Asymptotic behavior of Rauzy's sequence

David Moews
Center for Communications Research
4320 Westerra Court
San Diego, CA 92121
USA

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Abstract. For the sequence $u(n)$ defined by

$$u(1) = x, \qquad u(2) = y, \qquad u(n) = u(\lfloor n/3 \rfloor) + u(n - \lfloor n/3 \rfloor) \ (n \geq 3),$$

$\lim_{n\to\infty} u(n)/n$ exists and is approximately equal to $0.37512046x+0.31243977y$.

To prove the claimed result, we must make various estimates. First, however, we prove a lemma on equidistribution mod 1. For real x, let $\{x\} = x - \lfloor x \rfloor$ be the fractional part of x.

Lemma 1 *Let θ be irrational and I be a subinterval of $[0,1]$ with length L. For γ real and $k \in \mathbf{Z}$, define $z_{I,\theta}(\gamma,k)$ to be 1 if $\{\gamma + k\theta\} \in I$ and 0 otherwise. Then as $n \to \infty$, $\frac{1}{n}\sum_{0\leq k<n} z_{I,\theta}(\gamma,k) \to L$ uniformly in γ.*

Proof. Let $w(\gamma,n) = \sum_{0\leq k<n} z_{I,\theta}(\gamma,k)$. Since θ is irrational, for any real γ and $v \in [0,1]$, there is at most one $k \in \mathbf{Z}$ with $\{\gamma + k\theta\} = v$. Let I have left endpoint a and right endpoint b, so $L = b-a$. If $L = 0$, the previous remark then implies that $w(\gamma,n) \leq 1$ for all n, and if $L = 1$, it implies that $w(\gamma,n) \geq n-1$ for all n. Uniform convergence of $w(\gamma,n)/n$ is clear in both cases, so let $L \in (0,1)$. If $a = 0$, add $(1-b)/2$ to a and b. This replaces $w(\gamma,n)/n$ by $w(\gamma-(1-b)/2,n)/n$, and the uniform convergence of the latter clearly implies that of the former. If $b = 1$, similarly, subtract $a/2$ from a and b. After these changes we may assume $0 < a < b < 1$. Let $\min(L/2, a, 1-b) > \epsilon > 0$. Any continuous function on $\mathbf{R}/\mathbf{Z}$ can be approximated arbitrarily closely by a trigonometric polynomial [2, Theorem 2.5]. It follows that there are trigonometric polynomials $R_1(t)$ and

$R_2(t)$ such that

$$\begin{array}{rclcll} 1+\epsilon & \geq & R_1(t) & \geq & 1, & t \in [a,b]; \\ \epsilon & \geq & R_1(t) & \geq & 0, & t \in [0, a-\epsilon] \cup [b+\epsilon, 1]; \\ 1+\epsilon & \geq & R_1(t) & \geq & 0, & t \in (a-\epsilon, a) \cup (b, b+\epsilon); \\ 1 & \geq & R_2(t) & \geq & 1-\epsilon, & t \in [a+\epsilon, b-\epsilon]; \\ 0 & \geq & R_2(t) & \geq & -\epsilon, & t \in [0,a] \cup [b,1]; \\ 1 & \geq & R_2(t) & \geq & -\epsilon, & t \in (a, a+\epsilon) \cup (b-\epsilon, b). \end{array}$$

Let

$$R_l(t) = \sum_{-m \leq j \leq m} a_{lj} e^{2\pi i j t}, \qquad l \in \{1,2\}.$$

Observe that as θ is irrational, $e^{2\pi i j\theta} \neq 1$ for all $j \neq 0$. Therefore

$$\begin{aligned} \frac{1}{n} \sum_{0 \leq k < n} R_l(\{\gamma + k\theta\}) &= \frac{1}{n} \sum_{0 \leq k < n} \sum_{-m \leq j \leq m} a_{lj} e^{2\pi i j \{\gamma + k\theta\}} \\ &= \frac{1}{n} \sum_{0 \leq k < n} \left(a_{l0} + \sum_{-m \leq j \leq m, j \neq 0} a_{lj} e^{2\pi i j (\gamma + k\theta)} \right) \\ &= a_{l0} + \sum_{-m \leq j \leq m, j \neq 0} a_{lj} e^{2\pi i j \gamma} \frac{1}{n} \frac{e^{2\pi i j n\theta} - 1}{e^{2\pi i j \theta} - 1} \end{aligned}$$

