On Some Combinatorial Games Connected with Go

by

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Abstract
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Abstract. The two-person strategy game of *Go* has the feature that, with a simple set of rules, situations called *kos* allowing infinitely long play often arise. To prevent this, various *ko-ban* rules are adopted to prevent playing to previously occurring positions. Go also has the feature that endgame positions often decompose into subpositions, in such a way that these subpositions form an additive group. When kos and ko-ban are introduced, this decomposition may no longer behave well. However, we can improve the behavior slightly by working with bounds on Go positions, which we obtain by adding *ko-threats* for either player. We give an algorithm for computing such bounds, and give actual values for these bounds in certain special cases.

In the last part of this thesis, we define a simple game played with coins, and show that many Go endgame subpositions without kos are equivalent to properly placed coins in this game. This lets us determine how such subpositions behave under our group operation.

Chair Date

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Chapter 1

Introduction

1.1 Outline of this document

In the remainder of this chapter, Chapter 1, we discuss the rules of Go, including kos, ko-ban rules, and scoring rules, and how Go positions without kos or ko-ban can be mathematized. In §2.1, we explain the basic facts about loopy games, as given in [1, Chapter 11]. We will need these in order to analyze kos. In §2.2, we discuss sidling, which is a way of approximating loopy games with enders, and how it can be used to approximate the sorts of loopy games we will encounter. In Chapter 3, we define the mathematical notation we will use to talk about ko-ban rules and ko-threats. Adding arbitrarily many ko-threats for either player provides bounds on the values of games with ko-ban. We prove a few simple general facts about these bounds and about ko-ban in general. Chapter 4 deals with kos. In §4.1, we present the notation we use to mathematize kos. In §4.2, we give an example of how sidling is used to obtain the value of our bounds for kos or sums of kos. In §4.3 and §4.4, we derive formulas that tell us what these bounds are for a class of sums of kos. §4.5 discusses the application of this to Go.

Chapter 5 is independent of Chapters 2, 3, and 4. In §5.1, we define the coin-sliding game, which is a simple two-player strategy game played with coins. We provide a strategy for the coin-sliding game and a classification of which starting positions are won or lost for each player moving first. This provides a proof of Theorem 5 in [6] and [2]. The importance of the coin-sliding game is that many Go endgame subpositions correspond to coins in the coin-sliding game, so that Go endgames can be analyzed by

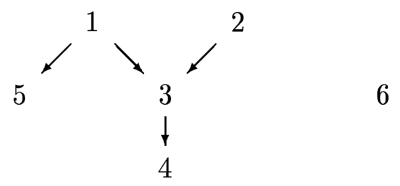


Figure 1.1: Order of dependency of chapters.

means of our strategy and classification. We explain this correspondence in §5.2.

Chapter 6 summarizes our results. The order of dependency of the chapters is as shown in Figure 1.1.

1.2 The rules of Go

1.2.1 Play and capturing

Go is played on a grid of horizontal and vertical lines. Customarily, a grid of 19 horizontal and 19 vertical lines is used, although Go can be played on differently sized grids, such as a grid with 9 horizontal and 9 vertical lines used by beginners. Each intersection of a horizontal and a vertical line, or *point*, can contain a *black stone*, a *white stone*, or be empty. If the two players are known to be of unequal skill, some black stones may be placed on the grid initially in order to handicap the stronger player, who then uses white stones. We will assume that this is not done. In this case, the grid is initially empty. The first player, or Black, then starts the game by placing a black stone on an empty intersection. The second player, or White, then places a white stone on an empty intersection. The first player then places another black stone, and so on. Either player can opt to pass at any time rather than place a stone. The game ends when both players pass successively, and the score of the game is then computed in a way we explain below.

A set of stones will be called *connected* if each pair of stones in the set can be joined by a path of stones in the set, each stone in the path being horizontally or vertically adjacent to the last. A set of intersections will be called connected under the same condition. A connected set of stones, all of the same color, is called a *string.* An empty intersection horizontally or vertically adjacent to a string is called a *liberty.*

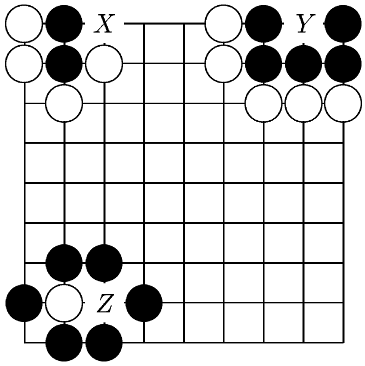


Figure 1.2: Examples of capturing and suicide in Go.

When a stone is placed that causes any string of stones of the opposite color to have no more liberties, that string is *captured* and its stones are removed from the board. The number of the opponent's stones a player has captured may affect the score; see below. A capturing move by White is illustrated at X of Figure 1.2.

Playing a stone may also create a string of your own color with no liberties, or cause a string of your own color to have no more liberties. If it does this as well as capturing a string, the capture takes precedence, as in White's move at Y of Figure 1.2. However if it captures no opponent's string, as in White's move at Z, the move is called *suicide* and results in the capture of your own string by the opponent. Such moves may or may not be legal, depending on the version of Go you are playing. This will not matter for us.

1.2.2 Kos and ko-ban

As well as a full 9 by 9 or 19 by 19 board as in Figures 1.2 and 1.5, we will often show positions representing a fragment of the board, as in the two pieces of Figure 1.3 illustrating a position called *ko*. In these partial positions, stones on top of grid lines extending beyond the edge of the position are assumed to be *immortal*, which means that we are assuming, for purposes of analysis, that they will never be captured and hence will always remain on the board.

Given this, we see that if Black moves on the partial board position in the left of Figure 1.3, Black will capture one of White's stones, and the result will be the

Figure 1.3: Ko.

position in the right of Figure 1.3. If White then moves on the partial position, White will capture one of Black's stones, and the result will be as in the left of Figure 1.3. We will see that the capture by each player of one of the opponent's stones has no net effect on the score. It follows that Black and White could move successively on Figure 1.3 forever to no result.

To prevent this, we have *ko-ban* rules. The *Japanese (J)* rule says that a player cannot move to the position that existed just before the previous player's move. The *North American (N)*, or *super-ko*, rule says that a player cannot move to any previously existing position. In this case, after Black has moved from the left to the right of Figure 1.3, either rule would prevent White from moving back to the left of Figure 1.3 until he has played somewhere else on the board, away from this partial position.

Ko can also occur at the edge of the board, as shown in Figure 1.4. We call the last player to move in a ko its *winner*. It may happen that the winner of a ko gains a positional advantage, and so both players may often want to respond in the ko. In this case, it is advantageous to have *ko-threats*; see below. The advantage gained in Figures 1.3 and 1.4 by winning the ko is not very big; once we define scoring, we will see that the total amount in dispute is about one point. However, if the stones on the edge of the ko are not immortal or effectively immortal, who wins the ko may determine whether they are eventually captured or not. In this case, the ko will be more important. See §4.5 for an example of this.

A *ko-threat* is a move on a piece of the board that forces a response on the same piece of the board. The point of playing a ko-threat is that once the opponent has played in it, the board position will have changed so that you can respond in the ko. An example of the use of ko-threats is shown in Figure 1.5, taken from [4]. Suppose that

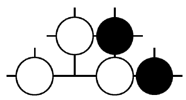


Figure 1.4: Ko at the edge.

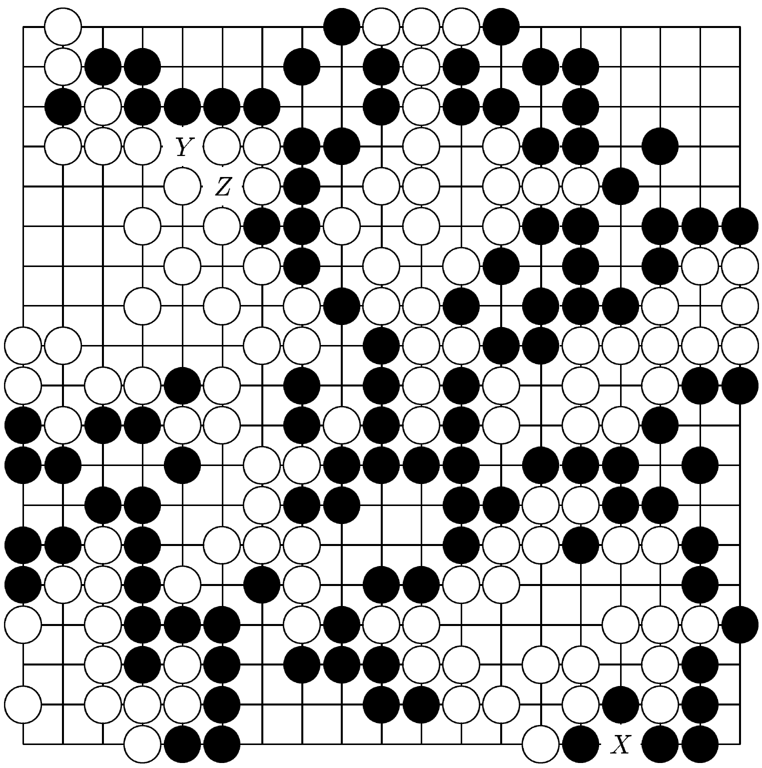
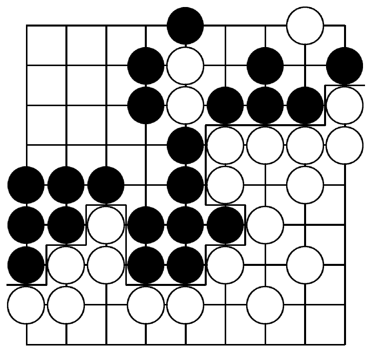


Figure 1.5: An example of the use of ko-threats.



White has captured 2 black stones.
Black has captured 1 white stone.

Figure 1.6: The end of a Go game.

White plays at X, in the ko shown. Black then has a ko-threat, consisting of playing at Y. This threatens the capture of the string of three white stones to the right of Y. White will then respond at Z, and Black can then recapture in the ko.

1.2.3 Scoring points

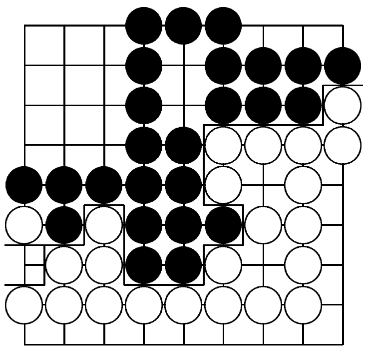
As we mentioned earlier, at the end of the game, after both players pass, a score is computed. To do this, first, *dead* strings are identified. Roughly speaking, dead strings are those which would be captured if play were to proceed. In Figure 1.6 [7], for example, the white stone on the top line and the string of two white stones in the second and third lines are dead. The stones in the dead strings are removed from the board and treated as the opponent's captives. After this, the board is partitioned into *Black territory*, which is those points of the board surrounded by black stones, and *White territory*, which is those points surrounded by white stones. In Figure 1.6, the Black territory is to the upper left of the line and the White territory is to the lower right. If a *seki* or *dame* is present, some empty points of the board may be neutral, and count towards neither Black nor White territory.

In *Chinese* scoring, each player is then given a score equalling the number of stones of his color on the board plus the number of points in his territory. The difference of Black's and White's scores is then the net score for the game. In *Japanese* scoring, each player is given a score equalling the number of stones of his color on the board plus the number of captives he has taken. In Figure 1.6, Chinese scoring gives Black a score of 23 points of territory plus 21 black stones, for a total of 44, and White a score of 19 points of territory plus 18 white stones, for a total of 37. The net score is then 7 in Black's favor. Japanese scoring gives Black a score of 23 points of territory plus 4 White captives (1 during the game and 3 dead stones,) for a total of 27. White is given a score of 19 points of territory plus 2 Black captives, for a total of 21. The net score is then 6 in Black's favor. The *North American (AGA)* rules specify that counting may be done either in the Chinese or the Japanese way. However, they require captive stones to be transferred during passes, and restrict when the game can end, in such a way that the score is almost always the same as under what we have called Chinese scoring.

The number of stones each player has placed on the board during the game will equal the number of stones remaining on the board at the end of the game plus the number of captives taken from him. If there are no passes until the end of the game, then the number of stones each player has placed on the board will equal the number of turns he has had in the game, and Black will have had either the same number of turns as White or one more. Hence, if there are no passes until the end of the game, and Chinese and Japanese scoring rules agree on the amount of each player's territory (they may not; see below,) the Chinese and Japanese scores will differ by at most 1. In our example Black made 23 moves and White made 22, so the Chinese and Japanese scores differ by 1. If there are passes during the game, then each pass will decrease the passing player's Chinese score by 1 relative to his Japanese score. Another way of putting this is that a Chinese pass scores the same as playing on a point of your territory, whereas a Japanese pass scores 1 point better.

Another difference between Chinese and Japanese scoring relates to the score given an unfilled ko at the end of the game. Suppose that instead of Figure 1.6 the game ends as in Figure 1.7, with first Black and then White passing. A ko remains at the left of the board. According to Chinese scoring, the point in the ko is part of White's territory, since it is surrounded by white stones. According to Japanese scoring, however, the point is part of neither Black's nor White's territory. Hence White's score is increased by 1 in Chinese scoring, relative to Japanese scoring.

As we have already noted, Chinese and Japanese scoring treat passes during the game differently. This is consistent with the difference in scoring kos in the following



A ko at the left remains unfilled.
Black has captured 2 White stones.

Figure 1.7: The end of another Go game.

sense: suppose that we continued the game in Figure 1.7 and did not permit it to end until the ko had disappeared. We assume that ko-ban prevents Black from playing on the empty point in the ko. With Japanese scoring, Black would pass, White would play on the empty point on the ko, and then Black and White would pass to end the game. With Chinese scoring, Black could pass, or he could place a stone on a point in his territory on in the opponent's territory. White would then play on the empty point in the ko, and after Black and White passed, the game would then end. In either case, the score would be the same as it was before. Hence we may require that all kos be played out before scoring.

1.3 Mathematization of Go

Go acquires greater mathematical interest when we observe that endgame positions can break up into sub-positions. For example, in Figure 1.8, all the stones outside the boxes are, effectively, immortal, and all the moves outside the boxes are bad. Hence we can consider the position of the combination of the 2 independent positions in Figure 1.9. Each turn, each player may choose to play on either one of the positions A and B. In A, Black can move to a terminal position, and so can White. We assume that Japanese scoring is used. Then the net score contribution from inside the box will be 0 if Black moves, and if White moves, it will be -2. Similarly, both Black and White can

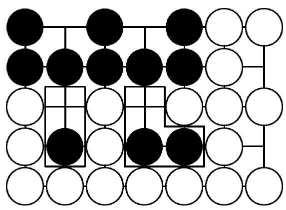


Figure 1.8: A Go position that decomposes.

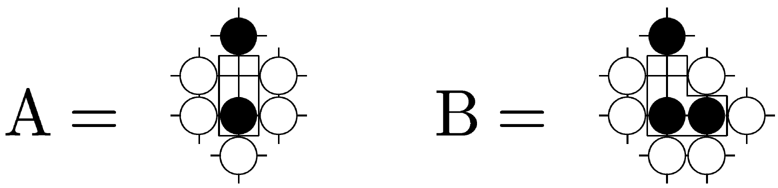


Figure 1.9: The result of decomposition.

move to a terminal position from B, with a score of 0 if Black moves and -4 if White moves.

In the first part of [1] (Chapters 1–8) and in [3] a theory, called *combinatorial game theory*, of games like this is developed. In combinatorial game theory, as here, there are two players who, in general, have different sets of moves, and games can be combined by placing them side by side and allowing each player to move in just one of the components at a time. In combinatorial game theory, it is assumed that the object of each player is to get the last move. However, as it turns out, games corresponding to integers occur naturally and behave like the scores in Go. Because of this, we can identify A and B with combinatorial games, which, using the notation in [1] or [3], we can write as $A = 0|-2$ and $B = 0|-4$. The game players in [1] or [3] are called Left and Right; we identify Left with Black and Right with White. We will use the notation and terminology used in the first part of [1] and in [3] freely; the reader is advised to make sure that he or she is familiar with this material before proceeding. See [6] and [2] for more on the application of combinatorial game theory to Go.

In the first part of [1], only *enders* are treated. Enders are games which do not permit infinite play. This means that in A and B, we do not allow players to pass. This will not usually be a problem since passing will not usually be advantageous. Also, we cannot have kos. We will relax both these restrictions in the sequel.

If we want to model Chinese instead of Japanese scoring, we can do so by starting with Japanese scoring and giving each player 1 extra point each time he moves. The result of this is that each player will, if two positions eventually resolve to the same Japanese score, prefer the one where he ends up moving last, and in fact, prefer it by 1 point. This preference will usually be compatible with combinatorial game theory. In combinatorial game theory getting the last move before the position becomes a number is worth only an infinitesimal amount, as far as stops of the game are concerned; however, we usually find that the games occurring in Go have integral stopping positions, which means that their Left and Right stops are infinitesimally above or below an integer, and in this case, if one stop exceeds another in Go, it will also do so in combinatorial game theory, although perhaps not vice versa. Another difference between Chinese and Japanese scoring comes from the fact that, when we are modelling Chinese scoring, we do not give a player 1 extra point every time he passes. This means that passing costs a player 1 point, and thus has the same effect as playing on a point of his territory, as we noted in the last section. Hence we will always assume that passing is disallowed when modelling Chinese scoring.

Chapter 2

Loopy games

2.1 Basic notions

We review the definitions of loopy games and basic theorems about them, as given by Conway in [1, Chapter 11, pp. 314–357].

2.1.1 Definition of a loopy game

We define a (partizan) loopy game G in the same sense as in [1]. This means that the main part of a loopy game is a directed pseudograph, also called G. G is allowed to have multiple edges from one vertex to another, or loops from a vertex to itself. Often, we will call G just a *graph*, instead of a directed pseudograph. The vertices of G are the *positions* of the game, and the directed edges of G are the *moves* of the game. As usual, we denote the set of positions, or vertices, of G by $V(G)$, and the multiset of moves, or edges, of G by $E(G)$. We also partition the edges of G into Left edges (for Left's moves) and Right edges (for Right's moves.) G will have a special vertex designated as the start vertex, which we call v_G. A *play* of G is a sequence of edges, finite or infinite, the first starting at v_G and each subsequent one starting at the last's end-vertex. The final part of the information associated with G is the designation of each infinite play as a win for Left or Right, or a draw. We assume that two infinite plays which are the same except for a finite initial segment will be similarly designated.

2.1.2 Strategies and outcomes

A play of G is *alternating* if the edges of the play are alternately Left's and Right's. A *strategy* for Left or Right is a rule which tells Left or Right which move to make, given the moves that have already been made. Formally, a strategy for Left moving first in G is a function from finite alternating plays of G with the first move Left's and last move Right's to moves for Left from the plays' final positions. Some plays may have no move for Left existing from the final position, and for these, the strategy of course specifies no move. Strategies for Left moving second and Right moving first or second are defined similarly. If we take a strategy for one player moving first and one for the other moving second, an alternating play will result in the obvious way; it will be finite if, at some point, the moving player is left without a move, and infinite otherwise. If the play is finite, the play will be won by the player with the last move, and otherwise, it will be won as specified in G. The game is then said to be *won* for Left moving first (say) if there is a strategy for Left moving first such that for all strategies for Right moving second, Left wins the resultant play, and *won or drawn* for Left moving first (say) if there is a strategy for Left moving first such that for all strategies for Right moving second, Left wins or draws the resultant play. The *outcome* of a game is the specification for each player of whether he can win moving first, whether he can win moving second, whether he can win or draw moving first, and whether he can win or draw moving second. For a non-loopy game, it is always the case that if one player does not win moving first, the other wins or draws moving second, and that if one player does not win or draw moving first, the other wins moving second. If this is true for a game, we say that the game is *determined.* This is not necessarily the case for infinite games; in fact, its truth is the Axiom of Determinacy (AD) which contradicts the Axiom of Choice. However, this will be true if the designation of infinite plays is relatively simple.

Let us make all plays of our loopy games infinite by adding dummy moves l and r and concatenating a sequence $(l, l, l, \ldots)$ to the end of each finite play where Left moves last and $(r, r, r, \ldots)$ to the end of each finite play where Right moves last. Then, for loopy games G that we consider, there will always be a finite partition $E_1, \ldots, E_n$ of $E(G)$ such that the designation of a play p as won, drawn or lost only depends on whether or not, for each j, p contains infinitely many moves from E_j. If we have made all our plays infinite, we can give the set of all plays of G a topology by taking an open base to be all sets of the form $\{P|P$ is a play of G with initial segment $f\}$. With this topology, provided that the sets of won, drawn or lost plays for G are Borel, G will be determined [5]. If the sets are as above, they will be Borel (in fact Δ^0_3), so our games will be determined.

2.1.3 Operators on loopy games

We define $-G$ to be the same game as G, but with the roles of Left and Right reversed. For graphs G and H, let $G \times H$ be the graph with vertex set $V(G) \times V(H)$, an edge from (a, b) to (a, c) for all $a \in V(G)$ and edges from b to c in H, and an edge from (a, b) to (c, b) for all $b \in V(H)$ and edges from a to c in G. We then define $G + H$ to have the graph $G \times H$ and to have start-vertex (v_G, v_H). Informally, $G + H$ is the game created by putting G and H side by side and allowing each player to move in just one of the games at a time. We often write the position (α, β) of $G + H$, where α is a position of G and β is a position of H, as $\alpha + \beta$. In this notation, and elsewhere, we may sometimes write a game H for its starting vertex, v_H. An infinite play in $G + H$ is won by Left (or Right) if, when we decompose the play into its component plays in G and H, all infinite component plays are won by Left (or Right.) Otherwise it is drawn. We write $G \equiv H$ if G and H have identical graphs, starting vertices, and designations of infinite plays as won, lost, or drawn.

G^+ (or G^-) is the same game as G, except that all draws in G are redefined as wins for Left (or Right.) Evidently, for all G, $(-G)^+ \equiv -G^-$ and $(-G)^- \equiv -G^+$. We will sometimes write $G^\pm$ to mean G^+ or G^-. We say that $G \geq H$ if Left wins or draws in both $G^+ - H^+$ and $G^- - H^-$, moving second. Obviously, $G \geq H$ iff $-H \geq -G$. It can be proved [1] that $\geq$ is reflexive and transitive, and that $G \geq H$ implies that $G + K \geq H + K$, for all loopy games G, H, and K. We write $G = H$ if $G \geq H$ and $H \geq G$. Our definitions of $\geq$ and $=$ evidently imply that $G \geq 0$ iff G is a second-player win for Left, $G \leq 0$ iff G is a second-player win for Right, and $G = 0$ iff G is a second-player win for both players. Also, $G^+ \geq 0$ iff G is a second-player win or draw for Left, and $G^- \leq 0$ iff G is a second-player win or draw for Right. If we observe that $G = H$ implies that $G^+ = H^+$ and $G^- = H^-$, it follows that the equality class of G determines the second-player outcome of G. Since all our games are determined, it follows that the equality class of G determines the outcome of G. If $\{G_\alpha\}$ and $\{G_\beta\}$ are sets of loopy games, let H be the graph with one vertex, v, and no edges, and take the disjoint union of graphs $K = (\cup_\alpha G_\alpha) \cup (\cup_\beta G_\beta) \cup H$. Add moves for Left from v to each v_{G_α} and moves for Right from v to each v_{G_β} to K. We can then make K into a loopy game by giving K the starting vertex v and designating an infinite play in K just as it is designated in whichever of the G_α's or G_β's it eventually ends up in. We will then call this loopy game $\{G_\alpha \mid G_\beta\}$. When no confusion results, we will omit the braces. We will also use double bars, triple bars, and so forth to indicate grouping: we write $M \mid N \parallel P$ for $\{\{M \mid N\} \mid P\}$, $M \mid\mid\mid N \mid P \parallel Q$ for $\{M \mid \{\{N \mid P\} \mid Q\}$, and so on. This notation is as in [1]. Let **on** be the loopy game with one position, p, and one move m for Left from p to p, where the infinite play $(m, m, m, \ldots)$ is considered a draw. (From now on, we will sometimes call moves in a loopy game from a position to itself *passes*.) Then, for all G, if Left always passes in $\mathbf{on}^+$, i.e., plays m, in $\mathbf{on}^+ - G$, he will at least draw, whether he moved first or second. Since $(\mathbf{on}^+)^+ \equiv (\mathbf{on}^+)^- \equiv \mathbf{on}^+$, this proves that $\mathbf{on}^+ \geq H$ for all loopy games H. If we define **off** to be $-\mathbf{on}$, it follows that $H \geq \mathbf{off}^-$ for all loopy games H.

