(\frac{1}{c}))\\
&= \mathcal{N}^2\frac{(cy)^c}{c!}2^{-c(c+1)/2}\mathcal{P}(y)(1+O(\frac{1}{c}))\\
&= \mathcal{N}^2\frac{(cy)^c}{c!}2^{-c(c+1)/2}y^{(1-\log_2 y)/2}\mathcal{Q}(\log_2 y)(1+O(\frac{1}{c})).
\end{aligned}$$

After some simplification, using Stirling's approximation, (11), $\mathcal{Q}(\log_2 y) = \mathcal{Q}(-\{c'\})(1+O(1/c))$, $O(1/c) = O(1/c')$, and the periodicity and evenness of $\mathcal{Q}$, we get

Theorem 12. *For real $c'\geq 1$,*

$$\mathbb{P}(Y_\infty \leq \frac{c'}{2^{c'}}) = \frac{\mathcal{N}^2}{\sqrt{2\pi c'}}e^{c'}2^{-c'(c'+1)/2}\mathcal{Q}(c')(1+O(\frac{1}{c'})).$$

It will be convenient later to find $\mathbb{P}(Y_\infty \leq c''y/2^{c''})$, where c'' and y are positive real and $(\log_2 y)^4 \leq c''$. We need then to find c' with

$$\frac{c'}{2^{c'}}\Big/\frac{c''y}{2^{c''}} = 1, \tag{12}$$

and if we set

$$c' := c'' - \log_2 y - \frac{\log_2 y}{c''\log 2} + \frac{K}{c''^{3/2}}, \tag{13}$$

we can verify that, if $c'' \geq 6$, the left-hand side of (12) is less than 1 if $K = 4$ and bigger than 1 if $K = -7$, so (12) must be satisfied with some $-7 \leq K \leq 4$. Substituting (13) into Theorem 12, we get

Theorem 13. *For real $c'' \geq 6$ and $2^{-c''^{1/4}} \leq y \leq 2^{c''^{1/4}}$,*

$$\mathbb{P}(Y_\infty \leq \frac{c'' y}{2^{c''}}) = \frac{\mathcal{N}^2}{\sqrt{2\pi c''}}(ey)^{c''} 2^{-c''(c''+1)/2} y^{(1-\log_2 y)/2} \mathcal{Q}(c'' - \log_2 y)(1 + O(\frac{1}{\sqrt{c''}})).$$

Finally, we observe that

$$\begin{aligned}
\mathbb{P}(Y_\infty > x) &= 1 - F_0(x) \\
&= 1 - \mathcal{N} \sum_{i \geq 0} \frac{(-1)^{i+1}(e^{-2^i x} - 1)}{(2^1 - 1) \cdots (2^i - 1)} \\
&= 1 + \mathcal{N} \sum_{i \geq 0} \frac{(-1)^{i+1}}{(2^1 - 1) \cdots (2^i - 1)} + \mathcal{N} \sum_{i \geq 0} \frac{(-1)^i e^{-2^i x}}{(2^1 - 1) \cdots (2^i - 1)}.
\end{aligned}$$

The last term is an alternating series whose terms decrease in magnitude, so its value is between 0 and the first term of the series, which is $\mathcal{N} e^{-x}$. Since $\mathbb{P}(Y_\infty > x)$ must approach 0 as $x \to \infty$, the first two terms must cancel, so we have

Theorem 14. *For $x \in \mathbb{R}_{>0}$,*

$$\mathbb{P}(Y_\infty > x) \leq \mathcal{N} e^{-x}.$$

6 The threshold of the sequence of paths, II

Returning to the situation of §4, we now let n be large and set $L := \lfloor \log_2 n \rfloor$, $c'' := \sqrt{\log_2 n}$, $p := (c'' + \sqrt{c''})/(e 2^{c''})$. If we use Theorem 13 to estimate q'', we will take $y := 2(1 - (r/2^L))\lambda(1 + c''^{-1/2})/(ep)$. Recalling that $r = (2 \log n)/p$ and, since $\lambda = -\log(1 - p)$, $|(\lambda/p) - 1| \leq p$ for all $0 \leq p \leq \frac{1}{2}$, we find that $\frac{1}{2} \leq y \leq 1$ for all sufficiently large n and so

$$q'' = \Theta(\frac{1}{\sqrt{c'' n}} 2^{c''/2} \Delta^{c''}), \qquad \text{where } \Delta := (1 - \frac{r}{2^L})\frac{\lambda}{p}(1 + \frac{1}{\sqrt{c''}}).$$

Then

$$p q''^2 \lfloor \frac{n}{2L+1} \rfloor = \Theta(\frac{\Delta^{2c''}}{L}) = \Theta(\frac{e^{2\sqrt{c''}}}{L})$$

will approach infinity as $n \to \infty$, so our random distribution is solvable with a probability that approaches 0 as $n \to \infty$. This means that, for this choice of p,

$$n(\frac{1}{p} - 1) = e \frac{2^{\sqrt{\log_2 n}}}{\sqrt{\log_2 n}} n(1 + O(\frac{1}{(\log n)^{1/4}}))$$

is below the geometric threshold of the sequence of families of solvable distributions of the n-paths.