so

$$\left| -a_{l0} + \frac{1}{n} \sum_{0 \leq k < n} R_l(\{\gamma + k\theta\}) \right| \leq \frac{2}{n} \sum_{-m \leq j \leq m, j \neq 0} \frac{|a_{lj}|}{|e^{2\pi i j\theta} - 1|} = \frac{Z_l}{n},$$

for some constants Z_1 and Z_2 independent of γ. Now

$$\int_0^1 R_l(t)dt = \sum_{-m \leq j \leq m} a_{lj} \int_0^1 e^{2\pi i j t} dt = a_{l0}$$

so

$$a_{10} = \int_0^1 R_1(t)dt \leq \epsilon + 1 \cdot (b - a + 2\epsilon) = L + 3\epsilon$$

and

$$a_{20} = \int_0^1 R_2(t)dt \geq -\epsilon + 1 \cdot (b - a - 2\epsilon) = L - 3\epsilon.$$

Now, it is clear that $R_1 \geq \chi_I \geq R_2$ on $[0,1]$, where χ_I is the indicator function of I. It follows that

$$\begin{aligned} L + 3\epsilon + \frac{Z_1}{n} \geq a_{10} + \frac{Z_1}{n} \geq \frac{1}{n} \sum_{0 \leq k < n} R_1(\{\gamma + k\theta\}) &\geq \frac{w(\gamma, n)}{n} \\ &\geq \frac{1}{n} \sum_{0 \leq k < n} R_2(\{\gamma + k\theta\}) \geq a_{20} - \frac{Z_2}{n} \geq L - 3\epsilon - \frac{Z_2}{n}. \end{aligned}$$

Letting $\epsilon \to 0$ concludes the proof. ∎

Lemma 2 *Fix irrational θ and a subinterval I of $[0,1]$ with length L. Allow $\nu \geq 0$, $\beta_1 \geq \beta_0$, and γ to vary in any way such that $\nu \to 0$ and $\nu(\beta_1-\beta_0)^2 \to \infty$. Then*

$$\sqrt{\nu} \sum_{\substack{\beta\in\mathbf{Z},\\ \beta_0\leq\beta\leq\beta_1}} z_{I,\theta}(\gamma,\beta)e^{-\nu(\beta-\beta_0)^2} \to \frac{\sqrt{\pi}L}{2}$$

and

$$\sqrt{\nu} \sum_{\substack{\beta\in\mathbf{Z},\\ \beta_0<\beta\leq\beta_1}} z_{I,\theta}(\gamma,\beta)e^{-\nu(\beta-\beta_0)^2} \to \frac{\sqrt{\pi}L}{2}.$$

Proof. We prove the first limit; since each term in the sum is bounded by 1, the second is then clear. Summation by parts gives, for any $\nu \geq 0$,

$$\sum_{\beta_0\leq\beta\leq\beta_1} z_{I,\theta}(\gamma,\beta)e^{-\nu(\beta-\beta_0)^2} = \sum_{\beta_0\leq\beta'\leq\beta_1}\left(\sum_{\beta_0\leq\beta\leq\beta'} z_{I,\theta}(\gamma,\beta)\right)(e^{-\nu(\beta'-\beta_0)^2}-e^{-\nu(\beta'+1-\beta_0)^2}) + \left(\sum_{\beta_0\leq\beta\leq\lfloor\beta_1\rfloor} z_{I,\theta}(\gamma,\beta)\right)e^{-\nu(\lfloor\beta_1\rfloor+1-\beta_0)^2}. \tag{1}$$

We now estimate $\sum_{\beta_0\leq\beta\leq\beta'} z_{I,\theta}(\gamma,\beta)$. If $\beta' \leq \beta_0+\nu^{-1/4}$, then the trivial bound $0 \leq z_{I,\theta}(\gamma,\beta) \leq 1$ gives $\sum_{\beta_0\leq\beta\leq\beta'} z_{I,\theta}(\gamma,\beta) = O(\nu^{-1/4})$. Otherwise, $\beta'-\beta_0 \geq \nu^{-1/4} \to \infty$, so we can apply Lemma 1 to find that $\sum_{\beta_0\leq\beta\leq\beta'} z_{I,\theta}(\gamma,\beta) = (\beta'-\lceil\beta_0\rceil+1)(L+o(1))$. Putting these estimates together yields

$$\sum_{\beta_0\leq\beta\leq\beta'} z_{I,\theta}(\gamma,\beta) = (\beta'-\lceil\beta_0\rceil+1)(L+o(1))+O(\nu^{-1/4}),$$

uniformly in β'. Substituting this into (1) gives

$$\sum_{\beta_0\leq\beta\leq\beta_1} z_{I,\theta}(\gamma,\beta)e^{-\nu(\beta-\beta_0)^2} =$$