An *ender* is a loopy game G such that every play of G is finite. It follows that the graph of G is acyclic. Playing an ender obviously never gives a draw. Also, when applied to enders, the definitions of addition, negation, and inequalities above are consistent with the usual definitions in the first part of [1] for non-loopy partizan games. For enders G, $G - G = 0$, so the enders form a commutative group under addition. This is not true in general; we saw that $\mathbf{on}^+ - G$ is always a win or draw for Left moving first, so $\mathbf{on}^+ - G \neq 0$ for all G. A loopy game G will be called *infinitesimal* if $-2^{-j} \leq G \leq 2^{-j}$ for all positive integers j.

2.2 Sidling

In [1], *sidling* is presented as a way of approximating loopy games by less loopy or non-loopy games. Sidling works as follows: suppose we have a loopy game G, and a set of moves M in G such that all infinite plays involving infinitely many moves in M are wins for Left. Then let G_0 be G with all moves in M deleted. If $H = (H_v | v \in V(G))$ is a sequence of loopy games with index set the positions of G, let $G_x^*(H)$ be G_0 with moves for Left (or Right) from $v \in V(G_0)$ to the starting vertex of H_w adjoined for each move from v to w for Left (or Right) in M, and with start-vertex changed to

$x \in V(G_0)$. Infinite plays in $G_x^*(H)$ must either have all moves in G_0, when they have the same outcome as in G_0 or G, or all but finitely many initial moves in some H_w, when their outcome is the same as the outcome of the piece of the play in H_w. Now let $G^*(H)$ be the sequence of loopy games $(G_v^*(H)|v \in V(G))$. Sidling then consists of starting with the sequence $K_0 = (\mathbf{on}^+|v \in V(G))$ and, for all positive integers i, setting $K_i = G^*(K_{i-1})$. If, for some m, $K_m = K_{m+1}$, then we have evidently found a fixed point of G^*. Moreover, if we define $A \leq B$ for two sequences A and B of loopy games to mean $A_v \leq B_v$ for all v, then $A \leq B$ implies that $G_x^*(A) \leq G_x^*(B)$, for all x, by an obvious reflection strategy. Hence $A \leq B$ implies that $G^*(A) \leq G^*(B)$. This implies that our fixed point K_m is maximum. But now let G'_v be the game G with start-vertex changed to v, and $G' = (G'_v|v \in V(G))$. Recalling that changing a finite initial segment of an infinite play doesn't change its outcome, we see that the obvious reflection strategy tells us that $G_v^*(G') = G'_v$ for all v. Hence $G^*(G') = G'$, so $G' \leq K_m$.

Theorem 1 *(Sidling Theorem) For all P with $P \leq G^*(P)$, we have $P \leq G'$.*

Proof. In [1], this is proven in the case where $M = E(G)$. The extension is easy. For all $w \in V(G)$, we need to prove that Left wins or draws moving second in $(G'_w)^+ - (P_w)^+$ and in $(G'_w)^- - (P_w)^-$. Consider the first of these two games. Evidently, $(G'_w)^+ = (G^+)'_w$, and if, for a sequence $H = (H_w|w \in V(G))$ of loopy games, we define $H^+ = ((H_w)^+|w \in V(G))$, then $(P_w)^+ = (P^+)_w$. We then need to show that Left wins or draws moving second in $(G^+)'_w - (P^+)_w$. Now $G_w^*(P)^+ = (G^+)_w^*(P^+)$ for all $w \in V(G)$, so $G^*(P)^+ = (G^+)^*(P^+)$. It follows that $(G^+)^*(P^+) = G^*(P)^+ \geq P^+$. We can now substitute G^+ for G and P^+ for P and observe that the hypotheses of the theorem are still satisfied, that the new G and the new P_w's have no drawn infinite plays, and that we wish to show that Left wins or draws moving second in $G'_w - P_w$. If we show this, we can also substitute in G^- and P^- for G and P, where we define H^- for a sequence of loopy games H as we did H^+, and our theorem will be proved.

So we wish to show that Left wins or draws moving second in $G'_w - P_w$. The strategy in [1, pp. 351–353] will almost suffice. The only change is that we might need to use a reflection strategy in $\alpha^+ - \alpha_m$ for a while. (The notation α^+ and α_m is as in [1]. In our notation, for some vertex v, $\alpha^+ = G'_v$ and $\alpha_m = G_v^*(P)$.) If we bring in infinitely many tables, we win the game as in [1]. If, after some point, we don't bring in any new tables, by our strategies for the difference games that make up $G^*(P) - P$,

$$\text{sign}([\alpha]) \leq \text{sign}(\alpha_0) \leq \text{sign}(\alpha_1) \leq \ldots \leq \text{sign}(\alpha_m),$$

but $\text{sign}(\alpha^+) = \text{sign}(\alpha_m)$, since by the reflection strategy, the plays in α^+ and α_m are the same, except for a finite initial segment which doesn't matter. Hence $\text{sign}([\alpha]) \leq \text{sign}(\alpha^+)$. If both of these signs are 0, we get the last move, as in [1]. ∎

This theorem tells us that if $K_m = K_{m+1}$, then $G' \geq K_m$. Combining this with our earlier observation, we have $G' = K_m$. Thus we have found a (hopefully) simplified form for G, namely $(K_m)_{v_G}$. Often, this will be an ender, and in this case, it is easy to determine the outcome of G summed with another ender.

We can also set $K_0 = (\mathbf{off}^- | v \in V(G))$, in which case if we find m with $K_m = K_{m+1}$, K_m will be a minimum fixed point of K. If we take negatives of everything, the Sidling Theorem implies that $G' = K_m$ if we treat all plays in G with infinitely many moves in M as wins for Right, instead of Left.

In [1], we only consider sidling when M is the entire set of moves for G. This is because in [1] we usually consider all plays of a loopy game G as being drawn. Then, to find the outcome of G summed with any other game, it suffices to find the outcomes of G^+ and G^- summed with any other game, and since G^+, for example, has all infinite plays won by Left, we can sidle G^+ using $M = E(G)$. However, in the sequel we will define, for example, loopy games H such that, for some proper subset M of $E(G)$, an infinite play of H will be lost for Left iff it has infinitely many moves from M, and otherwise drawn. Then to compute H^+, for example, we need to first sidle using M. Doing this will entail looking at many loopy games with all infinite plays won for Left, namely, the K_i's. We can then sidle the K_i's themselves to attempt to determine whether some $K_i = K_{i+1}$, using the whole set $E(K_i)$ as M for each K_i. We give an example of this process in §4.2. The process can be performed by machine; in fact, the author has written a program to do so in some cases of interest, including the case in §4.2.

Chapter 3

Ko-ban

3.1 History

The problem with loopy games as we have defined them is that there is no prohibition at all on moving to previous positions, i.e., no ko-ban. Let G be a loopy game with all infinite plays drawn. We then define $\phi^J(G)$ to be, roughly speaking, G with a Japanese ko-ban added and $\phi^N(G)$ to be G with a North American ko-ban added. We will take both $\phi^J(G)$ and $\phi^N(G)$ to have sets of positions that are subsets of $V(G) \times \mathcal{P}(V(G))$. Here, $\mathcal{P}(T)$ is the power-set of T, or the set of all subsets of T. We will write a position (v, S) in such a set as $v[S]$. S is called the *history set* of the position. A move from $v[S]$ to $w[T]$ will occur only if $w \notin S$. $\phi^J(G)$ will have a position $\alpha[\{\beta\}]$ for all positions α and $\beta \neq \alpha$ of G such that there is a move in G from β to α, as well as a position $\alpha[\emptyset]$ for each position α in G. $\phi^J(G)$ has a move for Left (or Right) from $\alpha[S]$ to $\beta[\{\alpha\}]$ for each position $\alpha[S]$ and $\beta \notin S \cup \{\alpha\}$ such that there is a move for Left (or Right) from α to β in G. In addition, for each $\alpha[S]$ such that α has a move by Left (or Right) to itself in G, $\phi^J(G)$ will have a move for Left (or Right) from $\alpha[S]$ to itself. Let α be a position of G and S be a sequence of moves in G from some position β to α, containing no moves from a position to itself and never moving to a previously moved-from position. For all such α's and S's, $\phi^N(G)$ will have a position $\alpha[T]$, where T is the set of positions that some move in S moves out of. ϕ^N will have a move for Left (or Right) from $\alpha[S]$ to $\beta[S \cup \{\alpha\}]$ for each position $\alpha[S]$ and $\beta \notin S \cup \{\alpha\}$ such that there is a move for Left (or Right) from α to β in G, and in addition, for each $\alpha[S]$ such that α has a move by Left (or Right) to itself in G, $\phi^N(G)$ will have a move for Left

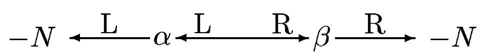


Figure 3.1: The graph of the game H.

(or Right) from $\alpha[S]$ to itself. All infinite plays in $\phi^J(G)$ and $\phi^N(G)$ are drawn. Both $\phi^J(G)$ and $\phi^N(G)$ have start-vertex $v_G[\emptyset]$.

The difficulty with using $\phi^N(G)$ and $\phi^J(G)$ directly is that they are highly dependent on the form of G, and not just its equality class. For example, let H be a game with graph as in Figure 3.1, start vertex β, and all infinite plays drawn; that is, in the notation of §4.1, $H = KO[-N, -N]$. Here, N is a large positive integer (or ordinal.) Then $\phi^N(H)$ and $\phi^J(H)$ are both first-player wins for Left, since Right cannot move from $\alpha[\{\beta\}]$. However, let K be an ender that has a move for Right. Then $\phi^N(H+K)$ and $\phi^J(H+K)$ are both second-player wins for Right, for N sufficiently large. In fact, if Left opens to $\beta + K^L$, Right can reply to $-N + K^L$, which we can assume to be negative. Otherwise Left opens to $\alpha + K$, and Right can move to some $\alpha + K^R$. If N is big we can assume $-N + K^R < 0$, so if Left moves to $-N + K^R$ he loses. Otherwise, Left can only move to some $\alpha + K^{RL}$. Right can then move back to $\beta + K^{RL}$; since the K-component of the position has changed, this does not violate the ko-ban. Left's only possible move is then to some $\beta + K^{RLL}$, and Right can then move to $K^{RLL} - N$ and win. By considering $-H$ as well as H, we can see that if K is an ender, the only way for $\phi^J(G)$ and $\phi^J(G+K)$ or $\phi^N(G)$ and $\phi^N(G+K)$ to always have the same outcome, for all loopy games G, is to have $K \equiv 0$. However, there are many enders K with $K = 0$ but $K \not\equiv 0$, and adding these will sometimes change the outcome, as above.

3.2 Ko-threats

To remedy this difficulty, we approximate $\phi^x(G)$ (where x can be either J or N) by $\phi^x_L(G)$ and $\phi^x_R(G)$. These both have the same set of positions as $\phi^x(G)$, and have all the moves $\phi^x(G)$ has, but differ as follows: the game $\phi^x_L(G)$ has an added move for Left from $\alpha[S]$ to $\beta[T]$, for all positions $\alpha[S]$ and all Left's moves from $\alpha[\emptyset]$ to some $\beta[T]$, and the game $\phi^x_R(G)$ has an added move for Right from $\alpha[S]$ to $\beta[T]$, for all positions $\alpha[S]$ and all Right's moves from $\alpha[\emptyset]$ to some $\beta[T]$. (Here, α could be equal to β.) An infinite play in ϕ_L^x (or ϕ_R^x) will be considered a loss for Left (or Right) if it contains an infinite number of these extra moves; otherwise, it will be a draw. The effect is the same as if Left (or Right) was allowed to, finitely often, reset the history set to the empty set at any desired time before his move. When this happens we say that Left (or Right) *uses a ko-threat.* This is essentially what Right accomplished with his move on K above. After a move on an ender K which is a summand of our position, from $K+G$ to K^R+G, say, there is no way to move back to a position involving K as a summand. Hence, as far as any further moves are concerned, we may treat the history set as being empty. This was also the effect of the ko-threat in Figure 1.5. In terms of Go, $\phi_L^x(G)$ represents the position G together with very many (but not infinitely many) ko-threats for Left, and $\phi_R^x(G)$ represents the position G together with very many ko-threats for Right.

We also define $\phi_{LR}^x(G)$ and $\phi_{RL}^x(G)$, for $x = J$ or N. These games both have extra ko-threat using moves as above for both Left and Right. However, the outcomes of infinite plays are defined differently. In $\phi_{LR}^x(G)$, if Right uses an infinite number of ko-threats, he loses. Otherwise, if Left uses an infinite number of ko-threats, he loses. Otherwise the play is a draw. In $\phi_{RL}^x(G)$, if Left uses an infinite number of ko-threats, he loses. Otherwise, if Right uses an infinite number of ko-threats, he loses, and otherwise the play is drawn. The effect is that in $\phi_{LR}^x(G)$, Right has very many ko-threats, but Left has very many more than Right, and similarly for $\phi_{RL}^x(G)$.

Theorem 2 *For $x = J$ or N, $\phi_L^x(G) \geq \phi^x(G) \geq \phi_R^x(G)$, and $\phi_L^x(G) \geq \phi_{LR}^x(G) \geq \phi_{RL}^x(G) \geq \phi_R^x(G)$.*

Proof. Since $\phi_{LR}^x(G)$ and $\phi_{RL}^x(G)$ are the same game, with infinite plays defined more favorably for Left in the first than in the second, the obvious reflection strategy gives $\phi_{LR}^x(G) \geq \phi_{RL}^x(G)$. For the other inequalities, it suffices to see that if H and K are the same loopy game except with moves added for Left (or Right) in H, and with infinite plays common to both H and K defined the same way in both, then $H \geq K$ (or $K \geq H$,) again by a reflection strategy. ∎

3.3 Identities

We say that $\phi_y^x(G)$ is *independent of history* if, for all positions $\alpha[S]$ and $\alpha[T]$ of $\phi_y^x(G)$, $\phi_y^x(G)'_{\alpha[S]} = \phi_y^x(G)'_{\alpha[T]}$.

Lemma 3 *If G and H are the same except that some positions of G possess moves by Right to* $\mathbf{on}^+$ *which are absent in H, then $G = H$. (By negation, we will also have $G = H$ if some positions of G possess moves by Left to* $\mathbf{off}^-$ *which are absent in H.)*

Proof. We need to show that the second player wins or draws in $G^+ - H^+$ and $G^- - H^-$. By replacing G and H by G^+ and H^+ or G^- and H^-, we see that it will do to show that the second player wins or draws in $G - H$. If we are Right we can use the reflection strategy. If we are Left we can use it until Right moves to $\mathbf{on}^+$ in G, but then we can move in $\mathbf{on}^+$ forever afterwards and hence win or draw. ∎

Theorem 4 *If $\phi^x_L(G)$ is independent of history, so is $\phi^x_{LR}(G)$, and in fact $\phi^x_L(G)'_v = \phi^x_{LR}(G)'_v$ for all positions v. Similarly, if $\phi^x_R(G)$ is independent of history, so is $\phi^x_{RL}(G)$, and $\phi^x_R(G)'_v = \phi^x_{RL}(G)'_v$ for all v.*

Proof. We prove the first half; the second follows by negating G. Let $H = \phi^x_L(G)$ and $I = \phi^x_{LR}(G)$. Then I has been defined to be the same as H with extra moves M for Right added, such that all play with infinitely many moves in M is a win for Left. As in our discussion of the Sidling Theorem, let $K_0 = (\mathbf{on}^+ | v \in V(I))$, and for all $i \geq 0$, let $K_{i+1} = I^*(K_i)$. Then $K_1 = H'$. This is because if we look at some $(K_1)_v$ and H'_v, they will be the same except that some positions in $(K_1)_v$ have extra moves by Right to $\mathbf{on}^+$, so we can use our Lemma. We claim that $I^*(H') = H'$, so $K_2 = K_1$. If this is true it will prove the desired result, by the Sidling Theorem. A reflection strategy in $I^*_v(H')^\pm - (H'_v)^\pm$ only fails for Left moving second, and then, only when Right uses a ko-threat in the first term and moves to $(H'_{\beta[S]})^\pm - (H'_{\alpha[T]})^\pm$, say. But $H'_{\alpha[T]} = H'_{\alpha[\emptyset]}$, so this game has the same outcome as $(H'_{\beta[S]})^\pm - (H'_{\alpha[\emptyset]})^\pm$, and Left has a move in $-(H'_{\alpha[\emptyset]})^\pm$ to $-(H'_{\beta[S]})^\pm$, so $(H'_{\beta[S]})^\pm - (H'_{\alpha[\emptyset]})^\pm$ is won or drawn by Left moving first. This proves the desired result. ∎

Theorem 5 *If $\phi^x_y(G)$ is independent of history, then $\phi^x_y(G + H) = \phi^x_y(G) + H$ for all enders H, and $\phi^x_y(G + H)$ is independent of history.*

Proof. We induce on H, taking our induction hypothesis to be that $\phi^x_y(G + H)'_{v+H} = \phi^x_y(G)'_v + H$ for all positions v of $\phi^x_y(G)$, where if $v = \alpha[\{\beta_1, \ldots, \beta_n\}]$, then we take $v + H$ to be $(\alpha + H)[\{\beta_1 + H, \ldots, \beta_n + H\}]$. If $H \equiv 0$, then the result is clear, since $G + H \equiv G$

and $\phi_y^x(G)+H \equiv \phi_y^x(G)$. Otherwise, assume the result is known for all options of H. We want to show that the second player wins or draws $(\phi_y^x(G+H)'_{v+H})^\pm - (\phi_y^x(G)'_v)^\pm - H$. We can play the reflection strategy until the first player moves in one of the H's. When this happens we will move in the other one. In the resultant position, the history in $\phi_y^x(G+H)$ will all be inaccessible to us, since H's graph is acyclic. Our position will then be equivalent to $(\phi_y^x(G+H)'_{(\alpha+I)[\emptyset]})^\pm - (\phi_y^x(G)'_{\alpha[S]})^\pm - I$ for some option I of H and position $\alpha[S]$ of $\phi_y^x(G)$. By history independence, this game will equal $(\phi_y^x(G+H)'_{(\alpha+I)[\emptyset]})^\pm - (\phi_y^x(G)'_{\alpha[\emptyset]})^\pm - I$. This will be a win or draw for the second player, by the induction hypothesis. This concludes the induction. History-independence of $\phi_y^x(G+H)$ follows once we remember that $\phi_y^x(G+H)'_{(\alpha+H)[S]}$ is certainly independent of the portion of S not involving H, as we have already remarked, and is independent of the portion involving H by the induction hypothesis. ∎

We say that G has no *bad Japanese (J) loops* if there are no positions α and $\beta \neq \alpha$ of G with moves by the same player from α to β and from β to α. Evidently, if G and H have no bad J loops, neither do $-G$ or $G+H$. Also, enders always have no bad J loops.

Theorem 6 *If G has no bad J loops, then $\phi_L^J(G)$ and $\phi_R^J(G)$ are independent of history.*

Proof. We prove this for $\phi_L^J(G)$; this will follow for $\phi_R^J(G)$ by negating G. It will do to show that $\phi_L^J(G)'_{\alpha[\emptyset]} = \phi_L^J(G)'_{\alpha[\{\beta\}]}$ for all positions α of G with moves from $\beta \neq \alpha$ to α in G. The reflection strategy for the second player in $(\phi_L^J(G)'_{\alpha[\{\beta\}]})^\pm - (\phi_L^J(G)'_{\alpha[\emptyset]})^\pm$ will work, unless the first player moves in the second term, from $\alpha[\emptyset]$ to $\beta[\{\alpha\}]$. But in this case, if we are Left, there is a move from α to β in G for Left, so we can use a ko-threat and move in the first term from $\alpha[\{\beta\}]$ to $\beta[\{\alpha\}]$, and use the reflection strategy thereafter. If we are Right, there is a move from α to β in G for Right, and there is a move from β to α in G for some player, which must be Left, by assumption. Then we can use a ko-threat in the second term and move from $\beta[\{\alpha\}]$ to $\alpha[\{\beta\}]$, and use the reflection strategy thereafter. ∎

Corollary 7 *If G has no bad J loops, then $\phi_L^J(G) = \phi_{LR}^J(G)$ and $\phi_R^J(G) = \phi_{RL}^J(G)$ are independent of history. Furthermore, if H is an ender, and y is L, R, LR, or RL, then $\phi_y^J(G+H) = \phi_y^J(G) + H$.*

Proof. This follows immediately from the preceding three theorems. ∎

We say that G has *no bad North American (N) loops* if, for any position v of G such that there is a sequence of an odd number of moves from v to v, there is a move from v to itself. Enders have no bad N loops, and the set of games having no bad N loops is closed under addition and negation.

Theorem 8 *Let G have no bad N loops. Then if $\alpha[\emptyset]$ is a position of $\phi_L^N(G)$, won (or won or drawn) by Left moving first or second, and $\beta[S]$ is a position moved on by Left (or Right) when Left plays his winning (or winning or drawing) strategy starting from $\alpha[\emptyset]$, then $\beta[T]$ is also won (or won or drawn) by Left when Left (or Right) moves first on it, for all $T \subseteq S$, and after each of Left's moves in his winning (or winning or drawing) strategy starting from $\beta[T]$, the position is some $\gamma[U]$, where for a position $\gamma[V]$ moved on by Right when Left plays his winning (or winning or drawing) strategy starting from $\alpha[\emptyset]$, $V \supseteq U$. (A similar statement holds for Right.)*

Proof. We prove this by induction on the size of S. If $S = \emptyset$ it is clear. Otherwise Left can play after $\beta[T]$ pretending it was $\beta[S]$. This will be unproblematic until Right moves to a position $\gamma[U]$ that was prohibited by history in the play from $\beta[S]$. But then $\gamma \in S$, and we reached $\beta[S]$ by playing our winning strategy from $\alpha[\emptyset]$. Hence someone must have moved from some $\gamma[V]$ in this play, where V is a proper subset of S. If it was Left, then we can win (or win or draw) $\gamma[V]$, moving first, and by induction, we can win (or win or draw) $\gamma[\emptyset]$, moving first. By using a ko-threat, we can play $\gamma[U]$ as if it were $\gamma[\emptyset]$. The desired result then follows.