We now need to find an upper bound for the geometric threshold. Suppose as before that we place Z_i pebbles on each vertex i, where the Z_i's are i.i.d. geometric random variables with parameter p, and let $M \le L$ be positive integers with $n \ge 2(L+M)+1$. If $i \le L+M$, if i is unpebblable, then $\sum_{0\le k\le L-1} Z_{i+k}2^{-k}$ must be less than 1, and if $i \ge n-(L+M)+1$, if i is unpebblable, then $\sum_{0\le k\le L-1} Z_{i-k}2^{-k}$ must be less than 1. In both cases, then, since the Z_i's are i.i.d.,

$$\mathbb{P}(i \text{ unpebblable}) \le X(1), \quad \text{where} \quad X(r) := \mathbb{P}(Z_1 + \frac{Z_2}{2} + \cdots + \frac{Z_L}{2^{L-1}} < r).$$

If $L+M+1 \le i \le n-(L+M)$, we observe that for i to be unpebblable, $Z_{i-M}, \ldots, Z_{i+M}$ must all be less than 2^M, and $\sum_{0\le k\le L-1} Z_{i+M+1+k}2^{-k}$ and $\sum_{0\le k\le L-1} Z_{i-M-1-k}2^{-k}$ must both be less than 2^{M+1}. In this case, then,

$$\mathbb{P}(i \text{ unpebblable}) \le \mathbb{P}(Z_1 < 2^M)^{2M+1} X(2^{M+1})^2.$$

Summing these probabilities, and using

$$\mathbb{P}(Z_1 < 2^M) = \sum_{0\le j<2^M} \mathbb{P}(Z_1 = j) \le 2^M p,$$

we find that the probability that our random distribution is unsolvable is no more than

$$\begin{aligned} & 2(L+M)X(1) + (n-2(L+M))(2^M p)^{2M+1} X(2^{M+1})^2 \\ \le\ & 4LX(1) + n(2^M p)^{2M+1} X(2^{M+1})^2. \end{aligned} \tag{14}$$

Realizing Z_i as $\lfloor W_i/\lambda \rfloor$ as before, we have

$$\begin{aligned} X(r) &= \mathbb{P}(\lfloor W_1/\lambda \rfloor + \cdots + \frac{\lfloor W_L/\lambda \rfloor}{2^{L-1}} < r) \\ &\le \mathbb{P}((W_1/\lambda) + \cdots + \frac{W_L/\lambda}{2^{L-1}} < r+2) \\ &\le \mathbb{P}(\sum_{i\ge 1} \frac{W_i}{\lambda 2^{i-1}} < r+3) + \mathbb{P}(\sum_{i\ge L+1} \frac{W_i}{\lambda 2^{i-1}} > 1) \\ &= \mathbb{P}(Y_\infty < (r+3)\lambda) + \mathbb{P}(Y_\infty > 2^L \lambda). \end{aligned} \tag{15}$$

We now let n be large, set $L := \lfloor \log_2 n \rfloor$, $M := \lfloor (\log_2 n)^{1/16} \rfloor$, $c'' := \sqrt{\log_2 n}$, and $p := (c'' - \sqrt{c''})/(e2^{c''})$, and estimate $X(r)$ using (15). For large n, $\mathbb{P}(Y_\infty > 2^L\lambda) < \mathbb{P}(Y_\infty > \sqrt{n})$, which is $O(e^{-\sqrt{n}})$ by Theorem 14. To estimate the first term in (15), we use Theorem 13, setting $y := (r+3)\lambda(1 - c''^{-1/2})/(ep)$. In the case $r = 1$ we have $1 \le y \le 2$ for all sufficiently large n so

$$X(1) = \Theta(\frac{1}{\sqrt{c'' n}} 2^{3c''/2} \Delta'^{c''}) + O(e^{-\sqrt{n}}), \quad \text{where } \Delta' := \frac{\lambda}{p}(1 - \frac{1}{\sqrt{c''}}). \tag{16}$$

In the case $r = 2^{M+1}$, $\log_2 y = O(M)$ so

$$X(2^{M+1}) = \Theta(\frac{1}{\sqrt{c''n}}(2^{M+1}+3)^{c''}2^{-c''/2}2^{O(M^2)}\Delta'^{c''}) + O(e^{-\sqrt{n}}). \qquad (17)$$

Combining (14), (16), (17), and $\Delta'^{c''} = \Theta(e^{-\sqrt{c''}})$ shows that our random distribution is unsolvable with a probability that approaches 0 at $n \to \infty$, so, for this choice of p,

$$n(\frac{1}{p} - 1) = e\frac{2^{\sqrt{\log_2 n}}}{\sqrt{\log_2 n}}n(1 + O(\frac{1}{(\log n)^{1/4}}))$$

is above the geometric threshold of the sequence of families of solvable distributions of the n-paths. Together with our previous lower bound on the geometric threshold, this proves

Theorem 15. *For all positive integers n, let the n-path have n vertices, 1, ..., n, and edges between vertices i and $i+1$ for $i = 1, \ldots, n-1$. Then the geometric threshold of the sequence of families of solvable distributions of the n-paths is*

$$e\frac{2^{\sqrt{\log_2 n}}}{\sqrt{\log_2 n}}n(1 + O(\frac{1}{(\log n)^{1/4}})).$$

Corollary 16. *The pebbling threshold of the sequence of n-paths is*

$$\Theta(\frac{2^{\sqrt{\log_2 n}}}{\sqrt{\log_2 n}}n).$$

Proof. Use Theorem 15, Theorem 11 and [1, Theorem 1.3]. □

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