$$\sum_{\beta_0\leq\beta'\leq\beta_1}((\beta'-\lceil\beta_0\rceil+1)(L+o(1))+O(\nu^{-1/4}))(e^{-\nu(\beta'-\beta_0)^2}-e^{-\nu(\beta'+1-\beta_0)^2})$$
$$+((\lfloor\beta_1\rfloor-\lceil\beta_0\rceil+1)(L+o(1))+O(\nu^{-1/4}))e^{-\nu(\lfloor\beta_1\rfloor+1-\beta_0)^2}$$

$$= (L+o(1))\left(\sum_{\beta_0\leq\beta'\leq\beta_1}(\beta'-\lceil\beta_0\rceil+1)(e^{-\nu(\beta'-\beta_0)^2}-e^{-\nu(\beta'+1-\beta_0)^2}) + (\lfloor\beta_1\rfloor-\lceil\beta_0\rceil+1)e^{-\nu(\lfloor\beta_1\rfloor+1-\beta_0)^2}\right)+O(\nu^{-1/4})$$

and applying summation by parts again gives

$$\sum_{\beta_0 \le \beta \le \beta_1} z_{I,\theta}(\gamma,\beta) e^{-\nu(\beta-\beta_0)^2} = (L+o(1)) \left(\sum_{\beta_0 \le \beta \le \beta_1} e^{-\nu(\beta-\beta_0)^2} \right) + O(\nu^{-1/4})$$

so it is enough to show that

$$\sqrt{\nu} \sum_{\beta_0+1 \le \beta \le \beta_1} e^{-\nu(\beta-\beta_0)^2} \to \frac{\sqrt{\pi}}{2}.$$

However, if $\beta \ge \beta_0 + 1$, then

$$\sqrt{\nu} \int_{\beta}^{\beta+1} e^{-\nu(\beta-\beta_0)^2} d\beta \le \sqrt{\nu} e^{-\nu(\beta-\beta_0)^2} \le \sqrt{\nu} \int_{\beta-1}^{\beta} e^{-\nu(\beta-\beta_0)^2} d\beta$$

and substituting $\gamma = \sqrt{\nu}(\beta - \beta_0)$ into the integrals yields

$$\int_{\sqrt{\nu}(\beta-\beta_0)}^{\sqrt{\nu}(\beta-\beta_0+1)} e^{-\gamma^2} d\gamma \le \sqrt{\nu} e^{-\nu(\beta-\beta_0)^2} \le \int_{\sqrt{\nu}(\beta-\beta_0-1)}^{\sqrt{\nu}(\beta-\beta_0)} e^{-\gamma^2} d\gamma.$$

Summing these inequalities and recalling that $\nu \to 0$ proves that

$$\sqrt{\nu} \sum_{\beta_0+1 \le \beta \le \beta_1} e^{-\nu(\beta-\beta_0)^2} - \int_0^{\sqrt{\nu}(\beta_1-\beta_0)} e^{-\gamma^2} d\gamma \to 0.$$

However, $\sqrt{\nu}(\beta_1 - \beta_0) \to \infty$, so the integral approaches $\int_0^\infty e^{-\gamma^2} d\gamma = \sqrt{\pi}/2$. This proves the lemma. ∎

Let $0 < p < 1$, $a > b \ge 1$, and for $n \in \mathbf{Z}_{>0}$, let

$$\Sigma_n = \{(\alpha,\beta) \mid \alpha,\beta \in \mathbf{Z}_{\ge 0}, \log n - \log a < -\alpha \log p - \beta \log(1-p) \le \log n - \log b\}$$

and $\sigma_n = \sigma_n(a,b) = \sum_{(\alpha,\beta)\in\Sigma_n} \binom{\alpha+\beta}{\alpha}$. Define the entropy function H by $H(x) = -x \log x - (1-x)\log(1-x)$.

Lemma 3 *If $\log p / \log(1-p)$ is irrational, $ap < b$, $a(1-p) < b$, and $\epsilon > 0$, then for all large enough n, σ_n/n is between $-\epsilon + (\log a - \log b)/(aH(p))$ and $\epsilon + (\log a - \log b)/(bH(p))$.*