If it was Right, then by hypothesis, there must be a move from γ to itself in G for Left or Right. If it was for Left, then we can move to $\gamma[\emptyset]$ by using a ko-threat, and induce as before. If it was for Right, then we can pretend that Right just passed after our move to $\gamma[V]$ in our winning strategy. Our strategy must then tell us how to win (or win or draw) $\gamma[V]$, moving first, so we can induce as before. ∎

Theorem 9 *Let G have no bad N loops. Then the outcomes of $\phi_{LR}^N(G)$ are at least as good for Left as those of $\phi_L^N(G)$. Hence $\phi_{LR}^N(G)$ and $\phi_L^N(G)$ have the same outcomes. Similarly, $\phi_{RL}^N(G)$ and $\phi_R^N(G)$ have the same outcomes.*

Proof. This is an immediate consequence of the preceding theorem, which we can apply every time Right uses a ko-threat. ∎

Theorem 10 *Let $H \geq 0$ be an ender. Then the outcomes of $\phi_y^N(G+H)$ are at least as good for Left as those of $\phi_y^N(G)$, where $y = LR$, R, or RL.*

Proof. To play $\phi_y^N(G+H)$, Left can play his second-player winning strategy in H, and attempt to play his hypothetical winning (or winning or drawing) strategy moving first or second in $\phi_y^N(G)$ as well. The only problem is that at any time, before Right's move, the history might be effectively reset to the empty set, because of an exchange in H. But this has the same effect as letting Right use finitely many ko-threats, and Right is already allowed to do this in all our cases. ∎

Corollary 11 *Let y be L, R, LR, or RL, and let G have no bad N loops. If $H \geq 0$ is an ender, the outcomes of $\phi_y^N(G+H)$ are at least as good for Left as those of $\phi_y^N(G)$; if $H \leq 0$ is an ender, the outcomes of $\phi_y^N(G+H)$ are at least as good for Right as those of $\phi_y^N(G)$; and if $H = 0$ is an ender, the outcomes of $\phi_y^N(G+H)$ are the same as those of $\phi_y^N(G)$.*

Proof. The third statement follows from the first two, and the second follows from the first by negation. The first statement follows from the last two theorems. ∎

Hence the use of ko-threats does at any rate solve the problem we noted in §3.1, namely, that the outcome of a game might change upon adding a zero ender.

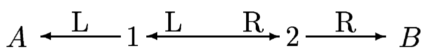
Chapter 4

Kos as combinatorial games

4.1 Definitions of kos

If A and B are loopy games with all infinite plays drawn, $m \geq 3$, and $1 \leq i \leq m-1$, a *ko*, $KO_i^m[A,B]$, is a loopy game with all infinite plays drawn. Aside from the positions in A and B, it has positions 1, 2, $\ldots$, $m-1$, moves for Right from i to $i+1$ and Left from $i+1$ to i for $i = 1, 2, \ldots, m-2$, a move for Left from 1 to v_A, and a move for Right from $m-1$ to v_B. The starting vertex is i. We write $KO[A,B] = KO_2^3[A,B]$ and $OK[A,B] = KO_1^3[A,B]$. We show the graph of $KO[A,B]$ and $OK[A,B]$ in Figure 4.1. The Go positions in Figure 1.3 will correspond to $KO[1,0]$ (on the left) and $OK[0,-1]$ (on the right), with Chinese scoring.

We write $KOL_i^m[A,B]$, $KOL[A,B]$, and $OKL[A,B]$ for games that are the same as the games we just defined except that each position 1, 2, $\ldots$, $m-1$ has a move by Left to itself, and similarly for $KOR_i^m[A,B]$, $KOR[A,B]$, and $OKR[A,B]$. Evidently, these games all have no bad J or N loops, assuming that A and B also do not. We will use KOL, OKL, KOR and OKR to represent kos in Japanese scoring. In Japanese scoring it is possible for players to pass. We usually don't model the pass in our theory because, although it is usually not a good move in Go, its presence complicates the mathematical theory. However, if a player has had a move banned by ko-ban, the pass may become a reasonable move. Hence we allow passes, but only when there is a ko present, and only for the player who does not have more ko-threats; for example, we would compute $\phi_y^x(KOR[1,0])$ for $y = L$ or LR and $\phi_y^x(KOL[1,0])$ for $y = R$ or RL, but not vice versa.

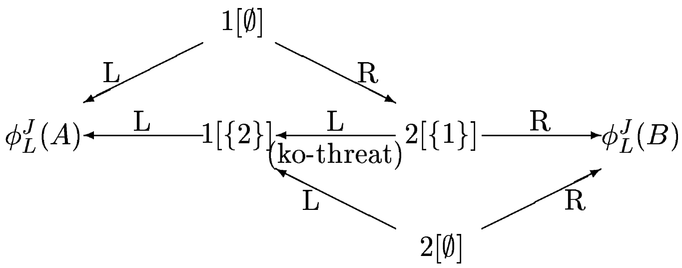


1 is the start-vertex for $OK[A, B]$.

2 is the start-vertex for $KO[A, B]$.

All infinite plays are drawn.

Figure 4.1: The graph of $KO[A, B]$ and $OK[A, B]$.



$1[\emptyset]$ is the start-vertex for $\phi_L^J(OK[A, B])$.

$2[\emptyset]$ is the start-vertex for $\phi_L^J(KO[A, B])$.

Figure 4.2: The graph of $\phi_L^J(KO[A, B])$ and $\phi_L^J(OK[A, B])$.

It is easy to see that J and N history behave the same for these games. Let A and B be loopy games; then $\phi^x_y(KO[A,B])$ and $\phi^x_y(OK[A,B])$ introduce no cycles other than those in $\phi^x_y(A)$ and $\phi^x_y(B)$. In fact, we illustrate the graph of $\phi^J_L(KO[A,B])$ and $\phi^J_L(OK[A,B])$ in Figure 4.2. We can then compute, for $x = J$ or N,

$$\begin{aligned}
\phi^x_y(KO[A,B]) &= \{\{\phi^x_y(A) \mid\} \mid \phi^x_y(B)\} && (y = L \text{ or } LR.)\\
\phi^x_y(KO[A,B]) &= \{\phi^x_y(A) \mid \{\mid \phi^x_y(B)\} \parallel \phi^x_y(B)\} = \{\mid \phi^x_y(B)\} && (y = R \text{ or } RL.)\\
\phi^x_y(OK[A,B]) &= \{\phi^x_y(A) \parallel \{\phi^x_y(A) \mid\} \mid \phi^x_y(B)\} = \{\phi^x_y(A) \mid\} && (y = L \text{ or } LR.)\\
\phi^x_y(OK[A,B]) &= \{\phi^x_y(A) \mid \{\mid \phi^x_y(B)\}\} && (y = R \text{ or } RL.)
\end{aligned}$$

The second equalities in some lines follow from Left's or Right's move being reversible in the more complicated position. It follows from these equalities that $\phi^x_y(KO[A,B])$ and $\phi^x_y(OK[A,B])$ are always independent of history, as long as $\phi^x_y(A)$ and $\phi^x_y(B)$ are. (This is essentially a special case of Theorem 6.) In this case, by Theorem 5, we can add an ender H to the left- and right-hand side of these equations. Similarly, for $y = L$ or LR, we have $\phi^x_y(KO^m_i[A,B]) = \{\ldots\{\phi^x_y(A) \mid\}\ldots \mid\}$, with i bars, if $1 \le i \le m-2$, and $\phi^x_y(KO^m_{m-1}[A,B]) = \{\phi^x_y(KO^m_{m-2}[A,B]) \mid \phi^x_y(B)\}$, and these games are independent of history, if $\phi^x_y(A)$ and $\phi^x_y(B)$ are. If $y = R$ or RL, we can negate these equalities.

If A is an integer with $A \ge -1$, then $\{A \mid\} = A + 1$, so if B is an ender, we can simplify these formulas to give $\phi^x_L(KO[A,B]) = \{A+1 \mid B\}$ and $\phi^x_L(OK[A,B]) = A+1$. Under the same conditions, we have $\phi^x_L(KO^m_i[A,B]) = A + i$ for $1 \le i \le m-2$ and $\phi^x_L(KO^m_{m-1}[A,B]) = \{A + m - 2 \mid B\}$.

If A and B are numbers, we can compute, for $x = J$ or N,

$$
\begin{array}{lll}
\phi^x_y(KOR[A,B]) = & \{(A+\frac{1}{\mathbf{on}}) \mid B\} & (y = L \text{ or } LR,\ A \geq B.) \\
\phi^x_y(KOR[A,B]) = & A+\frac{1}{\mathbf{on}} & (y = L \text{ or } LR,\ A < B.) \\
\phi^x_y(OKR[A,B]) = & A+\frac{1}{\mathbf{on}} & (y = L \text{ or } LR.) \\
\phi^x_y(KOL[A,B]) = & B-\frac{1}{\mathbf{on}} & (y = R \text{ or } RL.) \\
\phi^x_y(OKL[A,B]) = & B-\frac{1}{\mathbf{on}} & (y = R \text{ or } RL,\ A < B.) \\
\phi^x_y(OKL[A,B]) = & \{A \mid (B-\frac{1}{\mathbf{on}})\} & (y = R \text{ or } RL,\ A \geq B.)
\end{array}
$$

As before, the games will be independent of history in this case. $\frac{1}{\mathbf{on}}$ is the loopy game with two vertices, z and w, a move by Right from z to itself, a move by Left from z to w, starting vertex z, and all infinite plays drawn. It can be thought of as an infinitesimal positive number.

If a ko $KO^m_i[A,B]$, $KOL^m_i[A,B]$, or $KOR^m_i[A,B]$ has $A = L+\delta$ and $B = M+\epsilon$, where L and M are numbers and δ and ϵ are infinitesimal enders, we call it a $(L-M)$-*point* ko, and call its *mean value* $(iM+(m-i)L)/m$. When Left moves on a ko, we say he moves it *to the left*, and when Right moves on a ko, we say he moves it *to the right.*

4.2 An example of sidling

When we just have one ko, it is easy to compute its ϕ^x_y value, as we saw in the last section. For more complicated sums of kos, one way to compute the ϕ^x_y value is to use sidling (§2.2.) For example, let $I = KO[7,0]+KO[1,0]$. We wish to compute $\phi^J_L(I)$.

The graph of $\phi^J_L(I)$ will consist of a subgraph, N, where both kos are still present, the graphs of various subgames where one ko has been destroyed, namely $\phi^J_L(7+KO[1,0])$, $\phi^J_L(7+OK[1,0])$, $\phi^J_L(KO[1,0])$, $\phi^J_L(OK[1,0])$, $\phi^J_L(1+KO[7,0])$, $\phi^J_L(1+OK[7,0])$, $\phi^J_L(KO[7,0])$, and $\phi^J_L(OK[7,0])$, and some moves out of N into these graphs of subgames. We can replace these subgames with their values from §4.1 without changing the overall value of $\phi^J_L(I)$. This is from a reflection strategy like that used in §2.2 to show that $A \leq B$ implies $G^*_x(A) \leq G^*_x(B)$. The graph we will then be left with is N, with some moves out of N into various enders, which we collectively call $\mathcal{E}$. If we let N_h be the subgraph of N induced by the set of vertices $\eta[S]$ with $S \neq \emptyset$, then we see that in N, there are no moves out of N_h into the remaining vertices of N. By Theorem 6, finding the values of the vertices in N_h will be enough to find the values of all the vertices in N, where by the value of a vertex v we mean the value of $\phi_L^J(I)'_v$. We consider N_h, with its moves into $\mathcal{E}$, to be a game, with the same results for infinite play as $\phi_L^J(I)$, and some arbitrarily chosen starting vertex. Call this game P. It will then do to look at P.

In Figure 4.3, P is shown in the center. For reference, $\phi_L^J(KO[1,0])_h$, or the subgraph of $\phi_L^J(KO[1,0])$ with nonempty history, is shown at the top and lower left, and $\phi_L^J(KO[7,0])_h$ is shown at the upper left and bottom. In the top half of P, a move on the 1-point ko is shown by a horizontal line, and a move on the 7-point ko is shown by a vertical line. In the bottom half of P, this convention is reversed. The moves which correspond to Left using a ko-threat are indicated as ko-threat-using moves and are always shown as vertical lines going from one half to the other. (Technically, according to our definition in §3.2, Left has many more moves using ko-threats; in fact, he can make with a ko-threat any move he can make without a ko-threat. However, Left has no reason to use a ko-threat unless he has to, since changing any number of ko-threat moves to regular moves in a play will always improve its outcome for Left, so we can omit these moves.) We have assigned names to the positions in N_h so that we can write $N_h = \{A, B, C, D, E, F, G, H\}$.

We try to approximate P's value by sidling. We will start by approximating P^+. We will let M be the set of Left's ko-threat-using moves. Remembering that all plays using infinitely many moves in M are lost for Left, and referring to §2.2, we see that we have to start sidling from $\mathbf{off}^-$. Let K_0 be the sequence of games $(\mathbf{off}^- | v \in N_h)$. (Since we will never have a move in M entering $\mathcal{E}$, we can evidently omit the positions in $\mathcal{E}$ as subscripts of the K_i's.) The next step is to construct $(P^+)^*(K_0) = K_1$. K_1 is a sequence of games which themselves are loopy. In fact, if we let $\bar{P}$ (shown in Figure 4.4) be P with the moves in M removed, then any $(K_1)_v$ will be the same as $(\bar{P}^+)'_v$, by Lemma 3. So we need to sidle $\bar{P}$, and now we can let M be the set of all the moves in $\bar{P}$ that do not destroy a ko, that is, all those moves not into or within $\mathcal{E}$.

Our first approximation to K_1, K_{10}, will be $(\mathbf{on}^+ | v \in N_h)$. (We omit the positions in $\mathcal{E}$ as subscripts here too.) We must then set $K_{11} = (\bar{P}^+)^*(K_{10})$. Given a sequence of approximate values for the vertices of $\bar{P}$, applying $(\bar{P}^+)^*$ refines it by taking just one more move in $\bar{P}$. For example, we have $(\bar{P}^+)^*_A(Q) = \{\{9 \mid 7\}, Q_B \mid 8\}$, so $(K_{11})_A = \{\mathbf{on}^+ \mid 8\}$. The other $(K_{11})_v$'s are computed similarly. Their values are given

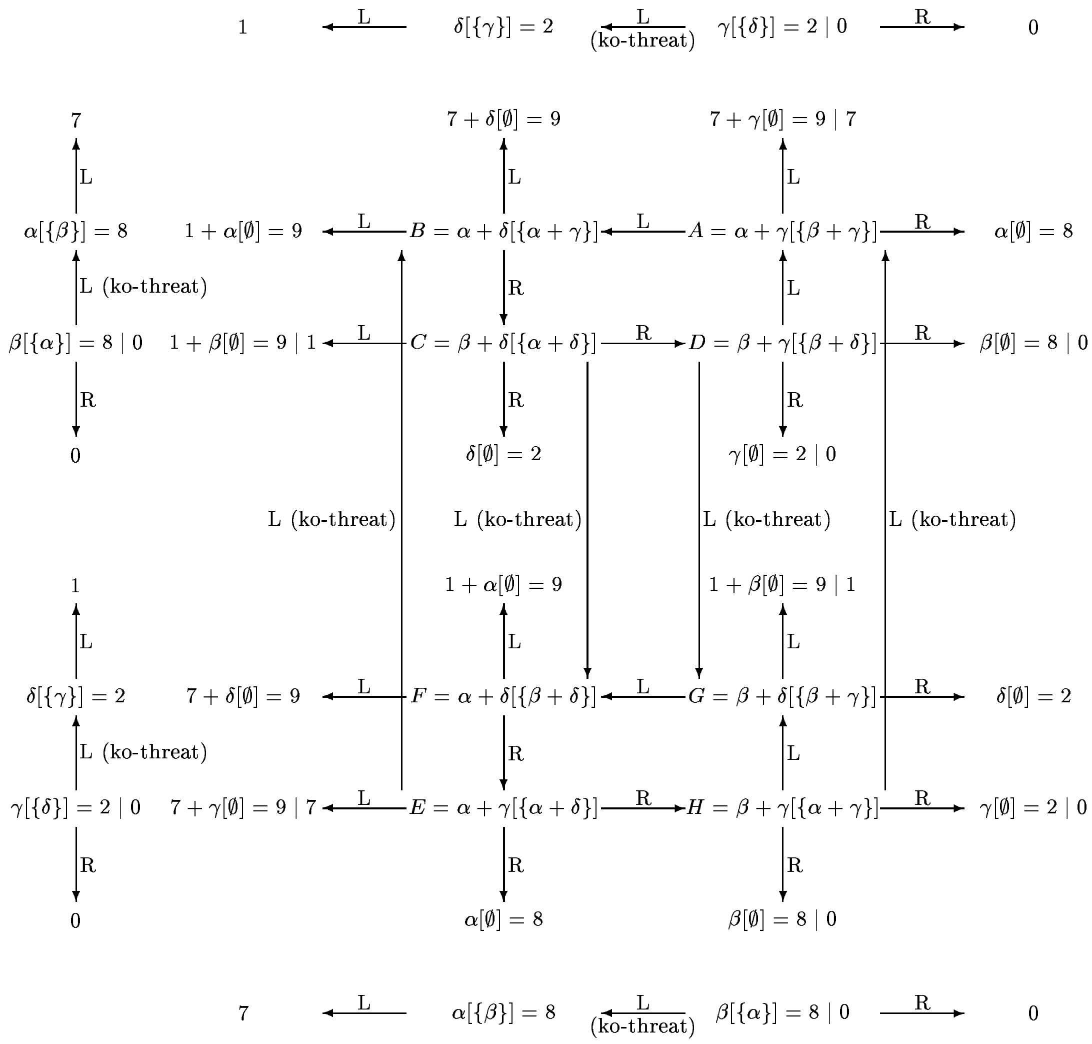


Figure 4.3: An example of sidling. P is shown in the center.

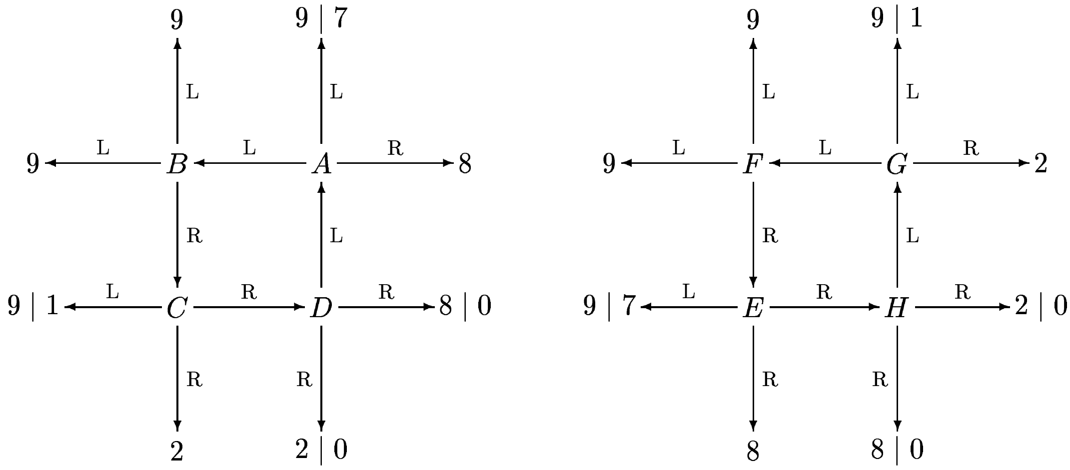


Figure 4.4: The graph $\bar{P}$.

in Table 4.1, and their graphs are shown in Figures 4.5 and 4.6.

We must then compute $K_{12} = (\bar{P}^+)^*(K_{11})$, $K_{13} = (\bar{P}^+)^*(K_{12})$, and so on. These values are also in Table 4.1; the graphs of the $(K_{12})_v$'s and $(K_{13})_v$'s are shown in Figures 4.5 and 4.6. Eventually, we find that $K_{14} = K_{15}$. We may then conclude, from the Sidling Theorem, that $(\bar{P}^+)' = K_1 = K_{15}$.

As we can see from Figure 4.5, the graphs of the $(K_{1j})_v$'s for $v \in \{A, B, C, D\}$ form a set of four spirals. To compute the $(K_1)_v$'s for $v \in \{A, B, C, D\}$, it would in fact suffice to look at any one of the four spirals. In general, we can sidle a game T as follows: we first let $(S_v | v \in V(T))$ be a sequence of games, all equal to $\mathbf{on}^+$ or $\mathbf{off}^-$, as appropriate, and pick an ordering $(v_1, \ldots, v_k)$ of $V(T)$. Then we set S_{v_1} to $T^*_{v_1}(S)$, set S_{v_2} to $T^*_{v_2}(S)$ (using our new value of S_{v_1},) set S_{v_3} to $T^*_{v_3}(S)$ (using our new values of S_{v_1} and S_{v_2},) and so on, finally setting S_{v_k} to $T^*_{v_k}(S)$. We repeat these k steps until none of the k steps alter S. At this point, we will have $S = T'$, assuming that an appropriate condition on infinite play in T is satisfied. This procedure is used in [1, Chapter 11]; it may save computational effort. However, we will continue to show all four spirals, in accordance with the way we presented sidling in §2.2.

We return to sidling with M equal to Left's ko-threat-using moves, and set $K_2 = (P^+)^*(K_1) = (P^+)^*((\bar{P}^+)')$. K_2 is also a sequence of loopy games. $(K_2)_v$ will

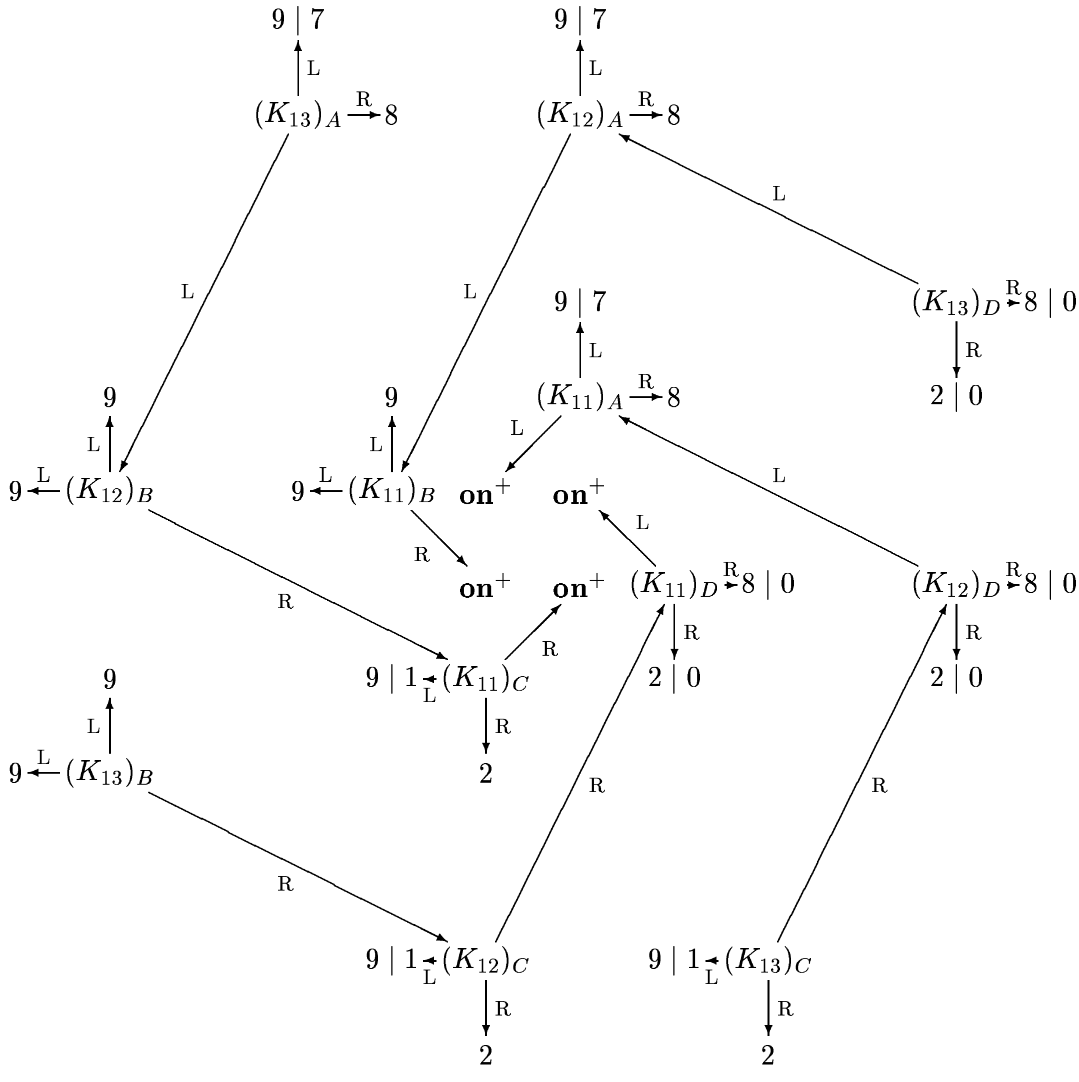


Figure 4.5: The graphs of $(K_{1i})_v$, for $i = 1, 2, 3$ and $v = A, B, C, D$.