Proof. We may interchange p and $1-p$ if necessary to find that, without loss of generality, $p < \frac{1}{2} < 1-p$. Also, let $n > \max(a,e)$. Now, since $ap < b$ and $a(1-p) < b$, fixing α for some (α,β) in Σ_n determines β, and vice versa. It follows that there are most two $(\alpha,\beta) \in \Sigma_n$ with $\alpha = 0$ or $\beta = 0$. Deleting these will change σ_n by at most 2, which is $o(n)$. We will therefore take the sum in σ_n to be over only positive α and β. We now have the estimate [1, (1.5)]

$$e^{-1/(6\min(\alpha,\beta))} \sqrt{\frac{\alpha+\beta}{2\pi\alpha\beta}} e^{\psi} \le \binom{\alpha+\beta}{\alpha} \le \sqrt{\frac{\alpha+\beta}{2\pi\alpha\beta}} e^{\psi}, \tag{2}$$

where

$$\begin{aligned}\psi = \psi(\alpha, \beta) &= (\alpha+\beta)\log(\alpha+\beta) - \alpha\log\alpha - \beta\log\beta \\ &= \alpha\log(1+\frac{\beta}{\alpha}) + \beta\log(1+\frac{\alpha}{\beta}).\end{aligned}$$

Let Q be $-\alpha\log p - \beta\log(1-p)$, α_0 be $pQ/H(p)$, and β_0 be $(1-p)Q/H(p)$. Observe that $Q_- \le Q \le Q_+$, where $Q_- = \log n - \log a$, $Q_+ = \log n - \log b$. Since $-\alpha_0 \log p - \beta_0 \log(1-p) = Q$, we can write, for some λ, $\alpha = \alpha_0 + \lambda\log(1-p)$ and $\beta = \beta_0 - \lambda\log p$. Now

$$\begin{aligned}\partial_\alpha \psi &= \log(\alpha+\beta) - \log\alpha = \log(1+\beta/\alpha), \\ \partial_\beta \psi &= \log(\alpha+\beta) - \log\beta = \log(1+\alpha/\beta), \\ \partial_{\alpha\alpha}\psi &= 1/(\alpha+\beta) - 1/\alpha, \\ \partial_{\alpha\beta}\psi &= 1/(\alpha+\beta), \\ \partial_{\beta\beta}\psi &= 1/(\alpha+\beta) - 1/\beta,\end{aligned}$$

so if we fix Q and treat ψ as a function of λ,

$$\partial_\lambda \psi = \log(1-p)\log(1+\frac{\beta}{\alpha}) - \log p \log(1+\frac{\alpha}{\beta}),$$

$$\partial_{\lambda\lambda}\psi = \frac{(\log p - \log(1-p))^2}{\alpha+\beta} - \frac{(\log(1-p))^2}{\alpha} - \frac{(\log p)^2}{\beta}.$$

We remark that

$$\begin{aligned}\psi|_{\lambda=0} &= Q, \\ \partial_\lambda\psi|_{\lambda=0} &= 0, \qquad \text{and} \\ \partial_{\lambda\lambda}\psi|_{\lambda=0} &= \frac{(\log p - \log(1-p))^2}{\alpha_0+\beta_0} - \frac{(\log(1-p))^2}{\alpha_0} - \frac{(\log p)^2}{\beta_0} \\ &= -\frac{H(p)^3}{Qp(1-p)}.\end{aligned}$$

.

Either $\alpha \le \alpha_0$ or $\beta \le \beta_0$ and therefore

$$\begin{aligned}\partial_{\lambda\lambda}\psi &\le -\frac{(\log p)^2 - (\log p - \log(1-p))^2}{\beta} - \frac{(\log(1-p))^2}{\alpha} \\ &\le \max(-\frac{(\log p)^2 - (\log p - \log(1-p))^2}{\beta_0}, -\frac{(\log(1-p))^2}{\alpha_0}) \\ &= -\frac{K}{Q}, \qquad \text{for some constant } K > 0.\end{aligned}$$

It follows that $\psi \le Q - K\lambda^2/(2Q)$, so $\psi \le Q_+ - K\lambda^2/(2Q_+)$. Let $\lambda' = (\log n)^{1/2}\log\log n$. If $|\lambda| > \lambda'$, then $\psi \le \log n - \log b - K(\log\log n)^2/2$, so $e^\psi = O(ne^{-K(\log\log n)^2/2})$. (This, and all succeeding o, O and Ω estimates, are uniform with respect to α, β, and λ.) But $\alpha \le -(\log n - \log b)/\log p$ and $\beta \le -(\log n - \log b)/\log(1-p)$, so the part of the sum in σ_n we are considering is over $O((\log n)^2)$ terms. As $\sqrt{(\alpha+\beta)/(2\pi\alpha\beta)}$ is clearly bounded, it follows that the portion of the sum in σ_n where $|\lambda| > (\log n)^{1/2}\log\log n$ is $O(n(\log n)^2 e^{-K(\log\log n)^2/2}) = o(n)$.

Assume from now on that $|\lambda| \le \lambda'$. If we take n sufficiently large, this will imply that $\alpha \ge \alpha_0/2$ and $\beta \ge \beta_0/2$. Then

$$\begin{aligned}
|\partial_{\lambda\lambda}\psi - \partial_{\lambda\lambda}\psi|_{\lambda=0}| &= |\lambda|\left(\frac{(\log(1-p)-\log p)^3}{(\alpha_0+\beta_0)(\alpha+\beta)} - \frac{(\log(1-p))^3}{\alpha_0\alpha} - \frac{(\log p)^3}{\beta_0\beta}\right) \\
&\le 2|\lambda|\left(\frac{(\log(1-p)-\log p)^3}{(\alpha_0+\beta_0)^2} - \frac{(\log(1-p))^3}{\alpha_0^2} - \frac{(\log p)^3}{\beta_0^2}\right) \\
&= 2|\lambda|O((\log n)^{-2}) \\
&= O(\log\log n(\log n)^{-3/2}),
\end{aligned}$$

so

$$\begin{aligned}
\psi &= Q - (H(p)^3/(Qp(1-p)) + O(\log\log n(\log n)^{-3/2}))\lambda^2/2 \\
&= Q - \lambda^2 H(p)^3/(2p(1-p)Q) + O((\log\log n)^3(\log n)^{-1/2}). \qquad (3)
\end{aligned}$$

Also, if $|\lambda| \le \lambda'$, we have

$$\begin{aligned}
\alpha/\alpha_0 &= 1 + \lambda\log(1-p)/\alpha_0 &&= 1 + O((\log\log n)(\log n)^{-1/2}), \\
\beta/\beta_0 &= 1 - \lambda\log p/\beta_0 &&= 1 + O((\log\log n)(\log n)^{-1/2}), \\
(\alpha+\beta)/(\alpha_0+\beta_0) &= 1 + \lambda(\log(1-p)-\log p)/(\alpha_0+\beta_0) &&= 1 + O((\log\log n)(\log n)^{-1/2}),
\end{aligned}$$

so

$$\begin{aligned}
\sqrt{\frac{\alpha+\beta}{2\pi\alpha\beta}} &= \sqrt{\frac{\alpha_0+\beta_0}{2\pi\alpha_0\beta_0}}(1 + O((\log\log n)(\log n)^{-1/2})) \\
&= \sqrt{\frac{H(p)}{2\pi p(1-p)Q}}(1 + O((\log\log n)(\log n)^{-1/2})). \qquad (4)
\end{aligned}$$

Finally, if $|\lambda| \le \lambda'$, then as we have already observed,

$$\min(\alpha,\beta) = \Omega(\log n). \qquad (5)$$

(2), (3), (4), and (5) then give

$$\sigma_n = \sqrt{\frac{H(p)}{2\pi p(1-p)}}\tau_n(1+o(1)) + o(n), \qquad (6)$$

where

$$\tau_n = \sum_{\substack{(\alpha,\beta)\in\Sigma_n \\ |\lambda|\le\lambda'}} Q^{-1/2}\exp(Q - \lambda^2 H(p)^3/(2p(1-p)Q)).$$

Observe that

$$Q_+^{-1/2} e^{Q_-} \phi(n, \frac{H(p)^3}{2p(1-p)Q_-}) \leq \tau_n \leq Q_-^{-1/2} e^{Q_+} \phi(n, \frac{H(p)^3}{2p(1-p)Q_+}), \quad (7)$$

where

$$\phi(n,\mu) = \sum_{\substack{(\alpha,\beta)\in\Sigma_n \\ |\lambda|\leq\lambda'}} e^{-\mu\lambda^2}.$$

Our task is now to estimate $\phi(n,\mu)$, where $\mu > 0$ and n is large. For $\beta \in \mathbf{Z}$, let $y(\beta)$ be 1 if there exists some $\alpha \in \mathbf{Z}$ with

$$\log n - \log a < -\alpha \log p - \beta \log(1-p) \leq \log n - \log b, \quad (8)$$

and 0 otherwise. Recalling that fixing β for some $(\alpha,\beta) \in \Sigma_n$ determines α and that $\lambda = (\beta_0 - \beta)/\log p$, we have

$$\phi(n,\mu) = \sum_{|\beta-\beta_0|\leq\lambda'|\log p|} y(\beta) e^{-\mu(\beta-\beta_0)^2/(\log p)^2}.$$