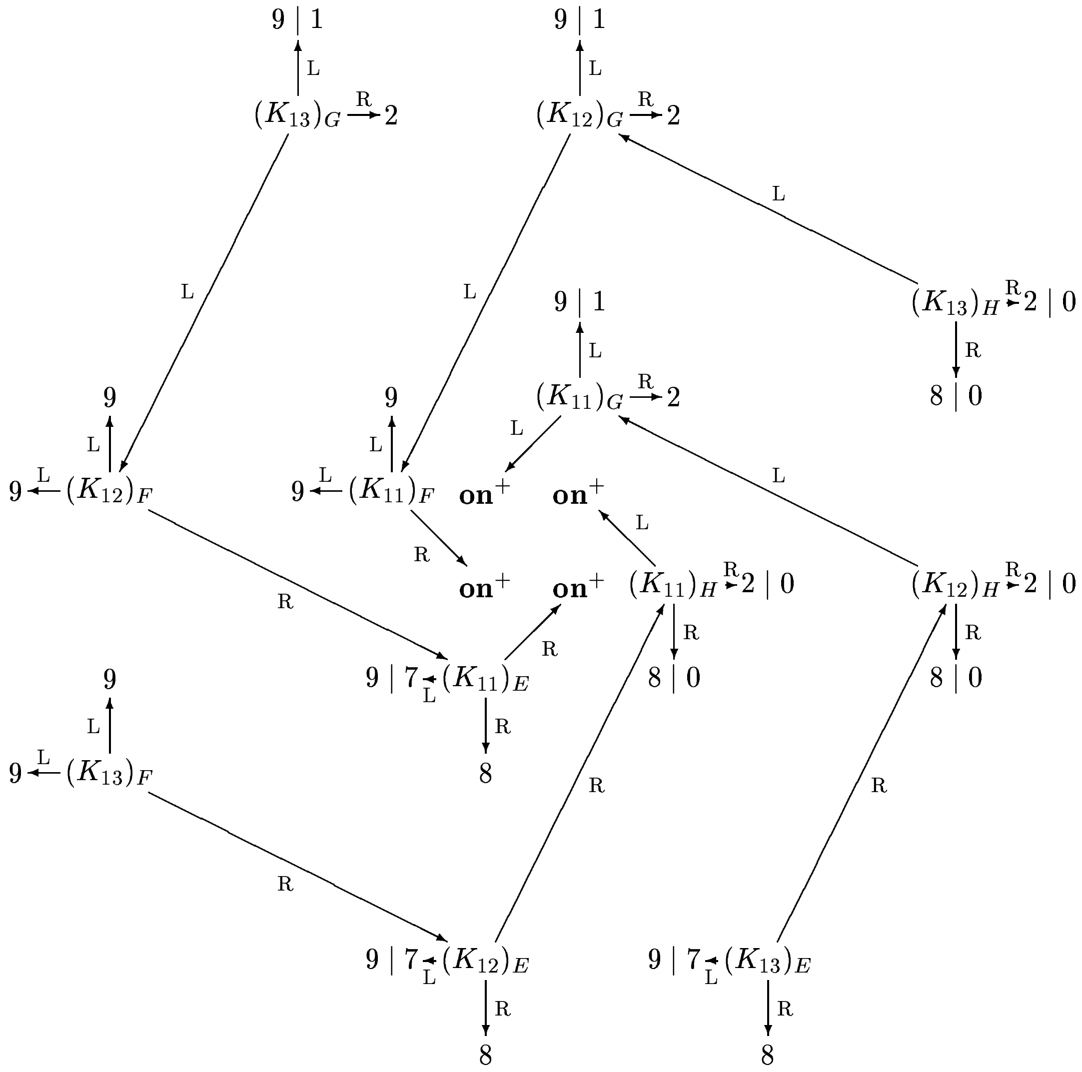


Figure 4.6: The graphs of $(K_{1i})_v$, for $i = 1, 2, 3$ and $v = E, F, G, H$.

be the same as $(\bar{P}^+)'_v$, except that extra moves are present: if P has a ko-threat-using move for Left going from E to B (say), then $(K_2)_v$ has a move for Left from E to $(K_1)_B = (\bar{P}^+)'_B$. In fact, we can write $(K_2)_v = Q_v$ for all $v \in N_h$, where Q is the graph in Figure 4.7. By replacing the bottom copy of $\bar{P}$ in Q with the enders in K_{15}, we get the simplified graph Q_2 in Figure 4.8. To approximate K_2, we can then sidle with all those moves in $\bar{P}$ that do not destroy a ko, as we did with K_1. We can then approximate K_2 by a sequence K_{20}, K_{21}, K_{22}, $\ldots$, as we did with K_1; we display graphs of some $(K_{2j})_v$'s in Figures 4.9 and 4.10. We find that $K_{24} = K_{25}$, so $K_2 = (P^+)^*(K_1) = K_{25}$.

We can compute K_3, and its approximations, K_{30}, K_{31}, K_{32}, $\ldots$, similarly. A graph R with $(K_3)_v = R_v$ for all $v \in N_h$ is shown in Figure 4.11, and a simplified graph R_2, obtained by replacing the copy of Q on the bottom of R_2 by the enders in K_{25}, is shown in Figure 4.12. We find that $K_{34} = K_{35}$ and that $K_3 = K_{35} = K_2 = K_{25}$, so that we can write $(P^+)' = K_2 = K_{25}$.

In an analogous manner, we may compute $(P^-)'$. In this case, we find that $(P^-)'$ is also equal to the sequence we just called K_{25}, so $(P^+)' = (P^-)' = K_{25}$. Every element of K_{25} is an ender, so it follows that $P' = K_{25}$. As we remarked earlier, by Theorem 6, $\phi^J_L(I) = P'_D = P'_H$, so $\phi^J_L(I) = 7 \parallel 2 \mid 0$.

As we will see in the next section, strategic arguments sometimes offer a simpler way of computing ϕ^x_y values.

4.3 Kos with integers

When more than one ko is involved, the details of history become more complicated. However, in some cases, we can fix a most valuable ko that we prefer to move on at all times. In this case, history will essentially be relevant only for the most valuable ko.

One such case is when we are dealing with $\phi^x_y(\sum_{j=1}^s KO^m_{i_j}[A_j, B_j])$, where each $A_j \geq -1$ and all A_j's and B_j's are integers. For a sequence $v_0, \ldots, v_n$ of integers, define the ender $H_{v_0,\ldots,v_n}$ as follows:

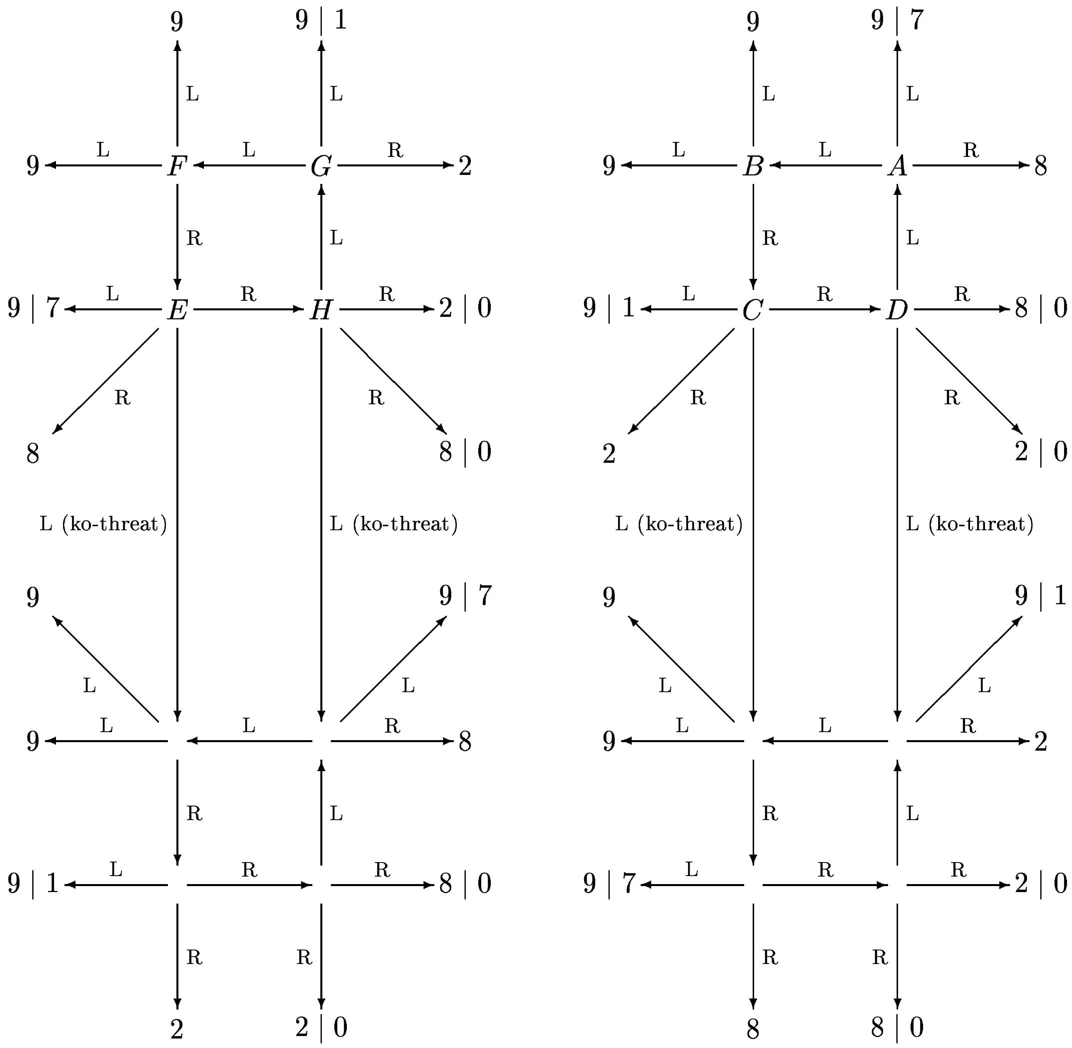


Figure 4.7: The graph Q.

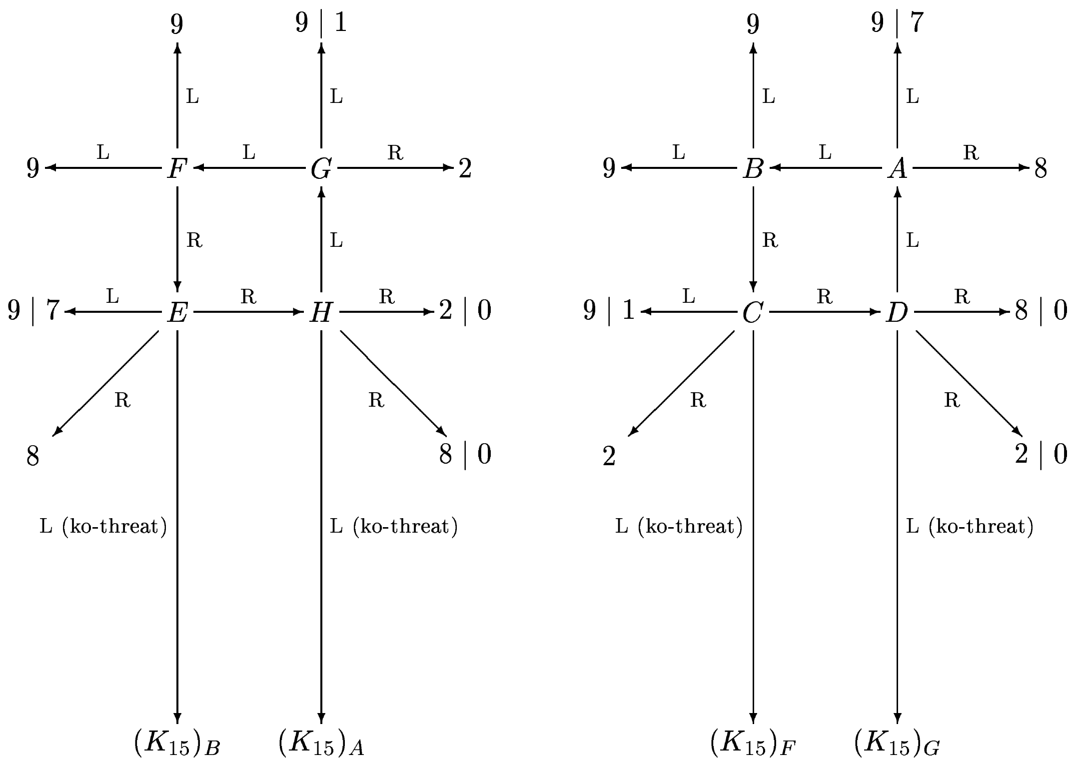


Figure 4.8: The graph Q_2.

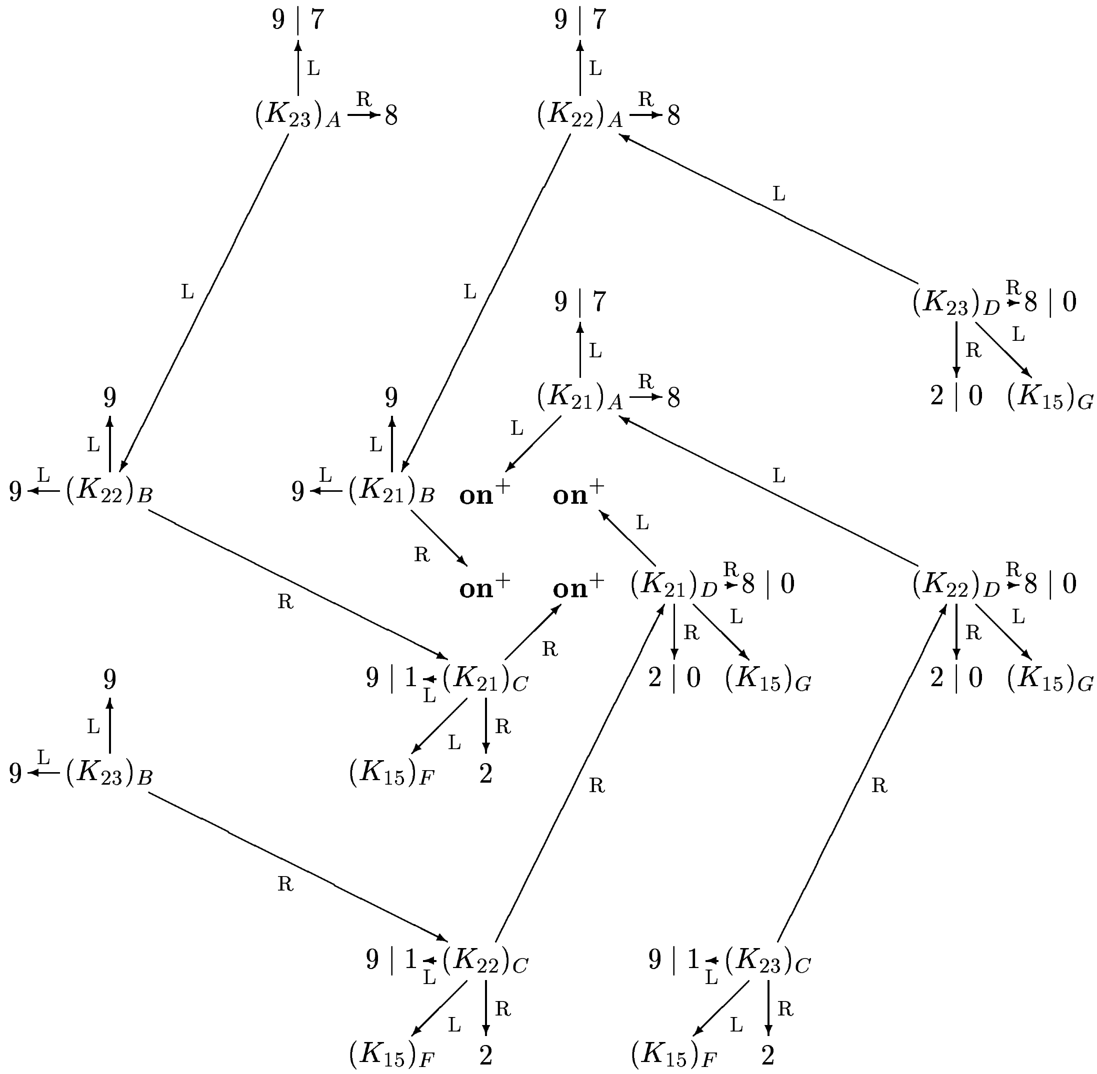


Figure 4.9: The graphs of $(K_{2i})_v$, for $i = 1$, 2, 3 and $v = A$, B, C, D.

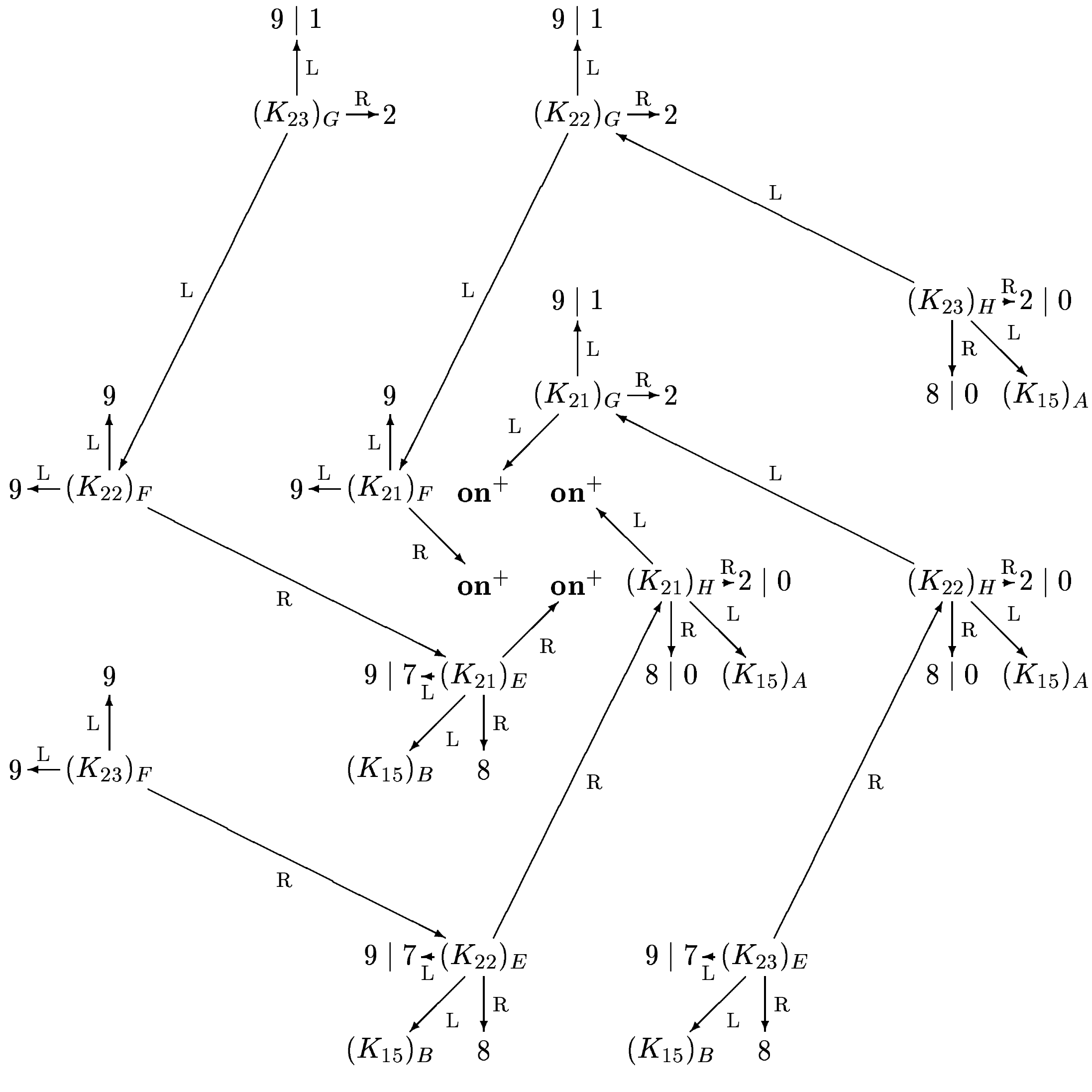


Figure 4.10: The graphs of $(K_{2i})_v$, for $i = 1, 2$ and $v = E, F, G, H$.

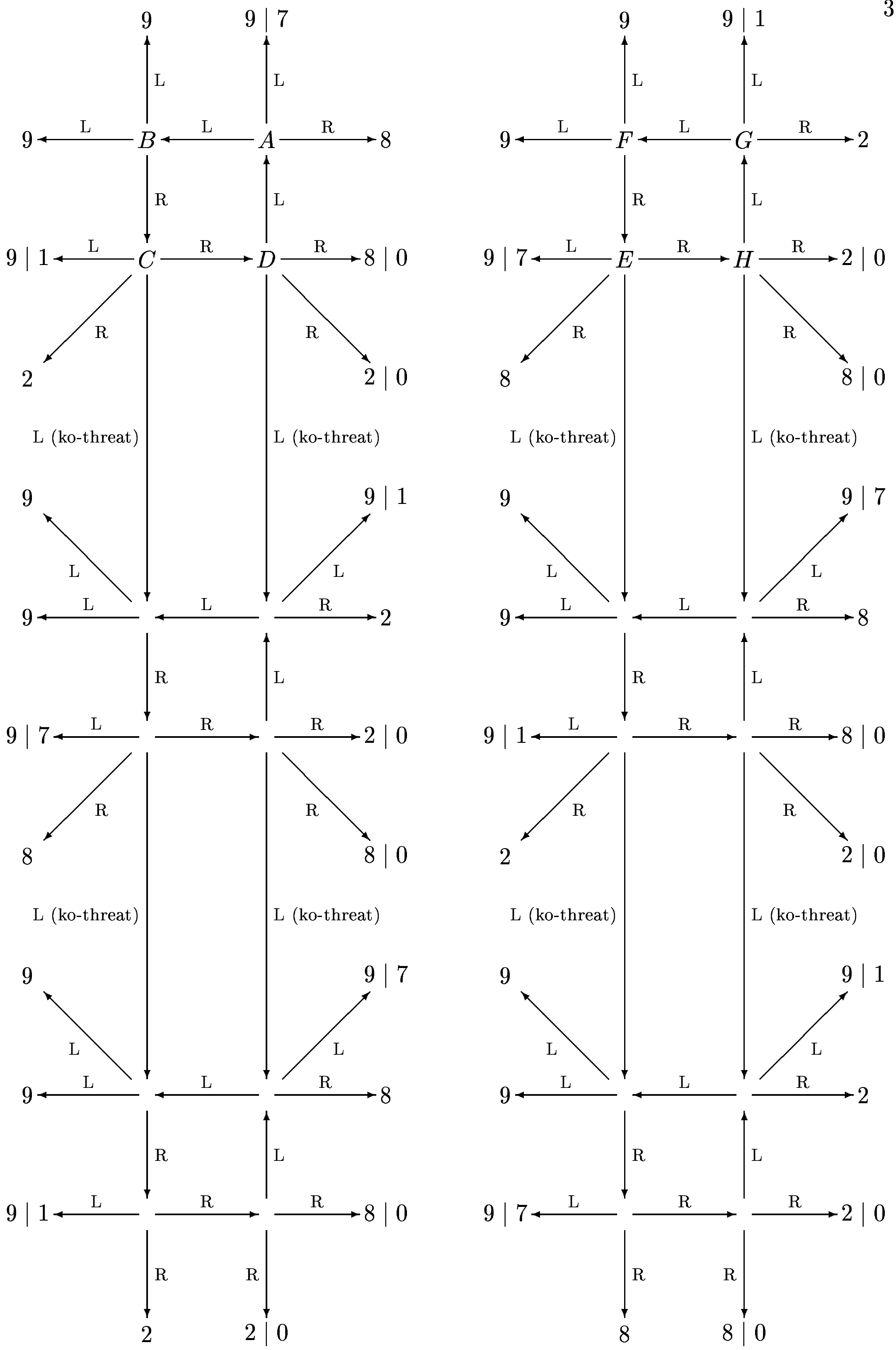


Figure 4.11: The graph R.