We can rewrite (8) as

$$-\alpha < \frac{\log n - \log a}{\log p} + \beta\frac{\log(1-p)}{\log p} \leq -\alpha + \frac{\log a - \log b}{-\log p} = -\alpha + \frac{\log(b/a)}{\log p},$$

and an α satisfying this will evidently exist just when

$$\left\{\frac{\log n - \log a}{\log p} + \beta\frac{\log(1-p)}{\log p}\right\} \in \left(0, \frac{\log(b/a)}{\log p}\right].$$

Therefore, if $I = (0, \log(b/a)/\log p]$, $\theta = \log(1-p)/\log p$, and $\gamma = (\log n - \log a)/\log p$, we have $y(\beta) = z_{I,\theta}(\gamma,\beta)$, so

$$\phi(n,\mu) = \sum_{|\beta-\beta_0|\leq\lambda'|\log p|} z_{I,\theta}(\gamma,\beta) e^{-\mu(\beta-\beta_0)^2/(\log p)^2}.$$

Now I has length $\log(b/a)/\log p$, so it follows from Lemma 2 that, provided that $\mu \to 0$ and $\mu\lambda'^2 \to \infty$,

$$\sqrt{\mu}\phi(n,\mu) \to \sqrt{\pi}\log(a/b). \quad (9)$$

If μ is a constant divided by either Q_- or Q_+, it is certainly true that $\mu \to 0$ and $\mu\lambda'^2 \to \infty$. Hence (7) and (9) yield

$$\begin{aligned} & Q_+^{-1/2} e^{Q_-} \sqrt{\pi}\log(a/b)\sqrt{\frac{2p(1-p)Q_-}{H(p)^3}}(1+o(1)) \\ \leq\ & \tau_n \\ \leq\ & Q_-^{-1/2} e^{Q_+} \sqrt{\pi}\log(a/b)\sqrt{\frac{2p(1-p)Q_+}{H(p)^3}}(1+o(1)). \quad (10) \end{aligned}$$

Substituting (10) into (6) then yields the desired result. ∎

Lemma 4 *If* $\log p/\log(1-p)$ *is irrational, then* $\lim_{n\to\infty} \sigma_n/n$ *exists and equals* $H(p)^{-1}(b^{-1} - a^{-1})$.

Proof. Fix some integer $m > 0$ such that $\log(a/b)/m < \min(-\log p, -\log(1-p))$, and set $c_i = b(a/b)^{i/m}$, $i = 0, \ldots, m$. It now follows from Lemma 3 that for all $i = 0, \ldots, m-1$ and for large enough n,

$$\frac{\log a - \log b}{mc_iH(p)} + \frac{1}{m^2} \geq \frac{\sigma_n(c_{i+1}, c_i)}{n} \geq \frac{\log a - \log b}{mc_{i+1}H(p)} - \frac{1}{m^2}.$$

However, $\sigma_n(a,b) = \sum_{0\leq i<m} \sigma_n(c_{i+1}, c_i)$, so summing these inequalities over i gives, for large n,

$$\frac{1}{m} + \frac{\log a - \log b}{mH(p)} \sum_{0\leq i<m} c_i^{-1} \geq \frac{\sigma_n(a,b)}{n} \geq -\frac{1}{m} + \frac{\log a - \log b}{mH(p)} \sum_{0\leq i<m} c_{i+1}^{-1}.$$

However, both $\frac{1}{m}\sum_{0\leq i<m} c_i^{-1}$ and $\frac{1}{m}\sum_{0\leq i<m} c_{i+1}^{-1}$ are Riemann sums of the integral

$$\int_0^1 \frac{1}{b(a/b)^x} dx = \frac{b^{-1} - a^{-1}}{\log a - \log b}$$

so letting $m \to \infty$ proves the lemma. ∎

We now proceed to examine Rauzy's sequence. For any x and y, let

$$\begin{aligned} u_{x,y}(1) &= x, \\ u_{x,y}(2) &= y, \\ u_{x,y}(n) &= u_{x,y}(\lfloor n/3 \rfloor) + u_{x,y}(n - \lfloor n/3 \rfloor), \qquad n \geq 3. \end{aligned}$$