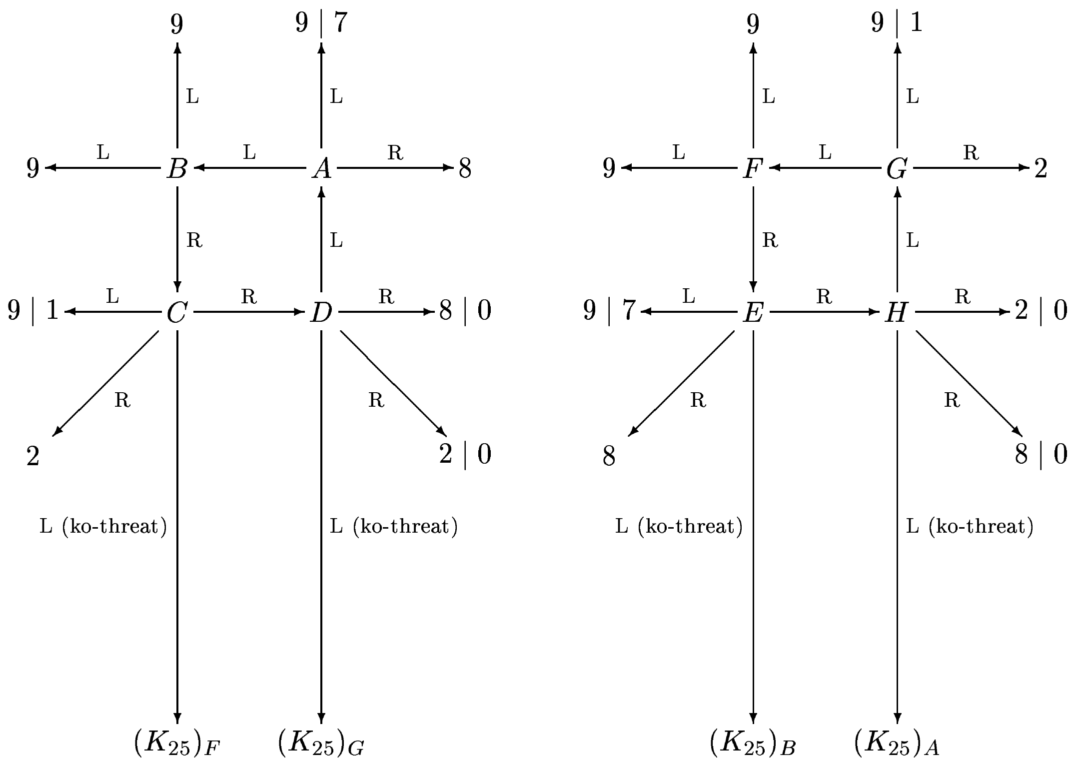


Figure 4.12: The graph R_2.

v	A	B	C	D	E	F	G	H
$(K_0)_v$	$\mathbf{off}^-$	$\mathbf{off}^-$	$\mathbf{off}^-$	$\mathbf{off}^-$	$\mathbf{off}^-$	$\mathbf{off}^-$	$\mathbf{off}^-$	$\mathbf{off}^-$
$(K_{10})_v$	$\mathbf{on}^+$	$\mathbf{on}^+$	$\mathbf{on}^+$	$\mathbf{on}^+$	$\mathbf{on}^+$	$\mathbf{on}^+$	$\mathbf{on}^+$	$\mathbf{on}^+$
$(K_{11})_v$	$\mathbf{on}^+ \mid 8$	10	1	$\mathbf{on}^+ \parallel 2 \mid 0$	7	10	$\mathbf{on}^+ \mid 2$	$\mathbf{on}^+ \|\|2\|0$
$(K_{12})_v$	$10 \mid 8$	$9 \mid 1$	1	$\mathbf{on}^+ \mid 8 \parallel 2 \mid 0$	7	$9 \mid 7$	$10 \mid 2$	2
$(K_{13})_v$	7	$9 \mid 1$	1	$10 \mid 8 \parallel 2 \mid 0$	$9 \mid 7 \parallel 2$	$9 \mid 7$	$9 \mid 7 \parallel 2$	2
$(K_{14})_v$	7	$9 \mid 1$	1	$7 \parallel 2 \mid 0$	$9 \mid 7 \parallel 2$	$9 \mid 7$	$9 \mid 7 \parallel 2$	2
$(K_{15})_v$	7	$9 \mid 1$	1	$7 \parallel 2 \mid 0$	$9 \mid 7 \parallel 2$	$9 \mid 7$	$9 \mid 7 \parallel 2$	2
$(K_{20})_v$	$\mathbf{on}^+$	$\mathbf{on}^+$	$\mathbf{on}^+$	$\mathbf{on}^+$	$\mathbf{on}^+$	$\mathbf{on}^+$	$\mathbf{on}^+$	$\mathbf{on}^+$
$(K_{21})_v$	$\mathbf{on}^+ \mid 8$	10	$9 \mid 7 \parallel 2$	$\mathbf{on}^+ \parallel 2 \mid 0$	7	10	$\mathbf{on}^+ \mid 2$	$\mathbf{on}^+ \|\|2\|0$
$(K_{22})_v$	$10 \mid 8$	$9 \mid 7$	$9 \mid 7 \parallel 2$	$\mathbf{on}^+ \mid 8 \parallel 2 \mid 0$	7	$9 \mid 7$	$10 \mid 2$	$7 \parallel 2 \mid 0$
$(K_{23})_v$	7	$9 \mid 7$	$9 \mid 7 \parallel 2$	$10 \mid 8 \parallel 2 \mid 0$	7	$9 \mid 7$	$9 \mid 7 \parallel 2$	$7 \parallel 2 \mid 0$
$(K_{24})_v$	7	$9 \mid 7$	$9 \mid 7 \parallel 2$	$7 \parallel 2 \mid 0$	7	$9 \mid 7$	$9 \mid 7 \parallel 2$	$7 \parallel 2 \mid 0$
$(K_{25})_v$	7	$9 \mid 7$	$9 \mid 7 \parallel 2$	$7 \parallel 2 \mid 0$	7	$9 \mid 7$	$9 \mid 7 \parallel 2$	$7 \parallel 2 \mid 0$
$(K_{30})_v$	$\mathbf{on}^+$	$\mathbf{on}^+$	$\mathbf{on}^+$	$\mathbf{on}^+$	$\mathbf{on}^+$	$\mathbf{on}^+$	$\mathbf{on}^+$	$\mathbf{on}^+$
$(K_{31})_v$	$\mathbf{on}^+ \mid 8$	10	$9 \mid 7 \parallel 2$	$\mathbf{on}^+ \parallel 2 \mid 0$	7	10	$\mathbf{on}^+ \mid 2$	$\mathbf{on}^+ \|\|2\|0$
$(K_{32})_v$	$10 \mid 8$	$9 \mid 7$	$9 \mid 7 \parallel 2$	$\mathbf{on}^+ \mid 8 \parallel 2 \mid 0$	7	$9 \mid 7$	$10 \mid 2$	$7 \parallel 2 \mid 0$
$(K_{33})_v$	7	$9 \mid 7$	$9 \mid 7 \parallel 2$	$10 \mid 8 \parallel 2 \mid 0$	7	$9 \mid 7$	$9 \mid 7 \parallel 2$	$7 \parallel 2 \mid 0$
$(K_{34})_v$	7	$9 \mid 7$	$9 \mid 7 \parallel 2$	$7 \parallel 2 \mid 0$	7	$9 \mid 7$	$9 \mid 7 \parallel 2$	$7 \parallel 2 \mid 0$
$(K_{35})_v$	7	$9 \mid 7$	$9 \mid 7 \parallel 2$	$7 \parallel 2 \mid 0$	7	$9 \mid 7$	$9 \mid 7 \parallel 2$	$7 \parallel 2 \mid 0$

Table 4.1: Sidling values from Figure 4.3.

$$\begin{aligned}
H_{v_0} &= -v_0 \\
H_{v_0,\ldots,v_{n-1},0} &= H_{v_0,\ldots,v_{n-1}} \\
H_{v_0,\ldots,v_{n-1},1} &= H_{v_0,\ldots,v_{n-1},2} \mid H_{v_0,\ldots,v_{n-1},0} \\
H_{v_0,\ldots,v_{n-1},b} &= H_{v_0,\ldots,v_{n-1}-1,b+1} \qquad (2 \le b \le m-1) \\
H_{v_0,\ldots,v_{n-1},cm+d} &= H_{v_0-c(n-m),v_1,\ldots,v_{n-1},d}
\end{aligned}$$

For the purpose of determining the options of $H_{v_0,\ldots,v_n}$, we always consider H_{v_0} to have options H_{v_0-1} for Right and H_{v_0+1} for Left, and otherwise, we consider $H_{v_0,\ldots,v_n}$ to have the options shown above. There is then always a unique option for both Left and Right in any $H_{v_0,\ldots,v_n}$.

If we write the sequence $(v_0+\delta, v_1, \ldots, v_{n-1}, v_n+\epsilon)$, and $n=0$, this will mean $(v_0+\delta+\epsilon)$. Similarly, if we write $(v_0+\delta, v_1, \ldots, v_{n-2}, v_{n-1}+\epsilon, v_n)$, and $n=1$, this will mean $(v_0+\delta+\epsilon, v_1)$.

Lemma 12 *If we write $G^{L\ldots L}$ with a L's, $a \ge 0$, as G^{L^a}, and $G^{R\ldots R}$ with b R's as G^{R^b}, and if $v = (v_0, v_1, \ldots, v_{n-1}, v_n + l)$ and $w = (v_0 + r, v_1, \ldots, v_{n-1}, v_n)$, then $H_v^{L^a} \ge H_w^{L^{a+l}}$ and $H_v^{R^{a+l}} \ge H_w^{R^a}$, for all $a, l, r \ge 0$.*

Proof. If $l=0$, the result is clear, so let $l>0$. We induce on n. If $n=0$, the result is clear. Otherwise, let $\bar{v} = (v_1, \ldots, v_{n-2})$, and let α and β be minimal so that $v_n + l + \alpha$ and $v_n + \beta$ are congruent to 0 or 1 modulo m.

If $v_n + l + \alpha \equiv v_n + \beta \equiv 0 \pmod{m}$, then

$$H_v = H_{v_0-(v_n+l+\alpha)(n-m)/m,\bar{v},v_{n-1}-\alpha} = H_{v_0-(v_n+l+\alpha)/m,\bar{v},v_{n-1}+v_n+l},$$

and similarly, $H_w = H_{v_0+r-(v_n+\beta)/m,\bar{v},v_{n-1}+v_n}$. Since $v_n + l + \alpha \ge v_n + \beta$, the desired result follows from the induction hypothesis.

If $v_n + l + \alpha \equiv v_n + \beta \equiv 1 \pmod{m}$, then we in fact have $\alpha = \beta = 0$, and $l \ge m$. Now

$$\begin{aligned}
H_v &= H_{v_0,\bar{v},v_{n-1},v_n+l+1} \mid H_{v_0,\bar{v},v_{n-1},v_n+l-1} \\
&= H_{v_0,\bar{v},v_{n-1}-(m-2),v_n+l+m-1} \mid H_{v_0,\bar{v},v_{n-1},v_n+l-1} \\
&= H_{v_0-(v_n+l+m-1)(n-m)/m,\bar{v},v_{n-1}-(m-2)} \mid H_{v_0-(v_n+l-1)(n-m)/m,\bar{v},v_{n-1}} \\
&= H_{v_0-(v_n+l+m-1)/m,\bar{v},v_{n-1}+v_n+l+1} \mid H_{v_0-(v_n+l-1)/m,\bar{v},v_{n-1}+v_n+l-1}.
\end{aligned}$$

Similarly,

$$H_w = H_{v0+r-(v_n+m-1)/m,\bar{v},v_{n-1}+v_n+1} \mid H_{v0+r-(v_n-1)/m,\bar{v},v_{n-1}+v_n-1}.$$

The result will then follow from the induction hypothesis if $a > 0$. If $a = 0$, we want to have

$$L_1 = \{H_{v0-(v_n+l+m-1)/m,\bar{v},v_{n-1}+v_n+l+1} \mid H_{v0-(v_n+l-1)/m,\bar{v},v_{n-1}+v_n+l-1}\} - H^{L^{l-1}}_{v0+r-(v_n+m-1)/m,\bar{v},v_{n-1}+v_n+1} \geq 0$$

and

$$L_2 = H^{R^{l-1}}_{v0-(v_n+l-1)/m,\bar{v},v_{n-1}+v_n+l-1} - \{H_{v0+r-(v_n+m-1)/m,\bar{v},v_{n-1}+v_n+1} \mid H_{v0+r-(v_n-1)/m,\bar{v},v_{n-1}+v_n-1}\} \geq 0.$$

In L_1, if Right moves first, Left should always respond in the first component. Recalling that $l \geq m$, this will guarantee him a win, by the induction hypothesis. In L_2, Left should always respond in the second component.

If $v_n + \beta \equiv 1 \pmod{m}$ and $v_n + l + \alpha \equiv 0 \pmod{m}$, then $\beta = 0$ and $l + \alpha \geq m - 1$. We will then want to have

$$M_1 = H^{L^a}_{v0-(v_n+l+\alpha)/m,\bar{v},v_{n-1}+v_n+l} - \{H_{v0+r-(v_n+m-1)/m,\bar{v},v_{n-1}+v_n+1} \mid H_{v0+r-(v_n-1)/m,\bar{v},v_{n-1}+v_n-1}\}^{L^{a+l}} \geq 0$$

and

$$M_2 = H^{R^{a+l}}_{v0-(v_n+l+\alpha)/m,\bar{v},v_{n-1}+v_n+l} - \{H_{v0+r-(v_n+m-1)/m,\bar{v},v_{n-1}+v_n+1} \mid H_{v0+r-(v_n-1)/m,\bar{v},v_{n-1}+v_n-1}\}^{R^a} \geq 0.$$

Recalling that $l > 0$, $M_1 \geq 0$ will follow from the induction hypothesis. If $a > 0$, $M_2 \geq 0$ will follow as well. Otherwise, if $a = 0$ and Right moves first on M_2, Left should move in the second component. The induction hypothesis then gives him a win.

If $v_n + \beta \equiv 0 \pmod{m}$ and $v_n + l + \alpha \equiv 1 \pmod{m}$, then $\alpha = 0$, and $l \geq \beta + 1$. We need

$$N_1 = \{H_{v0-(v_n+l+m-1)/m,\bar{v},v_{n-1}+v_n+l+1} \mid H_{v0-(v_n+l-1)/m,\bar{v},v_{n-1}+v_n+l-1}\}^{L^a} - H^{L^{a+l}}_{v0+r-(v_n+\beta)/m,\bar{v},v_{n-1}+v_n} \geq 0$$

and

$$N_2 = \{H_{v_0-(v_n+l+m-1)/m,\bar{v},v_{n-1}+v_n+l+1} \mid H_{v_0-(v_n+l-1)/m,\bar{v},v_{n-1}+v_n+l-1}\}^{R^{a+l}} -$$

$$H^{R^a}_{v_0+r-(v_n+\beta)/m,\bar{v},v_{n-1}+v_n} \geq 0.$$

Since $l > 0$, the induction hypothesis implies that $N_2 \geq 0$, and if $a > 0$, it does likewise for N_1. If $a = 0$, then Left should respond on the first component of N_1 moving second and will then win by induction. ∎

Lemma 13 *For all $v = (v_0, \ldots, v_n)$'s, we have $H_{v_0,\ldots,v_n} - H_{v_0,\ldots,v_n+1} \, || > 0$.*

Proof. If $n = 0$, this is clear, since the total game is just 1, so let $n > 0$. If $v_n \equiv 0 \pmod m$, Left's move is in the second component, to a total of 0, and if $v_n \equiv 1 \pmod m$, Left's move is in the first component, to a total of 0. If $v_n \equiv -1 \pmod m$, the difference equals $H_{v_0,\ldots,v_{n-1}-1,v_n+1} - H_{v_0,\ldots,v_{n-1},v_n+1}$, which equals $H_{v_0+\delta,\ldots,v_{n-1}-1} - H_{v_0+\delta,\ldots,v_{n-1}}$ for some δ, so we can induce on n. If v_n has some other residue mod m, we can add one to v_n and subtract one from v_{n-1} without changing our total game; do this repeatedly until $v_n \equiv -1 \pmod m$. ∎

Lemma 14 *For all $(v_0, \ldots, v_n)$'s, we have*

$$H_{v_0,v_1,\ldots,v_n} \geq H_{v_0+1,v_1,\ldots,v_n},$$

and

$$H_{v_0,\ldots,v_{j-1},v_j-1,v_{j+1},\ldots,v_{k-1},v_k+1,v_{k+1},\ldots,v_n} \geq H_{v_0,\ldots,v_n}.$$

Proof. The first fact is a special case of Lemma 12. For the second fact, let $\bar{v} = (v_0, \ldots, v_{n-2})$. It will do to prove that $H_{\bar{v},v_{n-1}-1,v_n+1} \geq H_{\bar{v},v_{n-1},v_n}$ for all v, and the remainder will follow by the definition of $H_{v_0,\ldots,v_n}$. Without loss of generality, we can assume that $0 \leq v_n < m$. This inequality then follows by definition for $v_n \geq 2$. We are left with the inequalities

$$H_{\bar{v},v_{n-1}-1,2} \geq H_{\bar{v},v_{n-1},1} = \{H_{\bar{v},v_{n-1},2} \mid H_{\bar{v},v_{n-1},0}\} \geq H_{\bar{v},v_{n-1}+1,0}.$$

In the difference between the left-hand side and middle, if Right moves on the first component, Left should respond on the second; the difference will be $H^R_{v_0-1,\ldots,v_{n-1}+1} -$

$H_{v_0,\ldots,v_{n-1}}$, and this is won by Left moving second, by Lemma 12. If Right moves on the second component, the difference will equal $H_{v_0-1,\ldots,v_{n-1}+1} - H_{v_0-1,\ldots,v_{n-1}+2}$, and Left wins this moving first by Lemma 13. In the difference between the middle and right-hand side, if Right moves on the first component, the difference will be $H_{\bar{v},v_{n-1}} - H_{\bar{v},v_{n-1}+1}$, which is won by Left moving first by Lemma 13. If Right moves on the second component, Left should respond in the first, to $H_{v_0-1,v_1,\ldots,v_{n-1}+2} - H^L_{\bar{v},v_{n-1}+1}$, which is won by Left moving second by Lemma 12. ∎

We call a sum of kos a $K_{v_0,\ldots,v_n}$ if it is of the form $\sum_{j=1}^{s} KO^m_{i_j}[A_j, B_j]$, where all the A_j's and B_j's are integers, all A_j's are at least -1, and if r_q is the sum of the $-i_j$'s for all q-point kos, then $v_0 = \sum_{q \le -m} r_q - \sum_{j=1}^{s} A_j$, $v_l = r_{l-m}$ for $l = 1, \ldots, n$, and $r_{l-m} = 0$ for $l > n$. We will also call sums of kos and integers $K_{v_0,\ldots,v_n}$'s; an integer T in the sum can either be treated as $KO^m_0[T, B]$ for any B, if $T \ge -1$, or as $KO^m_m[A, T]$ for any $A \ge \max(T - m, -1)$.

We can see from the definition of H_v that the Left option of $H_{v_0,v_1,\ldots,v_n}$, if it exists, will always be equal to some $H_{w_0,w_1,\ldots,w_n}$, where $w_k = v_k + 1$ for some k and otherwise $w_j = v_j$. The same thing is true for the Right option, except that it has $w_k = v_k - 1$ for some k. The same is true for games of the form $K_{v_0,\ldots,v_n}$; their Left and Right options will be $K_{w_0,\ldots,w_n}$'s, where w and v are related as above, except that Right's move from $KO^m_{m-1}[A, B]$ to B will decrease v_0 by more than 1 if $B - A > m$.

Theorem 15 *If G is a K_v, then $\phi^x_y(G + M) = H_v + M$ for all enders M, where $x = J$ or N and $y = L$ or LR.*

Proof. We claim that, if $K \le H_v$, then $\phi^x_y(G) - K$ is won by Left moving second, and similarly for $K \ge H_v$ and Right. Our strategy will always leave the game in the appropriate one of these forms, up to history, after the second player has moved. Left's strategy will eventually make G simpler; that is, we induce on the complexity of G. Right's strategy will either make G simpler or cause Left to lose by using infinitely many ko-threats. On the difference game $\phi^x_y(G + M) - H_v - M$, either player moving second can then use the strategy we just gave on the $G - H_v$ portion, and the reflection strategy on the $M - M$ portion.

We may as well assume that initially, before the first player moves, $K = H_w$, where $v = w$. Our second-player move will ensure that, at the end of play, H_w and H_v

have the appropriate order-relation; recall that we proved some order relations between the H_x's in Lemma 14.

The first case we consider is when G contains no kos. In this case G and H_v will be the same game, namely, some integer T. The result is then clear. The second case is everything else. Let $v = (v_0, \dots, v_n)$ and $w = (w_0, \dots, w_n)$, and let q be the maximum point-value of any ko present at the start of play. If Left moves second, he will usually move on a q-point ko. (When he does this, he should pick a particular q-point ko to move on. If Right tries to move it back, Left can use a ko-threat and reverse this move. Right's next move must then be elsewhere and, after this, Left can move the ko another step forward. In this manner, Left will eventually simplify the position.) Assuming this move is possible, it will definitely preserve the induction hypothesis. This is clear if Right's move was on G. If it was on H_w, and increased w_l by 1, then $w_l + \cdots + w_n \not\equiv 0 \pmod{m}$ at the start of play, assuming that $l > 0$, and hence $q \geq l - m$. If it increased w_0 by 1, then we need no restriction on q to preserve the induction hypothesis. The only exception is when there does not exist a q-point ko. The reason for this must be that Right has just moved on the unique q-point ko and destroyed it. If $q > -m$, Left will then move on H_w. This will evidently satisfy the induction hypothesis, and G will become simpler because Right made it so. If $q \leq -m$, Right's move from $KO^m_{m-1}[A, B]$ to B decreased v_0 by at least 1. Hence the game is now a difference $T - U$ of integers, and $T - U \geq 1$. We then clearly have a winning move.

Let Right move second. Right, too, will usually want to move on a q-point ko, and will pick a particular one to move on. The reasons he won't be able to move on it are more varied.

1. $q > -m$, and the only q-point ko is one that Left has just moved in, taking it from $KO^m_{m-1}[A, B]$ to $KO^m_{m-2}[A, B]$. In this case, we should move in H_w.

2. $q > -m$, Left has just moved in and possibly destroyed our q-point ko, we are not in case 1, and the next biggest ko has q' points, where $q' > -m$. In this case, we should observe that the value of H_w will not be changed if we increment w_{q+m} and decrement $w_{q'+m}$. We can thus move in the q'-point ko and preserve the induction hypothesis. (We should fix a particular q'-point ko to move in, as well as fixing a particular q-point ko to move in. This will ensure that Left will have to use infinitely many ko-threats if he wants the game to last forever. We might have

$q = q'$.)

3. Either $q > -m$, the only q-point ko is one that Left has just moved in and possibly destroyed, we are not in case 1, and there are no other kos with more than $-m$ points, or $q \leq -m$. In this case, at the start of the turn, G was of the form $T + \sum_{j=1}^{s} KO_{i_j}^{m}[A_j, B_j]$, and H_w was $T + \sum_{j=1}^{s}(A_j + i_j)$. If $T < 0$, then we can move on T (inside G,) and leave the forms of the games G and H_w as they were at the start of the turn, so assume that $T \geq 0$. If $T > 0$, or some $A_j + i_j > 0$, we can move in H_w. If $T = 0$ and $A_j + i_j = 0$ for all j, then we must have $A_j = -1$ and $i_j = 1$ for all j. Hence Left's move was from some $KO_1^m[-1, B]$ to -1, and we can move on the -1. These moves also leave our games in the same form as at the start of the turn.

∎

It may be wondered why we consider only kos $KO_i^m[A, B]$ with $A \geq -1$. The reason is that these are the only kos such that $\phi_L(KO_1^m[A, B]) = A + 1$, that is, the only kos such that moving from $KO_1^m[A, B]$ to A costs Left only one point, or the same as moving on an integer.

4.4 Kos with numbers and passes

If Japanese scoring is used, the results in the previous section will change because a player is allowed to pass. Let us say that Left has the ko-threat advantage, so that Right is the player we allow to pass. In this case, we will allow kos $KOR_i^m[A, B]$ to occur, where A and B can be arbitrary numbers, possibly shifted by infinitesimal enders. Suppose that we are dealing with kos of $P_0 = 0$, P_1, P_2, $\ldots$, P_{g-1}, or P_g points, where $0 = P_0 < P_1 < \cdots < P_g$. For any number C, and integers $0 \leq n \leq g$ and v_0, v_1, $\ldots$, v_n, we define $H_{v_0,\ldots,v_n}^{C}$ as follows:

$$
\begin{aligned}
H^C_{v_0} &= C \qquad \text{(independent of } v_0\text{)} \\
H^C_{v_0,\ldots,v_{n-1},0} &= H^C_{v_0,\ldots,v_{n-1}} \\
H^C_{v_0,\ldots,v_{n-1},1} &= H^C_{v_0,\ldots,v_{n-1},2} \mid H^C_{v_0,\ldots,v_{n-1},0} \\
H^C_{v_0,\ldots,v_{n-1},b} &= H^C_{v_0,\ldots,v_{n-1}-1,b+1} \qquad (2 \leq b \leq m-1) \\
H^C_{v_0,\ldots,v_{n-1},cm+d} &= H^{C+cP_n}_{v_0,\ldots,v_{n-1},d}
\end{aligned}
$$

These values were also found to be values of sums of kos by Yonghoan Kim.

If all numbers occurring in our kos are multiples of some power δ of 2, where $\delta \leq 1$, and ϵ is a power of 2 that is no more than δ, then we can consider $H^C_{v_0}$ to have Left option $H^{C-\epsilon}_{v_0}$ and Right option $H^{C+\epsilon}_{v_0}$. Otherwise, we consider $H^C_{v_0,\ldots,v_n}$ to have options as above.

Lemma 16 *If $v = (v_0, v_1, \ldots, v_{n-1}, v_n + l)$, and $w = (v_0, v_1, \ldots, v_{n-1}, v_n)$, then $(H^{C+D}_v)^{L^a} \geq (H^C_w)^{L^{a+l}}$ and $(H^{C+D}_v)^{R^{a+l}} \geq (H^C_w)^{R^a}$, for all integers $a, l \geq 0$ and numbers $D \geq 0$ and C.*

Proof. If $l = 0$, this is obvious from the definition of H^C_v, so let $l > 0$. We induce on n. If $n = 0$, the result is clear. Otherwise, we can proceed exactly as in Lemma 12. ∎

Lemma 17 *For all C's and $v = (v_0, \ldots, v_n)$'s, we have $H^C_{v_0,\ldots,v_n} - H^C_{v_0,\ldots,v_n+1} \not< 0$. If the difference is 0, then in fact both H^C_v's are the same number.*

Proof. If $n = 0$, then the difference is 0 and both H^C_v's are the same number. Otherwise, we can proceed exactly as in Lemma 13, repeatedly decreasing n by 1 until we either get $n = 0$ or a good first move for Left in the difference game. ∎

Lemma 18 *If $D > C$ are numbers, $v = (v_0, \ldots, v_n)$, $w = (v_0, \ldots, v_{n-1}, v_n + 1)$, and $x = (v_0, \ldots, v_{n-1}, v_n + 2)$, then $R(H^D_w) > R(H^C_v)$, $L(H^D_w) > L(H^C_v)$, and $R(H^D_x) > L(H^C_v)$. Also, $L(H^C_{v,1}) = R(H^C_{v,2}) > R(H^C_{v,1}) = L(H^C_{v,0})$—that is, $H^C_{v,1}$ is a hot game. Finally, $H^{C+E}_v = H^C_v + E$ for all numbers E.*

Proof. We induce on n. If $n = 0$, the statements in the first sentence are obvious. Also, if the statements in the first sentence are all known for $n \leq n'$, the statements in the second statements all are as well. In fact, we have that $R(H^C_{v,2}) = R(H^{C+P_n-P_{n-1}}_{v_0,\ldots,v_{n-1},v_n+2})$ and $L(H^C_{v,0}) = L(H^C_v)$, so it follows from $R(H^D_x) > L(H^C_v)$ for $D > C$ that $R(H^C_{v,2}) > L(H^C_{v,0})$ for $n \leq n'$. From the definition of Left and Right stops, we then see that $L(H^C_{v,1}) = R(H^C_{v,2})$ and that $R(H^C_{v,1}) = L(H^C_{v,0})$, and since $H^C_{v,1}$ has its Left stop greater than its Right stop, it is hot. Hence, if $n \leq n'$, the game $H^C_{v,1}$ satisfies a translation property:

$$H^C_{v,1} + E = \{H^C_{v,2} + E \mid H^C_{v,0} + E\}.$$

We can apply this inductively to derive the third sentence, which will in fact, then, be valid for all $n \leq n' + 1$.

It remains to prove the first sentence for $n > 0$. Let $\bar{v} = (v_0, \ldots, v_{n-1})$. We first prove $R(H^D_x) > L(H^C_v)$. Add one to v_n and subtract one from v_{n-1} until either v_n or $v_n + 2$ is congruent to 0 or 1 modulo m. Suppose that $v_n \equiv 1 \pmod{m}$. We can reduce v_n modulo m, and then we will want $R(H^F_{\bar{v},3}) > L(H^E_{\bar{v},1})$ for some $F > E$. From the first paragraph, it follows that $L(H^E_{\bar{v},1}) = R(H^E_{\bar{v},2})$, so we want to have $R(H^F_{\bar{v},3}) > R(H^E_{\bar{v},2})$. This is the same as saying that $R(H^{F+P_n-P_{n-1}}_{v_0,\ldots,v_{n-1}+3}) > R(H^{E+P_n-P_{n-1}}_{v_0,\ldots,v_{n-1}+2})$, which follows from the induction hypothesis. If $v_n + 2 \equiv 1 \pmod{m}$, we want $R(H^F_{\bar{v},1}) > L(H^E_{\bar{v},-1})$, or $L(H^F_{\bar{v},0}) > L(H^E_{\bar{v},-1})$, which follows from $L(H^F_{\bar{v}}) > L(H^E_{v_0,\ldots,v_{n-1}-1})$. This also follows from the induction hypothesis. If $v_n \equiv 0 \pmod{m}$, we want to have $R(H^F_{\bar{v},2}) > L(H^E_{\bar{v},0})$. This follows from the first paragraph of our proof. Finally, if $v_n + 2 \equiv 0 \pmod{m}$, we want to have $R(H^F_{\bar{v},0}) > L(H^E_{\bar{v},-2})$. In this case we must have $m > 3$ since otherwise we would be in the case $v_n \equiv 1 \pmod{m}$, which we have already dealt with. Hence the inequality will follow from $R(H^F_{\bar{v}}) > L(H^E_{v_0,\ldots,v_{n-1}-2})$, which follows from the induction hypothesis.

We also need to prove that $R(H^D_w) > R(H^C_v)$ and that $L(H^D_w) > L(H^C_v)$. We prove only the first statement, the second being wholly analogous. We can increase v_n and decrease v_{n-1} until v_n or v_n+1 is congruent to 0 or 1 modulo m. If $v_n \equiv 1 \pmod{m}$, we will want to have, for some $F > E$, $R(H^F_{\bar{v},2}) > R(H^E_{\bar{v},1})$, or $R(H^F_{\bar{v},2}) > L(H^E_{\bar{v},0})$. This follows from the first paragraph of our proof. If $v_n \equiv 0 \pmod{m}$ and $v_n + 1 \equiv 1 \pmod{m}$, we want $R(H^F_{\bar{v},1}) > R(H^E_{\bar{v},0})$, or $L(H^F_{\bar{v}}) > R(H^E_{\bar{v}})$. With the aid of the induction hypothesis, we can apply our translation property to find that $H^F_{\bar{v}} = F - E + H^E_{\bar{v}}$. Recalling that $L(G) \geq R(G)$ for all games G and that $R(G + N) = R(G) + N$ for all games G and numbers N, the desired result follows. Finally, if $v_n + 1 \equiv 0 \pmod{m}$, we want to have $R(H^F_{\bar{v},0}) > R(H^E_{\bar{v},-1})$. This will follow from $R(H^F_{\bar{v}}) > R(H^E_{v_0,\ldots,v_{n-1}-1})$, which follows from the induction hypothesis. ∎

Lemma 19 *For all C's and $(v_0, \ldots, v_n)$'s, we have that, for some positive infinitesimal enders η and η',*

$$H^C_{v_0,\ldots,v_{j-1},v_j+1,v_{j+1},\ldots,v_n} + \eta' \geq H^C_{v_0,\ldots,v_n},$$

and

$$H^C_{v_0,\ldots,v_{j-1},v_j-1,v_{j+1},\ldots,v_{k-1},v_k+1,v_{k+1},\ldots,v_n} + \eta \geq H^C_{v_0,\ldots,v_n}.$$

The games η and η' will depend on C, j, k, and $(v_0, \ldots, v_n)$.

Proof. Since $H^C_{v_0,\ldots,v_n}$ is independent of v_0, the second statement implies the first. If all options of enders G and G' are the same, up to infinitesimals, and neither G nor G' is a number, then $G - G'$ is infinitesimal. Lemma 18 proves that no $H^C_{v,1}$ is a number; hence if $H^C_{v,2} + \eta \geq H^C_{w,2}$ and $H^C_{v,0} + \eta' \geq H^C_{w,0}$ for some positive infinitesimal enders η and η', then $H^C_{v,1} + \eta'' \geq H^C_{w,1}$ for some infinitesimal ender η'', and since any infinitesimal ender is exceeded by a positive infinitesimal ender, we can assume that η'' is positive. By the definition of H^C_v, then, if we let $\bar{v} = (v_0, \ldots, v_{n-2})$, to prove our second statement, we only have to prove that

$$H^C_{\bar{v},v_{n-1}-1,2} + \eta \geq H^C_{\bar{v},v_{n-1},1} = \{H^C_{\bar{v},v_{n-1},2} \mid H^C_{\bar{v},v_{n-1},0}\} \geq H^C_{\bar{v},v_{n-1}+1,0} - \eta'.$$

The first inequality can be proved as in Lemma 14, without requiring the η, unless $H^C_{\bar{v},v_{n-1}-1,2}$ and $H^C_{\bar{v},v_{n-1},2}$ are the same number. In this case, the difference game is $E - \{E \mid K\}$, where E is a number, and K is an ender with, from Lemma 18, $E > L(K)$. It follows that there is a positive infinitesimal ender η with $E - \{E \mid K\} + \eta \geq 0$. The proof of the second inequality is similar. ∎

We call a sum of kos a $K^C_{v_0,\ldots,v_n}$ if it is of the form $\sum_{j=1}^s KOR^m_{i_j}[A_j, B_j]$, where for each j, $A_j = L_j + \delta_j$ and $B_j = M_j + \epsilon_j$, where L_j and M_j are numbers and δ_j and ϵ_j are infinitesimal enders, and if r_q is the sum of the $-i_j$'s for all q-point kos, then $C = \sum_{j=1}^s L_j$, $v_0 = \sum_{q \leq 0} r_q$, $v_k = r_{P_k}$ for $k = 1, \ldots, n$, and $r_q = 0$ (that is, there are no q-point kos present) unless $q \leq 0$ or $q = P_k$ for some $1 \leq k \leq n$. We will also call sums of kos, numbers, and infinitesimal enders $K^C_{v_0,\ldots,v_n}$'s; a number T in the sum can be treated as $KOR^m_0[T, M]$ for any number M, or as $KOR^m_m[L, T]$ for any number $L \geq T$;

an infinitesimal ender is ignored. Also, we call some sums of kos and enders $\hat{K}^C_{v_0,\ldots,v_n}$'s. Here, kos are treated as usual, and an ender G in the sum can be treated as a number L is in $K^C_{v_0,\ldots,v_n}$, for any $L \leq R(G)$ (so that $G - L + \eta$ is positive for some infinitesimal ender η.) We also call certain sums of kos and enders $\check{K}^C_{v_0,\ldots,v_n}$'s; these are the same as $\hat{K}^C_{v_0,\ldots,v_n}$'s, except that we treat an ender G as a number $L \geq L(G)$ is in $K^C_{v_0,\ldots,v_n}$. Evidently, if G is a $K^C_{v_0,\ldots,v_n}$, then G is both a $\hat{K}^C_{v_0,\ldots,v_n}$ and a $\check{K}^C_{v_0,\ldots,v_n}$.

The Left option of $H^C_{v_0,\ldots,v_n}$ will be some $H^C_{w_0,\ldots,w_n}$, where $w_k = v_k + 1$ for some k and otherwise $w_j = v_j$, unless $H^C_{v_0,\ldots,v_n}$ is a number, when its Left option will be less than $H^C_{v_0,\ldots,v_n}$. A similar statement is true for Right.

For a game of the form $\check{K}^C_{v_0,\ldots,v_n}$, if Left moves on a ko, the resultant game will be of the form $\check{K}^C_{w_0,\ldots,w_n}$, where $w_k = v_k + 1$ for some k and otherwise $w_j = v_j$. If Right moves on a ko, the resultant game will also be of the form $\check{K}^C_{w_0,\ldots,w_n}$, where $w_k = v_k - 1$ for some k and otherwise $w_j = v_j$, unless Right moves from some $KOR^m_{m-1}[A, B]$ to B, where $B - A$ exceeds some positive number. If Left moves on a non-ko, that is, some ender E in the sum, treated as L, say, the result will still be something of the form $\check{K}^C_{v_0,\ldots,v_n}$, as long as $L(E^L) \leq L$. If $L(E^L) > L$, we must have $R(E^L) \leq L$, since $L(E) \leq L$, and then Right will have a reverting move in E^L to something of the form $\check{K}^C_{v_0,\ldots,v_n}$.

For a game of the form $\hat{K}^C_{v_0,\ldots,v_n}$, if Left or Right moves on a ko, the resultant game will be of the form $\hat{K}^C_{w_0,\ldots,w_n}$, where $w_k = v_k \pm 1$ for some k and otherwise $w_j = v_j$. If Right moves on some ender in the sum, the result will either be something of the form $\hat{K}^C_{v_0,\ldots,v_n}$, or Left will have a reverting move in the ender to a $\hat{K}^C_{v_0,\ldots,v_n}$.

Theorem 20 *If G is a K^C_v, then there is a positive infinitesimal ender η such that $H^C_v + M - \eta \leq \phi^x_y(G + M) \leq H^C_v + M + \frac{1}{\mathbf{on}}$ for all enders M, where $x = J$ or N and $y = L$ or LR.*

Proof. We claim that, if $K \leq H^C_v$ and G is a $\hat{K}^C_v$, then $\phi^x_y(G) - K + \eta$ is won by Left moving second, and if $K \geq H^C_v$ and G is a $\check{K}^C_v$, then $\phi^x_y(G)^{\pm} - K - (\frac{1}{\mathbf{on}})^{\pm}$ is won or drawn by Right moving second, for both choices of the superscript, $+$ and $-$. Our strategy will always leave the game in the appropriate one of these forms, up to history, after the second player has moved. Left's strategy will eventually make G simpler; that is, we induce on the complexity of G. Right's strategy will either make G simpler, cause Left to lose by using infinitely many ko-threats, or draw by having infinitely many passes in both $-\frac{1}{\mathbf{on}}$ and G. On the difference game $\phi_y^x(G+M) - H_v^C - M + \eta$ or $\phi_y^x(G+M)^{\pm} - H_v^C - M - (\frac{1}{\mathbf{on}})^{\pm}$, the appropriate player moving second can then use the strategy we just gave on the $G - H_v^C$ portion, and the reflection strategy on the $M - M$ portion.

We will pick η to be a positive integral multiple of $\Uparrow$. These are the largest infinitesimal enders, in the sense that any other infinitesimal ender is less than some $N \cdot \Uparrow$. Since we have $N \cdot \Uparrow = \{0 \mid (N-1)\cdot\Uparrow + *\}$, this gives η the property that a move by Right on it can decrease it by at most $\Downarrow *$. We remark that $0 \leq N \cdot \Uparrow \leq \frac{1}{\mathbf{on}}$ for all multiples $N \cdot \Uparrow$ of $\Uparrow$, that $\frac{1}{\mathbf{on}} + \frac{1}{\mathbf{on}} = \frac{1}{\mathbf{on}}$, and that, consequently, $\frac{1}{\mathbf{on}} + \eta' = \frac{1}{\mathbf{on}}$ for all positive infinitesimal enders η'. This is why Right does not need to have an infinitesimal ender added when he plays second on the difference game. We will assume that initially, before the first player moves, $K = H_w^C$, where $v = w$.

If G contains no kos and G is a $\hat{K}_v^C$, then H_w^C will be a number, L say, and G will be an ender satisfying $R(G) \geq L$. Consequently, there will be a positive infinitesimal ender η with $\phi_y^x(G) - L + \eta \geq 0$, which is what we want. If G contains no kos and G is a $\check{K}_v^C$, then H_w^C will be a number, L say, and G will be an ender satisfying $L(G) \leq L$. Then there will be a positive infinitesimal ender η' with $\phi_y^x(G) - \eta' - L \leq 0$, which implies what we want.

Suppose that G has kos, and that Left is moving second from $\phi_y^x(G) - H_w + \eta$. If Right makes one of the moves in G that do not keep G a $\hat{K}_v^C$ but have a reverting move to a $\hat{K}_v^C$, we should make the reverting move. Otherwise, let $v = (v_0, \ldots, v_n)$, and let $w = (w_0, \ldots, w_n)$. We say that a ko has *advantage* k if it has P_k points, or, if $k = 0$, if it has no more than 0 points. Let k be the maximum advantage of any ko present at the start of the turn. As in Theorem 15, Left should usually pick a specific advantage k ko to move on, and then move on it whenever possible, using a ko-threat if necessary. If this is possible, Left will have incremented v_k by 1. If Right moves in G, he either left v unchanged or decremented some v_l by 1 with $l \leq k$. If Right moves in η, he leaves v unchanged, but decreases η by $\Downarrow *$. If Right moves in H_w^C, he either leaves w unchanged and decreases C or increments some w_l by 1, with $l \leq k$. If moving on an advantage k ko is not possible, Right just moved on the unique advantage k ko and destroyed it, and Left should move on H_w^C, assuming that $k > 0$. (If $k = 0$, then the resultant G will have no kos, and we can easily see that the resultant $\phi_y^x(G) - H_w^C$ exceeds some negative infinitesimal ender, so we need only pick η to have a positive sum with this infinitesimal, which condition we add to the ones about to follow.) At the conclusion of the turn, then, the game will be, up to history, greater than or equal to one of a finite number of forms $\phi_y^x(G_i) - H_{z_i}^C + \eta + \downarrow * $ or $\phi_y^x(G_i) - H_{z_i}^C + \eta$, where each G_i is a $\hat{K}_{w_i}^C$, and, from Lemma 19, each $H_{w_i}^C - H_{z_i}^C$ exceeds some negative infinitesimal ender, which we call $-\theta_i$. But each G_i will also be simpler than G, in the sense that it either has a simplified ender portion (in the case where we reverted Right's move,) has fewer kos, has our biggest ko moved further to the left, or has Right prohibited by ko-ban from moving on our biggest ko, which will enable us to move on it next turn. It then follows by induction that, for each i, $\phi_y^x(G_i) - H_{w_i}^C + \eta_i \geq 0$ for some positive infinitesimal ender η_i. If we let our η exceed each $\eta_i + \theta_i + \uparrow *$ and $\eta_i + \theta_i$, then, it will follow that the game will be nonnegative at the conclusion of the turn, and hence that Left has won.

Right's strategy is also analogous to that in Theorem 15; typically, he will pick a specific advantage k ko and move on it whenever possible, including when Left has just moved on $-\frac{1}{\mathbf{on}}$, but with the following exceptions:

1. $k > 0$, and the only advantage k ko is one that Left has just moved in, taking it from $KO_{m-1}^m[A, B]$ to $KO_{m-2}^m[A, B]$. In this case, we should move in H_w^C.

2. $k > 0$, Left has just moved in our advantage k ko, possibly destroying it, we are not in case 1, and the next biggest ko has advantage k', where $k' > 0$. In this case, the value of H_w^C will not be changed if we increment w_k and decrement $w_{k'}$. We can then move in the advantage k' ko. (We should fix a particular advantage k' ko to move in. This will ensure that Left must use infinitely many ko-threats if we move back and forth forever on the kos.)

3. Left has made a move in an ender within G, to a game G that is not a $\check{K}_v^C$ but has a reverting move to a $\check{K}_v^C$. In this case, we should make the reverting move.

4. Either $k > 0$, the only advantage k ko was one Left has just moved in and possibly destroyed, we are not in case 1, and there are no other kos with positive advantage, or $k = 0$. In this case, at the start of the turn, G was of the form $A + \sum_{j=1}^s KOR_{i_j}^m[A_j, B_j]$, and H_w^C was $L + \sum_{j=1}^s K_j$, where $L \geq L(A)$ and each A_j equals K_j shifted by an infinitesimal ender. Left's moves will then be to move on the kos, on $-\frac{1}{\mathbf{on}}$, on A, and on L and the K_j's. We should always respond by passing, except to revert Left's moves on A where appropriate, so that at the end of

the turn, the game always remains in the same form it was at the start, except for a possible increase of the K_j's relative to the A_j's. This may go on indefinitely, if Left chooses to pass on $-\frac{1}{\mathbf{on}}$ forever; in this case, we will get a drawn game. If not, Left must at some point remove the last ko. At this point the game will become of the form $A - L - (\frac{1}{\mathbf{on}})^{\pm}$, where $L \geq L(A)$. Then some positive infinitesimal ender E must exceed $A - L$. The position is then less than or equal to $E - (\frac{1}{\mathbf{on}})^{\pm}$, but as we have remarked, $E + \uparrow - (\frac{1}{\mathbf{on}})^{\pm} \leq 0$, so $E - (\frac{1}{\mathbf{on}})^{\pm} \leq \downarrow < 0$, and we have won.

∎

4.5 Application to Go; examples

The value of m most useful in Go is $m = 3$. Positions looking something like the $KO_i^m[A, B]$'s for $m \geq 4$ do occur, but they are different in that there is a move out of the ko from all the positions 1, 2, …, $m - 1$, and not just the end positions 1 and $m - 1$.

We saw in §4.1 how $KO[1, 0]$ and $OK[0, -1]$ can be realized as partial board positions. Bigger kos with an odd number of points, such as $KO[6, -1]$, can be represented as in the corner position Figure 4.13; here the life or death of the two White stones in the upper-right hand corner depends on who wins the ko—that is, who moves on the ko and destroys it. The position in Figure 4.13 is equal to $KO[6, -1]$ with Chinese scoring. With Japanese scoring, if Left has an excess of ko-threats, it will equal $KOR[6, -1]$, or if Right has an excess of ko-threats, it will equal $KOL[6, -1]$. Still bigger kos with 11 and 13 points are represented in Figure 4.14. These kos are equal to $KO[10, -1]$ and $KO[12, -1]$, respectively, with Chinese scoring. With Japanese scoring, if Left has many ko-threats, they will equal $KOR[10, -1]$ and $KOR[12, -1]$, or if Right has many ko-threats, they will equal $KOL[10, -1]$ and $KOL[12, -1]$. It should be clear how to extend the pattern to give any larger odd number of points.