It is immediately clear that $u_{1,2}(n) = n$ for all n and so

$$\begin{aligned} u_{x,y}(n) &= (x - \frac{y}{2})u_{1,0}(n) + \frac{y}{2}u_{1,2}(n) \\ &= (x - \frac{y}{2})u_{1,0}(n) + \frac{yn}{2} \end{aligned}$$

for all n. To prove that $u_{x,y}(n)/n$ approaches a limit, it will therefore do to prove that $u_{1,0}(n)/n$ approaches a limit. From now on, call $u_{1,0}(n)$ $u(n)$. Then for positive integers n,

$$\begin{aligned} u(3n) &= u(n) + u(2n), \\ u(3n+1) &= u(n) + u(2n+1), \\ u(3n+2) &= u(n) + u(2n+2), \end{aligned}$$

so if we write $(\delta u)(m) = u(m+1) - u(m)$, then for positive integers n,

$$\begin{aligned} (\delta u)(3n) &= (\delta u)(2n), \\ (\delta u)(3n+1) &= (\delta u)(2n+1), \\ (\delta u)(3n+2) &= (\delta u)(n), \end{aligned}$$

and $u(1) = 1$, $u(2) = 0$, $u(3) = u(1)+u(2) = 1$, so $(\delta u)(1) = -1$ and $(\delta u)(2) = 1$. It follows by induction that $|(\delta u)(n)| = 1$ for all n; together with $u(1) = 1$, this implies that $u(n) \leq n$ for all n.

For all nonnegative real x, define $g_0(x) = \lfloor x/3 \rfloor$ and $g_1(x) = \lceil 2x/3 \rceil$, let the set of finite length words of 0s and 1s be $\{0,1\}^*$, and for each $w = w_1 \cdots w_k \in \{0,1\}^*$, define $g_w = g_{w_1} \circ \cdots \circ g_{w_k}$. Then for all positive integers $n > m \geq 2$,

$$u(n) = \sum_{\substack{w \in \{0,1\}^* \\ g_w(n) > m \\ g_{0w}(n) \leq m}} u(g_{0w}(n)) + \sum_{\substack{w \in \{0,1\}^* \\ g_w(n) > m \\ g_{1w}(n) \leq m}} u(g_{1w}(n)). \tag{11}$$

Fix n, and for all nonnegative real x, set $h_0(x) = x/3$, $h_1(x) = 2x/3$, and $h_w = h_{w_1} \circ \cdots \circ h_{w_k}$ for $w \in \{0,1\}^*$. We have $|g_j(x) - h_j(x)| \leq 1$ for $j \in \{0,1\}$. It follows by induction on the length of w that $|g_w(x) - h_w(x)| \leq 3$ for $w \in \{0,1\}^*$. Also, if m is an even integer, then for integral n, $g_{0w}(n) \leq m$ iff $g_w(n) \leq 3m+2$ and $g_{1w}(n) \leq m$ iff $g_w(n) \leq 3m/2$, so we can rewrite (11) as

$$\begin{aligned} u(n) &= \sum_{\substack{w \in \{0,1\}^* \\ 3m+2 \geq g_w(n) \geq m+1}} u(g_0(g_w(n))) + \sum_{\substack{w \in \{0,1\}^* \\ 3m/2 \geq g_w(n) \geq m+1}} u(g_1(g_w(n))) \\ &= S_0(3m+2, m+1) + S_1(3m/2, m+1), \end{aligned} \tag{12}$$

where we write

$$S_j(a,b) = \sum_{\substack{w \in \{0,1\}^* \\ a \geq g_w(n) \geq b}} u(g_j(g_w(n))).$$

Now if we also write

$$T_j(a,b) = \sum_{\substack{w \in \{0,1\}^* \\ a > h_w(n) \geq b}} u(g_j(g_w(n))).$$

then for all $a \geq b+6$,

$$S_j(a,b) = T_j(a-3, b+3) + \sum_{\substack{w \in \{0,1\}^* \\ a+3 \geq h_w(n) \geq a-3 \\ a \geq g_w(n) \geq b}} u(g_j(g_w(n))) + \sum_{\substack{w \in \{0,1\}^* \\ b+3 > h_w(n) \geq b-3 \\ a \geq g_w(n) \geq b}} u(g_j(g_w(n)))$$

so

$$|S_j(a,b) - T_j(a-3, b+3)| \leq T_j(a+4, a-3) + T_j(b+3, b-3). \tag{13}$$

Now if $w \in \{0,1\}^*$ and $h_w(x) \geq 6$, as $|h_w(x) - g_w(x)| \leq 3$, we have $g_w(x) \geq 3$. It follows that if $j \in \{0,1\}$, then $|g_j(h_w(x)) - g_j(g_w(x))| \leq 2$. Now since $g_j(h_w(x)) \geq g_j(6) \geq 2$ and $g_j(g_w(x)) \geq g_j(3) \geq 1$, $u(g_j(h_w(x)))$ and