A parity principle in Go [6, 2] tells us that, in a partial board position, the number of moves taken until the position becomes an integer plus the eventual integer it becomes must usually have constant parity, i.e., always be odd or always be even. This prevents us from realizing kos with an even number of points directly. However, Figure 4.15 shows games that are equivalent to $KO[4, *]$ and $KO[6, *]$ under Chinese

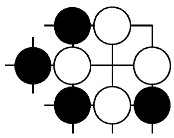


Figure 4.13: A bigger ko.

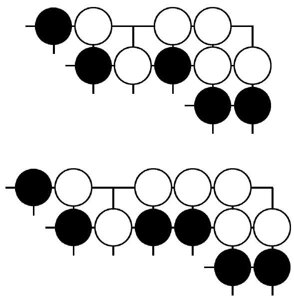


Figure 4.14: Still bigger kos.

scoring and Japanese ko-ban. Under Japanese scoring and ko-ban, these positions will become $KOL[4,*]$ and $KOL[6,*]$ if Right has an excess of ko-threats, or $KOR[4,*]$ and $KOR[6,*]$ if Left has an excess of ko-threats. Again, it should be clear how to extend the pattern.

Take $g = m = 3$, $P_1 = 11$, $P_2 = 13$, and $P_3 = 15$. We then give the values of

$$H_{-11i-13j-15k,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,-2i,0,-2j,0,-2k} = H_{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,i,0,j,0,k}$$

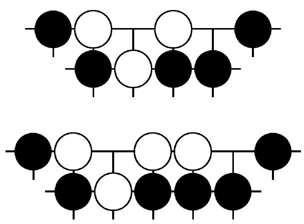


Figure 4.15: More ko-like positions.

below, in Table 4.2. These will be the values of enders corresponding to $i \cdot KO[11,0] + j \cdot KO[13,0] + k \cdot KO[15,0]$, where Left has a large number of ko-threats. If any of i, j, and k are bigger than 3, the value can be found by reducing i, j, and k modulo 3, looking in the table, and then adding $(11i + 13j + 15k)/3$ to the result. We show the values of $H^{11i+13j+15k}_{0,-2i,-2j,-2k} = H^0_{0,i,j,k}$ in Table 4.3. These will be the values of enders corresponding, up to an infinitesimal, to $i \cdot KOR[11,0] + j \cdot KOR[13,0] + k \cdot KOR[15,0]$, where Left has a large number of ko-threats; that is, it will be what Table 4.2 would become if Japanese instead of Chinese scoring were used.

$j,k\downarrow$ $\quad i \rightarrow$	0	1	2	3
$0,0$	0	$12 \mid 0$	12	11
$1,0$	$14 \mid 0$	$13 \parallel 12 \mid 0$	$25 \mid 13 \parallel 12$	$25 \mid 11$
$2,0$	14	13	$25 \mid 13$	25
$3,0$	13	$25 \mid 13$	25	24
$0,1$	$16 \mid 0$	$15 \parallel 12 \mid 0$	$27 \mid 15 \parallel 12$	$27 \mid 11$
$1,1$	$15 \parallel 14 \mid 0$	$27 \mid 15 \vert\vert\vert 13 \parallel 12 \mid 0$	$27 \vert\vert\vert 25 \mid 13 \parallel 12$	$26 \parallel 25 \mid 11$
$2,1$	$29 \mid 15 \parallel 14$	$28 \parallel 27 \mid 15 \vert\vert\vert 13$	$40 \mid 28 \parallel 27 \vert\vert\vert 25 \mid 13$	$40 \mid 26 \parallel 25$
$3,1$	$29 \mid 13$	$28 \parallel 25 \mid 13$	$40 \mid 28 \parallel 25$	$40 \mid 24$
$0,2$	16	15	$27 \mid 15$	27
$1,2$	15	$27 \mid 15$	27	26
$2,2$	$29 \mid 15$	$28 \parallel 27 \mid 15$	$40 \mid 28 \parallel 27$	$40 \mid 26$
$3,2$	29	28	$40 \mid 28$	40
$0,3$	15	$27 \mid 15$	27	26
$1,3$	$29 \mid 15$	$28 \parallel 27 \mid 15$	$40 \mid 28 \parallel 27$	$40 \mid 26$
$2,3$	29	28	$40 \mid 28$	40
$3,3$	28	$40 \mid 28$	40	39

Table 4.2: ϕ_L values of $i \cdot KO[11,0] + j \cdot KO[13,0] + k \cdot KO[15,0]$.

<table>
<tr><th>$j,k\downarrow$ $i\rightarrow$</th><th>0</th><th>1</th><th>2</th><th>3</th></tr>
<tr><td>0, 0</td><td>0</td><td>11 | 0</td><td>11</td><td>11</td></tr>
<tr><td>1, 0</td><td>13 | 0</td><td>13 || 11 | 0</td><td>24 | 13 || 11</td><td>24 | 11</td></tr>
<tr><td>2, 0</td><td>13</td><td>13</td><td>24 | 13</td><td>24</td></tr>
<tr><td>3, 0</td><td>13</td><td>24 | 13</td><td>24</td><td>24</td></tr>
<tr><td>0, 1</td><td>15 | 0</td><td>15 || 11 | 0</td><td>26 | 15 || 11</td><td>26 | 11</td></tr>
<tr><td>1, 1</td><td>15 || 13 | 0</td><td>26 | 15 ||| 13 || 11 | 0</td><td>26 ||| 24 | 13 || 11</td><td>26 || 24 | 11</td></tr>
<tr><td>2, 1</td><td>28 | 15 || 13</td><td>28 || 26 | 15 ||| 13</td><td>39 | 28 || 26 ||| 24 | 13</td><td>39 | 26 || 24</td></tr>
<tr><td>3, 1</td><td>28 | 13</td><td>28 || 24 | 13</td><td>39 | 28 || 24</td><td>39 | 24</td></tr>
<tr><td>0, 2</td><td>15</td><td>15</td><td>26 | 15</td><td>26</td></tr>
<tr><td>1, 2</td><td>15</td><td>26 | 15</td><td>26</td><td>26</td></tr>
<tr><td>2, 2</td><td>28 | 15</td><td>28 || 26 | 15</td><td>39 | 28 || 26</td><td>39 | 26</td></tr>
<tr><td>3, 2</td><td>28</td><td>28</td><td>39 | 28</td><td>39</td></tr>
<tr><td>0, 3</td><td>15</td><td>26 | 15</td><td>26</td><td>26</td></tr>
<tr><td>1, 3</td><td>28 | 15</td><td>28 || 26 | 15</td><td>39 | 28 || 26</td><td>39 | 26</td></tr>
<tr><td>2, 3</td><td>28</td><td>28</td><td>39 | 28</td><td>39</td></tr>
<tr><td>3, 3</td><td>28</td><td>39 | 28</td><td>39</td><td>39</td></tr>
</table>

Table 4.3: ϕ_L values of $i \cdot KOR[11,0] + j \cdot KOR[13,0] + k \cdot KOR[15,0]$, up to an infinitesimal.

Chapter 5

Corridors in Go

5.1 The coin-sliding game

5.1.1 Definition of the game

Consider the following game: Coins of various (nonnegative) monetary values, colored red or blue, are placed on a (quarter-infinite) checkerboard. Two players—Left and Right—move the coins as follows. Left can move a red coin left one square, or a blue coin from a square to the leftmost square one row down; Right, similarly, can move a blue coin left one square, or a red coin from a square to the leftmost square one row down. If a red (or blue) coin is on the leftmost column, Left (or Right) can move it off the left edge of the board, and if a blue (or red) coin is on the bottom row, Left (or Right) can move it off the bottom of the board. Once coins have been moved off the board, they cannot be moved again. Both Left and Right get to keep all the money they move off the bottom edge of the board. The player who ends up with more money is the winner; if both players get the same amount of money, the player who made the last move is the winner. Given some starting position and player to move, who wins, and what is the optimal strategy? We give nearly complete answers to both these questions here.

This game is simplest when all coins are on the leftmost column. In this case each player will just try to move coins down the board one square at a time until they end up in his pocket, subject to the condition that the other player may remove a coin at any time.

5.1.2 The playing field in detail

Figure 5.1 shows the checkerboard playing field marked for analysis. We have omitted the bottom row squares other than the leftmost. This causes no loss of generality, since a red (or blue) coin n squares to the right of the bipolar square would behave the same in play as a blue (or red) coin of the same monetary value n squares above the bipolar square together with a red (or blue) coin of the same monetary value that Right (or Left) already has.

We have also created an extra strip at the top to place the coins of zero value on. The strip behaves peculiarly in that both Left and Right are allowed to remove both red and blue coins from the strip at any time, as well as Left (or Right) being able to move red (or blue) coins left one square. It can be shown that a coin of zero value on the main board, m rows above and n columns to the right of the bipolar square, behaves the same as one coin on the neutral square, one coin $n+1$ squares to the right of the neutral square (if $m > 0$), and $m-1$ coins one square to the right of the neutral square (if $m > 1$).

5.1.3 Description of strategy and classification of positions

Each red or blue coin on a square of the playing field, or in Left's or Right's possession, gives us a *term*, or component combinatorial game, in our position. Monetary values are assumed to be numbers. A blue (or red) coin of monetary value x that Left (or Right) already has is thus identified with the term x (or $-x$,) which is then a number which is viewed as a combinatorial game. The squares in Figure 5.1 are marked with names, from [6] or [2], of terms for coins on the playing field, where coins of positive monetary value are assumed to have monetary value x. The term that a red coin gives is in the upper left, and the term a blue coin gives is in the lower right. We also use labels from Figure 5.1 to refer to certain kinds of terms (small, big, tiny, and so forth.) If G is a term coming from some coin on some square, the negative of G, $-G$, will be the term coming from a coin of the same monetary value and opposite color on the same square. Also, if G is a term coming from a coin in the left-hand column of the main region or the next-to-leftmost square of the top strip, we let $G^{\to k}$ ($k \geq 1$) be the term coming from the same coin when it is located $k-1$ squares to the right. Throughout play, we always assume that pairs of terms G and $-G$ have been removed, since such pairs sum to 0.

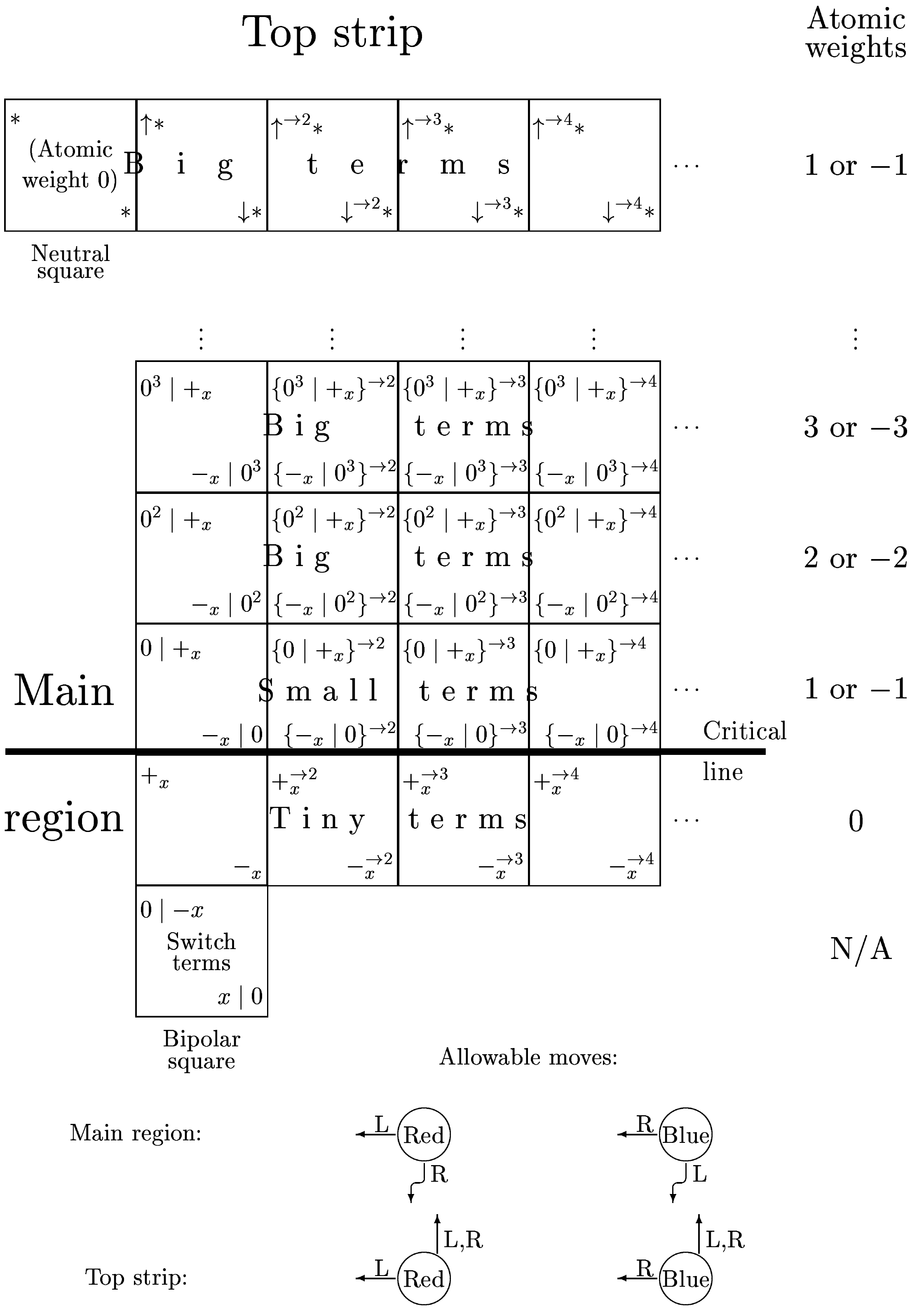


Figure 5.1: The playing field for coin-sliding.

In the same way, we cancel $*$ with itself if it appears more than once. For the initial position, we assume that no switch terms are present, and that no coins have yet been moved off the bottom of the board. Positions like this will be infinitesimal.

We give non-switch terms an *atomic weight* as shown in Figure 5.1; blue coins give terms with nonpositive atomic weight and red coins give terms with nonnegative atomic weight. The atomic weight of an infinitesimal position or group of non-switch terms is the sum of the atomic weights of the terms present. This atomic weight will in fact be that defined in [1], although we will not need this fact. The *monetary value* of a term is the monetary value of the coin it comes from. Red coins give terms called Right's and blue coins give terms called Left's, except that switch terms are called both Left's and Right's terms and $*$ is called neither.

We introduce a linear ordering for terms, called *desirability*. The following rules tell when a term coming from coin A is more desirable than a term coming from coin B. Lower-numbered rules take precedence over higher-numbered rules. If no rules apply, the terms come from coins of the same monetary value on the same square, and are equally desirable.

1. Coin A is below the critical line and coin B is above the critical line.

2. Coin A is in the main region and coin B is in the top strip.

3. Coin A is of higher monetary value than coin B, if both coins are below the critical line, or of lower monetary value than coin B, if both coins are above the critical line.

4. Coin A is closer to the critical line than coin B.

5. Coin A is further to the right than coin B.

The desirability ordering is summarized in Figure 5.2.

We will usually prefer to move on our most desirable term. In fact, if (in the original playing field) all coins start out in the leftmost column, we can win by always playing on our most desirable term (after pairs of terms summing to 0 have been removed) when some terms are ours, and avoiding a few elementary blunders when no terms are ours. If all coins do not start out in the leftmost column, there may be some exceptions to this strategy. We will prove that the strategy we just outlined works by determining precisely which games are won and lost by each player moving first. This classification requires more terminology, which we now introduce. This terminology will also be used to describe when we do not wish to move on our most desirable term.

A player is said to have the least desirable term advantage (LDTA) in a group of terms if the least desirable term (LDT) in the group belongs to him.

We group the terms in an infinitesimal position into *segments.* A small, or tiny, segment is a group of all the small, or tiny, terms with the same monetary value. A big segment is just a single big term. We call a tiny segment strongly Left's if there are more $+_x^{\to k}$'s than $-_x^{\to l}$'s, strongly Right's if there are more $-_x^{\to l}$'s than $+_x^{\to k}$'s, and weakly Left's (or Right's) if it has the same number of $+_x^{\to k}$'s and $-_x^{\to l}$'s and Left (or Right) has the LDTA in the segment. For purposes of desirability comparison and monetary value, small and tiny segments should be viewed as one of their member terms (it is irrelevant which one is used.)

We partition the small and big segments into four classes.

Class α includes all atomic weight 0 small segments in which Right has the LDTA.

Class β includes all Left's big segments, all small segments in which Left has the LDTA, all small segments with atomic weight -1 or less, and $*$.

Class γ includes all atomic weight 1 small segments in which Right has the LDTA.

Class δ includes all small segments of atomic weight 2 or greater in which Right has the LDTA, and all Right's big segments.

We now define a finite state machine with 4 states, called 1, -1, 2, and -2, and feed our big and small segments into it in order, from least to most desirable.

When we are in state 1, we stay in state 1 with a segment of class α, move into state 2 with class δ, move into state -2 with class β, and move into state -1 with class γ.

The negative of a segment is the segment consisting of the negatives of all the terms appearing in the original segment. (When we negate a segment, its atomic weight will be negated, and the LDTA will change hands.) If a segment of class X moves us from state 1 to state Y, then we let a segment of class $-X$, by which we mean a segment whose negative is of class X, move us from state -1 to state $-Y$.

When we are in state 2, we always remain in state 2, and likewise for state -2.

$$\ldots;$$

$$\ldots;\ \ +_3^{\to 3},\ -_3^{\to 3};\ \ +_3^{\to 2},\ -_3^{\to 2};\ \ +_3,\ -_3;\ \ 0 \mid -3,\ 3 \mid 0;$$

$$\ldots;$$

$$\ldots;\ \ +_2^{\to 3},\ -_2^{\to 3};\ \ +_2^{\to 2},\ -_2^{\to 2};\ \ +_2,\ -_2;\ \ 0 \mid -2,\ 2 \mid 0;$$

$$\ldots;$$

$$\ldots;\ \ +_1^{\to 3},\ -_1^{\to 3};\ \ +_1^{\to 2},\ -_1^{\to 2};\ \ +_1,\ -_1;\ \ 0 \mid -1,\ 1 \mid 0;$$

$$\ldots;$$

$$\ldots;\ \ \{0 \mid +_1\}^{\to 3},\ \{-_1 \mid 0\}^{\to 3};\ \ \{0 \mid +_1\}^{\to 2},\ \{-_1 \mid 0\}^{\to 2};\ \ 0 \mid +_1,\ -_1 \mid 0;$$

$$\ldots;\ \ \{0^2 \mid +_1\}^{\to 3},\ \{-_1 \mid 0^2\}^{\to 3};\ \ \{0^2 \mid +_1\}^{\to 2},\ \{-_1 \mid 0^2\}^{\to 2};\ \ 0^2 \mid +_1,\ -_1 \mid 0^2;$$

$$\ldots;\ \ \{0^3 \mid +_1\}^{\to 3},\ \{-_1 \mid 0^3\}^{\to 3};\ \ \{0^3 \mid +_1\}^{\to 2},\ \{-_1 \mid 0^3\}^{\to 2};\ \ 0^3 \mid +_1,\ -_1 \mid 0^3;$$

$$\ldots;$$

$$\ldots;\ \ \{0 \mid +_2\}^{\to 3},\ \{-_2 \mid 0\}^{\to 3};\ \ \{0 \mid +_2\}^{\to 2},\ \{-_2 \mid 0\}^{\to 2};\ \ 0 \mid +_2,\ -_2 \mid 0;$$

$$\ldots;\ \ \{0^2 \mid +_2\}^{\to 3},\ \{-_2 \mid 0^2\}^{\to 3};\ \ \{0^2 \mid +_2\}^{\to 2},\ \{-_2 \mid 0^2\}^{\to 2};\ \ 0^2 \mid +_2,\ -_2 \mid 0^2;$$

$$\ldots;\ \ \{0^3 \mid +_2\}^{\to 3},\ \{-_2 \mid 0^3\}^{\to 3};\ \ \{0^3 \mid +_2\}^{\to 2},\ \{-_2 \mid 0^3\}^{\to 2};\ \ 0^3 \mid +_2,\ -_2 \mid 0^3;$$

$$\ldots;$$

$$\ldots;$$

$$\ldots;\ \ \uparrow^{\to 3}*,\ \downarrow^{\to 3}*;\ \ \uparrow^{\to 2}*,\ \downarrow^{\to 2}*;\ \ \uparrow*,\ \downarrow*;$$

$$*$$

Reading in the normal way, from left to right and from top down,
reads from the most to the least desirable term.
A comma separates terms of equal desirability, and
a semicolon separates terms of unequal desirability.

Figure 5.2: Terms in desirability order.

Atomic weight of position	State machine classification (start state →end state)	Tiny advantage	Outcome for Right (if Right starts)		Outcome for Left (if Left starts)	
≤ -2			win	A	lose	$-J$
-1	$-1\to1$	Left	win	A	win	$-F$
-1	$-1\to1$	Not Left	win	A	lose	$-G$
-1	$-1\to2$		win	A	lose	$-I$
-1	$-1\to-2$		win	A	win	$-H$
0	$-1\to-1$	Right	win	B		
0	$-1\to-1$	Not Right	lose	C		
0	$-1\to2$		win	D		
0	$-1\to-2$		lose	E		
0	$1\to1$	Left			win	$-B$
0	$1\to1$	Not Left			lose	$-C$
0	$1\to2$				lose	$-E$
0	$1\to-2$				win	$-D$
1	$1\to-1$	Right	win	F	win	$-A$
1	$1\to-1$	Not Right	lose	G	win	$-A$
1	$1\to2$		win	H	win	$-A$
1	$1\to-2$		lose	I	win	$-A$
≥ 2			lose	J	win	$-A$

Table 5.1: Outcomes of infinitesimal positions in coin-sliding.

We always start either in state 1 or in state -1. Notice that if we start in the same state we end in, the total atomic weight must be 0; if we start in state -1 and end in state 1, the total atomic weight must be -1; and if we start in state 1 and end in state -1, the total atomic weight must be 1.

We say that Right has the tiny advantage (TA) if the least desirable tiny segment is strongly Right's and its monetary value is no more than the monetary value of any small segment, or if it is weakly Right's and its monetary value is less than any of these monetary values. We define Left's having the TA similarly.

Theorem 21 *We can classify infinitesimal positions as follows, in terms of the outcome for the player who starts. The italic letters next to the outcomes reference the section of the proof where the outcome is proved to be as claimed.*

In cases D, $-D$, E, $-E$, H, $-H$, I, and $-I$, we call the segment responsible for sending the state machine into state 2 or -2 *crucial*, and in cases B, $-B$, F, and $-F$, we call the least desirable tiny segment crucial, provided that it is only weakly Left's or Right's. In all other cases, or if the position is not infinitesimal, no segment is considered crucial. If a segment is crucial, its LDT is also called crucial.

We then claim that the strategy below wins when it is consistently applied starting from a won infinitesimal position.

(1) If we have any terms, move in our most desirable term (OMDT), unless it is crucial, equalling $G^{\to k}$, say, and there is a $-G^{\to l}$ present, for some $l > k \geq 1$.

(1a) In this case, move in our next-to-most desirable term, if it exists.

(1b) If we have no other terms, move from $-G^{\to l}$ to $-G^{\to l-1}$. (In this case, $-G^{\to l}$ must be unique.)

(2) If we have no terms, the position is a group of our opponent's terms, perhaps along with $*$. In this case, it doesn't really matter what we do, provided that we do not leave a position where there is only one non-tiny term, or that if we do, this term is neither $*$, a $\uparrow^{\to k}*$, nor a $\downarrow^{\to l}*$.