$u(g_j(g_w(x)))$ are defined, and since $|(\delta u)(n)| = 1$ for all positive integral n, $|u(g_j(g_w(x))) - u(g_j(h_w(x)))| \leq 2$. Therefore, if we set

$$\begin{aligned} U_j(a,b) &= \sum_{\substack{w\in\{0,1\}^* \\ a>h_w(n)\geq b}} u(g_j(h_w(n))), \\ V(a,b) &= \sum_{\substack{w\in\{0,1\}^* \\ a>h_w(n)\geq b}} 1, \end{aligned}$$

we have, for $b \geq 6$,

$$|T_j(a,b) - U_j(a,b)| \leq 2V(a,b). \tag{14}$$

Combining (12), (13), and (14) now yields, if $m \geq 14$,

$$|u(n) - U_0(3m-1, m+4) - U_1(3m/2-3, m+4)| \leq$$

$$U_0(3m+6, 3m-1) + U_0(m+4, m-2) + U_1(3m/2+4, 3m/2-3) + U_1(m+4, m-2) +$$
$$2V(3m+6, 3m-1) + 4V(m+4, m-2) + 2V(3m/2+4, 3m/2-3) +$$
$$2V(3m-1, m+4) + 2V(3m/2-3, m+4). \tag{15}$$

Now if $x \leq a$, $j \in \{0,1\}$, $u(g_j(x))$ is defined, and a is integral, then $u(g_j(x)) \leq g_j(x) \leq g_j(a) \leq a$, so

$$U_j(a,b) \leq aV(a,b) \qquad (j \in \{0,1\},\ a,\ b \text{ integral},\ b \geq 3.) \tag{16}$$

Also, if $j \in \{0,1\}$, i is a positive integer and $x \in [i, i+1)$, then $|g_j(x) - g_j(i)| \leq 1$, so if $u(g_j(i))$ is defined, $|u(g_j(x)) - u(g_j(i))| \leq 1$. This means that

$$\left| U_j(a,b) - \sum_{a>i\geq b} u(g_j(i))V(i+1,i) \right| \leq V(a,b) \qquad (j \in \{0,1\},\ a,\ b \text{ integral},\ b \geq 3.) \tag{17}$$

Substituting (16) and (17) into (15) yields

$$\left| u(n) - \sum_{3m-1>i\geq m+4} u(g_0(i))V(i+1,i) - \sum_{3m/2-3>i\geq m+4} u(g_1(i))V(i+1,i) \right| \leq$$

$$(3m+8)V(3m+6, 3m-1) + (2m+12)V(m+4, m-2) + (3m/2+6)V(3m/2+4, 3m/2-3)$$
$$+3V(3m-1, m+4) + 3V(3m/2-3, m+4). \tag{18}$$

Now observe that, if the word w has α 0s and β 1s, $h_w(x) = (\frac{1}{3})^\alpha(\frac{2}{3})^\beta x$. Therefore, if we set $p = \frac{1}{3}$, $V(a,b) = \sigma_n(a,b)$. Now fix $m \geq 50$, divide (18) by n and let n tend to infinity. We can then apply Lemma 4 to find that for any $\epsilon > 0$,

$$\left| \frac{u(n)}{n} - \sum_{3m-1>i\geq m+4} \frac{u(g_0(i))}{H(\frac{1}{3})i(i+1)} - \sum_{3m/2-3>i\geq m+4} \frac{u(g_1(i))}{H(\frac{1}{3})i(i+1)} \right| \leq \epsilon + \frac{23}{H(\frac{1}{3})m} \tag{19}$$

for sufficiently large n. Letting $\epsilon = 1/m$ and $m \to \infty$ now immediately proves that $\lim_{n\to\infty} u(n)/n$ exists, as claimed. Furthermore, it follows immediately from (19) that if this limit is $\mathcal{L}$, then

$$\left| \mathcal{L} - \sum_{3m-1>i\geq m+4} \frac{u(g_0(i))}{H(\frac{1}{3})i(i+1)} - \sum_{3m/2-3>i\geq m+4} \frac{u(g_1(i))}{H(\frac{1}{3})i(i+1)} \right| \leq \frac{23}{H(\frac{1}{3})m} \qquad (m \geq 50). \tag{20}$$

Obviously, this allows us to compute $\mathcal{L}$ to any desired degree of accuracy. In fact, taking $m = 10^9$, we find that $\mathcal{L} = 0.37512046 \pm 4 \cdot 10^{-8}$. Finally we remark that for all x and y,

$$\lim_{n\to\infty} \frac{u_{x,y}(n)}{n} = (x - \frac{y}{2})\mathcal{L} + \frac{y}{2} = (0.37512046 \pm 4 \cdot 10^{-8})x + (0.31243977 \pm 2 \cdot 10^{-8})y.$$

References

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