Notice that if all coins are in the leftmost column of the main region or the left 2 squares of the top strip, i.e., there are no $G^{\to k}$'s with $k \geq 2$, part (1) of this strategy reduces to always playing on OMDT until we have no terms left, since the caveat in (1) never applies. This proves Theorem 5 in [6] or [2].

5.1.4 Proof of strategy and classification

Proof. We first claim that our strategy always lets Right win positions that are *monetarily won*, by which we mean positions with negative Right stop. This is because our strategy always gets Right at least his Right stop, since if there are switch terms present we always move on the hottest one, or make a move from one of our tiny terms with a greater incentive.

We now prove that when Right applies his strategy to an infinitesimal position we claim is won, he either ends up moving from a monetarily won position, or reaches an infinitesimal position that we claim is lost for Left, with Left to move.

First take the case where Right's strategy tells him to move on one of his tiny terms, say on $+_x^{\to k}$. In this case $+_x^{\to k}$ must be OMDT, since all terms less desirable than the crucial term are always non-tiny. The result will be infinitesimal except for a switch term $0 \mid -x$. If our opponent leaves it in this form, we will win since this is monetarily won. Hence our opponent must either move from $0 \mid -x$ to 0—by *moving to 0* we mean moving on a term and removing it completely, by moving its coin off the left-hand edge of the main region or removing its coin from the top row—or from some $-_y^{\rightarrow l}$ to $y \mid 0$. If he moved to $y \mid 0$ and $y < x$, the position is monetarily won; if he moves to $y \mid 0$ and $y > x$, our strategy will tell us to move from $y \mid 0$ to 0 and our opponent will then be in the same position as before. If our opponent moves from $0 \mid -x$ to 0 or from $-_x^{\rightarrow l}$ to $x \mid 0$, the position again becomes infinitesimal, and when this happens, the position will be as it was, except that the term $+_x^{\rightarrow k}$ will be missing, along with perhaps some $-_y^{\rightarrow l}$ terms with $y > x$ and possibly a $-_x^{\rightarrow l}$ term. If Right could win the position before, he can win it now, unless he needs the TA to win, started with the TA, and lost it. We can't possibly lose the TA unless the the minimum-monetary-value tiny segment in the original position had monetary value x, so say it did. The monetary value x tiny segment always remains strongly Right's if it started strongly Right's. If it started weakly Right's, it will remain strongly or weakly Right's if we need the TA to win, since if we need the TA to win our initial move cannot have been on the LDT of the tiny segment, which is then crucial.

Also, suppose that there are none of our terms present, i.e., we are using clause (2). There must be some terms present, since otherwise we'd be in case C. We can and must, then, move to a position with none of our terms present, which will then have atomic weight ≤ 0, and in which Left will then not have the TA. If it has atomic weight ≤ -2, it is already lost for Left, by case $-J$. If $*$ is present and it has atomic weight -1, it is lost for Left, by case $-I$. If $*$ is not present and it has atomic weight 0, it is lost for Left by $-C$. If $*$ is not present and it has atomic weight -1, it is type $-G$ and hence lost for Left, unless the only one of Left's non-tiny terms present is a big term, i.e., a $\downarrow^{\rightarrow k}*$. This covers all cases except those we have been directed not to move to. It is easy to avoid these. For example, we can move arbitrarily if there is no $*$ or $\downarrow^{\rightarrow k}*$, on the $*$ if there is a $*$ and zero or two $\downarrow^{\rightarrow k}*$'s, from the $\downarrow^{\rightarrow k}*$ to 0 if there is one $\downarrow^{\rightarrow k}*$ and no $*$, and from a $\downarrow^{\rightarrow k}*$ to $\downarrow^{\rightarrow k-1}*$ (if $k \geq 2$) or $*$ (if $k = 1$) in all other situations.

So we can suppose that we are using clause (1) and not moving on one of our tiny terms. We must, then, move to another infinitesimal position. We split up into cases labelled as in Table 5.1.

A. In this case, the atomic weight is ≤ -1, and we move on one of our own terms, to a position with atomic weight ≤ -2, which is lost for Left by $-J$.

D. Since the crucial term is not a tiny, we can assume that we have no tiny terms, or else we would move on one. Call the non-tiny terms more desirable than the crucial segment the *postfix,* and call the terms less desirable than the crucial segment the *prefix.* If there is some term of ours in the postfix, we will move on it, to a position of atomic weight -1. Moving on one of our terms always makes it more desirable, or removes it, so the state-machine classification will remain the same and we will move to a position lost for Left by $-I$.

If we have no term in the postfix, there are 2 cases. Let x be the crucial segment's monetary value, if this value exists.

(1) We entered state 2 from state 1 by means of a class δ segment. In this case the prefix must have atomic weight -1. Hence the crucial segment together with the postfix has atomic weight 1.

Say the crucial term is a big term. If it is a $\uparrow^{\rightarrow k}*$, the postfix must be empty, so moving from $\uparrow^{\rightarrow k}*$ to 0 gives a position lost for Left by $-G$. (Remember that we have no tiny terms, so Left cannot have TA.) If it is a $\{0^n \mid +_x\}^{\rightarrow l}$ for some $n \geq 3$, we move on it, to a position of type $-I$, unless the postfix is $-_x \mid 0^{n-1}$, in which case the crucial term will cancel with the postfix, leaving a type $-G$ position as before. If it is a $\{0^2 \mid +_x\}^{\rightarrow l}$, the postfix must be $\{-_w \mid 0\}^{\rightarrow m}$ for some $w \leq x$. Then moving on the crucial term either cancels with the postfix, leaving a type $-G$ position, or leaves a class γ segment followed by a class $-\gamma$ segment, also leaving a type $-G$ position.

If the crucial segment is a small segment, moving on OMDT in it leaves a type $-I$ position if it is of atomic weight ≥ 3. If it is of atomic weight 2, the postfix must be $\{-_w \mid 0\}^{\rightarrow l}$ for some $w < x$, which is then of class $-\gamma$. Then moving from OMDT in the crucial segment leaves a class γ segment followed by a class $-\gamma$ segment, as well as introducing $+_x$; but this does not give Left the TA, since $w < x$, so this leaves a type $-G$ position.

(2) We entered state 2 from state -1 by means of a class $-\beta$ segment. In this case the prefix must have atomic weight 0, so the crucial segment together with the postfix also has atomic weight 0.

Say we end up using clause (1a) or (1b) in our strategy. Since the crucial segment together with the postfix has atomic weight 0 it must be just $G^{\rightarrow k} - G^{\rightarrow l}$, where

$l > k$ and G is of the form $\uparrow *$ or $0^n \mid +_x$. If we use clause (1b), there are none of our terms in the prefix, so there can be none of our opponent's terms either. Hence the entire position consists of the crucial segment together with the postfix, plus perhaps some $-_y^{\rightarrow m}$'s, and making the move directed by our strategy leaves a position of type either $-C$ or $-E$. If we use clause (1a), we will move on OMDT in the prefix. Our most desirable segment in the prefix will have class $-\alpha$ or class γ, and in either case, must contain a term of ours, so we in fact move in OMDT in this segment. If class $-\alpha$, the segment becomes class $-\gamma$ upon moving on OMDT in it, and if class γ, it either becomes class α or disappears upon moving on OMDT in it. Also, a $+_y$ is produced for some $y > x$. If G is not of the form $0 \mid +_x$, our crucial term is big, so it is of class δ and we leave a position of type $-I$. If G is of the form $0 \mid +_x$, the crucial segment is of class α and the postfix is evidently empty, and the $+_y$ does not give Left the TA since $y > x$, so we leave a position of type $-G$.

We may then assume that clauses (1a) and (1b) are not used, so that we move on OMDT. Say the crucial term is a big term. It can't be $*$ since then we would have no terms. If it is a $\uparrow^{\rightarrow k}*$, the postfix must be a $\{-_x \mid 0\}^{\rightarrow l}$ or a $\downarrow^{\rightarrow l}*$. If it is the former, moving in the $\uparrow^{\rightarrow k}*$ leaves the postfix a class $-\gamma$ segment, so we leave a type $-G$ position. If it is the latter we do not move on OMDT. If the crucial term is a $\{0^n \mid +_x\}^{\rightarrow k}$, $n \geq 3$, and we move on it, it will remain of class $-\beta$ and more desirable than the postfix, and we will leave a type $-I$ position, unless there is a $-_x \mid 0^{n-1}$ in the postfix, and in this case, the postfix must equal $-_x \mid 0^{n-1} + \{-_w \mid 0\}^{\rightarrow l}$ for some $w \leq x$ and l. Then moving makes the postfix of class $-\gamma$, and the position becomes of type $-G$. If the crucial term is a $\{0^2 \mid +_x\}^{\rightarrow k}$, and we move on it, the resultant term $0 \mid +_x$ will be a segment of class $-\beta$ more desirable than the postfix unless the postfix is $\{-_x \mid 0\}^{\rightarrow l} + \{-_w \mid 0\}^{\rightarrow m}$ for some $w \leq x$, where we let $l \leq m$ if $w = x$. If then $l > 1$, after moving, we have the LDT in the least desirable segment in the postfix, which is then of class $-\beta$, so the position is of type $-I$. If $l = 1$, the postfix becomes $\{-_w \mid 0\}^{\rightarrow m}$ after moving, which is of class $-\gamma$, so the position is of type $-G$.

If the crucial segment is not big, it must have atomic weight ≥ 0 since the postfix has none of our terms and, together with the crucial segment, it has atomic weight 0. If the crucial segment has atomic weight ≥ 2, moving on OMDT in it leaves it of class $-\beta$ and gives a type $-I$ position. If it has atomic weight 1, the postfix must be of the form $\{-_w \mid 0\}^{\rightarrow l}$, $w < x$, so moving on OMDT in the crucial segment either leaves the segment of class $-\beta$ and gives a type $-I$ position, or produces a class $-\alpha$ segment followed by a class $-\gamma$ segment and an additional $+_x$, making a type $-G$ position, or wipes out the crucial segment completely, leaving a class $-\gamma$ segment and an additional $+_x$, again a type $-G$ position. Finally, if it has atomic weight 0, the postfix must be empty and Right must have the LDTA in the segment. Then moving in OMDT will leave a class $-\beta$ segment and thus a type $-I$ position unless OMDT in the segment is also the LDT in the segment, in which case we use clause (1a) or (1b).

H. This is analogous to *D*, except that the total atomic weight and prefix atomic weight are each increased by 1. Also, types $-G$ and $-I$ are replaced by types $-C$ and $-E$, respectively, and we cannot end up using clause (1b) in our strategy.

B. In this case, if any term is crucial, the only tiny term of ours present is, so we must move in our most desirable non-tiny term, if it exists. If there is no non-tiny term of ours present, the position must be of the form $+_x^{\rightarrow k} + -_x^{\rightarrow l}$ with $l > k$ plus other $-_y^{\rightarrow m}$'s, $y > x$. Our strategy then directs us to move to $+_x^{\rightarrow k} + -_x^{\rightarrow l-1}$ plus the other $-_y^{\rightarrow m}$'s. This is then lost for Left by $-C$.

Otherwise, the most desirable non-tiny segment is either of class γ or class $-\alpha$, since we finish in state -1. Let it be of monetary value x, say. If it is of class $-\alpha$, we move in OMDT in it to a segment of class $-\gamma$ plus a $+_x$. However, adding $+_x$ does not give Left the TA, since we started with the TA, so the resultant position is of type $-G$, which Left must lose. If it is of class γ, we move in OMDT in it, to a segment which is either of class α or vanishes, plus a $+_x$. If it is of class α, we leave a position of type $-G$, as before. If it vanishes, the $+_x$ is still present, and x is now smaller than the minimum weight of all small segments. However, our TA is still secure if it was caused by a tiny segment of monetary value below x. If not, the smallest tiny segment must have been strongly ours, so no term was crucial. Consequently there are no tiny terms of ours left, but there is at least one $-_x^{\rightarrow k}$ left, and adding $+_x$ to this does not give Left the TA. So in any case, we leave a position of type $-G$.

F. As in *B*, but we always move in a non-tiny term, and instead of type $-G$, we leave a position of type $-C$.

We now prove that when Left is to move from any of the infinitesimal positions we have claimed are lost for Left, he cannot win, because he must reach a position that is monetarily won, or an infinitesimal position which we have claimed is won for Right, with Right to move.

First consider the case where Left moves on one of his tiny terms, from $-_x^{\to k}$ to $x \mid 0$, say. In this case, Right will reply from some $+_y^{\to l}$ to $0 \mid -y$ if such a term is present with $y > x$. Then if we move on $x \mid 0$, on a non-switch term that is not one of our tiny terms, or on some $-_w^{\to m}$ with $w < y$, the resulting position is monetarily won. Otherwise we must move on $0 \mid -y$ or some $-_y^{\to m}$, reaching a position like that after our first move, or from some $-_z^{\to m}$ to $z \mid 0$, where $z > y$; if we do the latter, Right will immediately reply from $z \mid 0$ to 0, so we will be in the same position as before. When Right has to move from a position whose only switch term is $x \mid 0$ and there are no $+_y^{\to l}$'s present with $y > x$, he will reply on the $+_x^{\to n}$ with biggest n if one is present, or if not, on $x \mid 0$. The position will then again be infinitesimal, and when this happens, it will be the same as at the outset, except that $-_x^{\to k}$ will be gone, all $+_y^{\to l}$'s with $y > x$ will be gone, the $+_x^{\to n}$ with biggest n will be gone, if any $+_x^{\to n}$'s were present, and some $-_z^{\to m}$'s with $z > x$ may also be gone. We claim that this does not give us the TA unless we started with the TA; this will show that if we lost the original position we must lose this one as well. We could not possibly get the TA unless the minimum-monetary-value tiny segment in the original position had monetary value x. If this segment started weakly Right's it remains weakly Right's or disappears, and if it started strongly Right's it remains strongly Right's or disappears. If it disappears we do not get the TA because all the $+_y^{\to l}$'s with $y > x$ have been removed.

Now consider the cases where Left does not move on one of his tiny terms, and hence moves to an infinitesimal position. Again, we label these cases with letters.

$-J$. In this case, the atomic weight is ≤ -2. We can increase the atomic weight by at most 1, so we must move to a position with atomic weight ≤ -1, which Right can win, by A.

$-G$. We must increase the atomic weight from -1 to 0, so we must move on one of our terms. If we move on a class γ segment, it becomes class δ, so the position becomes type D and is won by Right; if we move on a class $-\alpha$ segment, it becomes class $-\beta$, so the position becomes type D. If we we move on a segment of class α, it leaves a class γ segment, plus a tiny term of ours. If there is a segment following the class α segment, it was of class α or class γ, and is hence of class $-\beta$; so the position becomes of type D after our move and is won by Right. If there was no segment following the class α segment, our extra tiny term suffices to give Right the TA, since we did not originally have the TA. Hence we leave a position of type B which is won by Right. If we move on a class $-\gamma$ segment, it either becomes class $-\alpha$ or class $-\beta$ or disappears. The same reasoning as before then shows that the position becomes of type B or D and is won by Right.

$-I$. We must move on one of our terms and make the atomic weight 0. The reasoning in case $-G$ shows that we cannot move in the prefix, unless perhaps we move on the segment immediately prior to the crucial segment and change it from class α to class γ, change it from class $-\gamma$ to class $-\alpha$, or remove it, when it is of class $-\gamma$. However, in all these cases we must have entered state 2 from state 1 by means of a class δ segment, and class δ segments are class $-\beta$, so we then leave a position of type D which Right can win. Moving in the postfix just makes one of our terms more desirable, or removes it, so the position becomes of type D. Finally, suppose we move in the crucial segment. If state 2 is reached from state 1, the crucial segment is of class δ, and moving in it leaves it of class δ. If state 2 is reached from state -1, the crucial segment is of class $-\beta$, and moving in it leaves it of class $-\beta$. The position is then of type D and Right can win it.

$-C$, $-E$. Analogously to cases $-G$ or $-I$, if we move on one of our own terms, the position becomes of type F or H and is then won by Right. However, we have the additional option of not moving on one of our own terms. If we are not to immediately lose by A, we must either move from a $+_x$ to 0 or from a $G^{\to k}$ to $G^{\to k-1}$, where $k \geq 2$ and G is of the form $+_x$, $\uparrow *$, or $0^n \mid +_x$. The term $*$ cannot appear since we would then be in case $-D$. Suppose that there are non-tiny segments present. The least desirable segment is then either of class α, class γ, or class δ, and is hence of class $-\beta$. If we do not move on the least desirable segment, the least desirable segment will remain unchanged and we will leave a type D position. If we do move in the least desirable segment, its atomic weight will be unchanged and Right will still have the LDTA in it if he started with it, so it remains of class $-\beta$ and we leave a type D position. Finally, if there are no non-tiny segments present, we must be in case $-C$. If the position is nonzero, then, Right must have the TA. Then moving from $+_x$ to 0 or from $+_x^{\to k}$ to $+_x^{\to k-1}$ will not destroy Right's TA, so we will end up in a type B position which Right can win.

This verifies that our strategy wins Right all the positions we have claimed are won for him, and that Left loses all the positions we have claimed are lost for him. To prove that Left can win all the supposedly won positions by using our strategy, and that Right loses his supposedly lost positions, we negate the position—that is, negate every term in the position—and apply the above arguments. (We have set things up so that if position P is in case X, then $-P$ is in case $-X$. This is so because negating a position negates its atomic weight and makes the TA change hands, and because if position P takes us from state Y to state Z, then $-P$ takes us from state $-Y$ to state $-Z$.) ∎

5.1.5 Generalizations of the coin-sliding game

In our analysis of the coin-sliding game, we saw that in a well-played infinitesimal game, no coin actually moves off the bottom row; all coins are slid off the left before then. Because of this, we may hope that our analysis does not depend very much on the fact that the monetary values of the coins in the main region are numbers. We could make them other (non-loopy) combinatorial games instead (added to the sum of terms upon the turn the coin is slid off.)

Theorem 22 *The analysis of the coin-sliding game in the last section still holds provided that the following conditions are true:*

1. *The nonzero monetary values, or x's, are totally ordered. (This implies that the desirability ordering still exists as before.)*
2. *For all monetary values x, $R(x)$, the Right stop of x, is positive.*
3. *$R(x^R) > 0$ for all x^R. (This condition is automatically satisfied if x is a positive number.)*
4. *$2 \cdot \{x \mid 0\} \leq x$ for all x. ($x^{RR} \geq 0$ for all x^{RR} is a sufficient condition for this. This is also satisfied if x is a positive number.)*
5. *For any $x_2 > x_1$, $x_2 - x_1$ is greater than any sum $+_{x_3}^{\to a_1} + \cdots + +_{x_r}^{\to a_{r-2}}$, where x_3, $\ldots$, x_r can stand in any order-relation to x_1 and x_2.*

Proof. We look at the portion of the proof of Theorem 21 where Left or Right moves on one of his tiny terms. We claim that it is still the best strategy to move on your most desirable tiny term or switch term until there are none left, unless your opponent makes a blunder.

Suppose that we are Right, moving in a sum G on one of our tiny terms, from $+_x^{\to k}$ to $0 \mid -x$. Play will proceed as in the analysis in Theorem 21 until Left makes a blunder by moving to a position we claimed was monetarily won. Such moves were either to $0 \mid -x$ plus an infinitesimal or to $\{0 \mid -x\} + \{w \mid 0\}$ plus an infinitesimal, with $w < x$. In the first case, we can move to $-x$ plus an infinitesimal. This then has negative Left stop, by 2, so we have won. In the second case we reply on $0 \mid -x$ to $-x$.

After we have replied, the position will be of the form $\{w \mid 0\} - x$ plus an infinitesimal. If Left moves on the infinitesimal and leaves it infinitesimal, we can respond on $w \mid 0$ and win by 2. If Left moves on $-x$ we can respond on $w \mid 0$, to a position which has negative Left stop, by 3, so we have won. If Left moves on some $-_a^{\to m}$ to $a \mid 0$, we can respond from $a \mid 0$ to 0. If Left moves on $w \mid 0$, we are at $w - x$ plus an infinitesimal. For our original position G, we can write $G = +_x^{\to k} + -_w^{\to l} +$ (other tiny terms) $+ H$. Then the position we are in is less than $K = H + -_v$, for all v, by 5. We can pick v small enough to give ourselves the TA in K. Hence, since our criterion gave us the win in G, it will also do so in K. Since K is simpler than G, we can now apply this Theorem inductively to show that we have won K and hence our position.

Suppose that we are Left, moving in a sum G on one of our tiny terms, from $-_x^{\to k}$ to $x \mid 0$, say. Right will either respond from $x \mid 0$ to 0, in which case the position remains infinitesimal, or from some $+_{y'}^{\to l}$ to $0 \mid -y'$ with $y' > x$. In this second case, the analysis will then proceed as before, until Left makes some move that would have, in Theorem 21, resulted in a monetarily won position for Right. Such a move must be made from a position $\{x \mid 0\} + \{0 \mid -y\}$ plus an infinitesimal, for some $y > x$, and will be one of the following:

1. From $x \mid 0$ to x. Right can reply from $0 \mid -y$ to $-y$. In this case, if we write $G =$ (tiny terms) $+ H$, we will be in a position which is less than $K = H + -_v$, for all v, by 5. We can pick v small enough to give Right the TA in K; our criterion will then make K lost by us to move, and since K is simpler than G, we can apply this Theorem inductively, so we have lost K and hence our position.

2. In the infinitesimal portion of the game, leaving it infinitesimal. Suppose that we had made this infinitesimal move to start with, from G to G', say. Since our criterion tells us that G is lost with us to move, Right should be able to move on G' and win. In fact, if we write $G' = +_y^{\to m} + -_x^{\to n} +$ (other tiny terms) $+ H$, our criterion says that Right will win $H + -_v$, if v is small enough. It follows that if Right, in our position $\{x \mid 0\} + \{0 \mid -y\}$ plus an infinitesimal, moves from $\{0 \mid -y\}$

to $-y$, he can win as in the third paragraph of this proof. Hence our position is lost.

3. From $-_{w}^{\to p}$ to $w \mid 0$, where $w < y$. In this case, Right can respond from $0 \mid -y$ to $-y$, to a position that is $\{x \mid 0\} + \{w \mid 0\} - y$ plus an infinitesimal, H. If v is the maximum of w and x, this is no more than $2 \cdot \{v \mid 0\} - y + H$, and this is no bigger than $v - y + H$, by 4. This is then lost by us to move as in the first case.

∎

Hypothesis 5 is necessary because of, for example, the identity $+_{1_{-1}} = 3 \cdot +_1$.

5.2 Applications to corridors and rooms in Go

We will call a connected set of intersections, adjacent to no empty intersections or non-immortal stones outside the set, a *room*. If all intersections in the room are on the border of the room, and all intersections in the room are adjacent to only immortal Black stones outside the room, except for one intersection, which is adjacent to (perhaps) some immortal Black stones outside and just one immortal White stone outside, we will call the room a *hyper-Black room*. In this case, the only non-dominated move for both Black and White will be to play next to the White stone, assuming that this intersection is empty. If Black plays here, the empty intersections inside the room will all become Black's territory, any White stones inside the room will become dead, and the value of the position will become a nonnegative integer.

The simplest hyper-Black room is a corridor, as shown in Figure 5.3. In this case, after any number of White's moves, the remaining room will still be hyper-Black. Hence the value of the Go position in the figure will be $8 \mid\mid\mid 7 \mid\mid 6 \mid 0$. If cooled by 1 point [1, 3], or chilled [2, 6], this will become $7 + \{0 \mid +_4\}$. This is, up to an integer, the value of the coin in Figure 5.3. In general, a Black corridor as in Figure 5.3 with $r > 0$ empty spaces and $n > 0$ stones will correspond [6, 2], after chilling, to the integer $2n + r - 2$ plus a coin of value $2n - 2$, $r - 1$ squares above the bottom row.

Coins with an integer value that are not in the left-hand column also correspond to something that can occur in Go. See [6, Section 3.10] or [2, Section 4.11] for an example involving a set of interacting corridors.

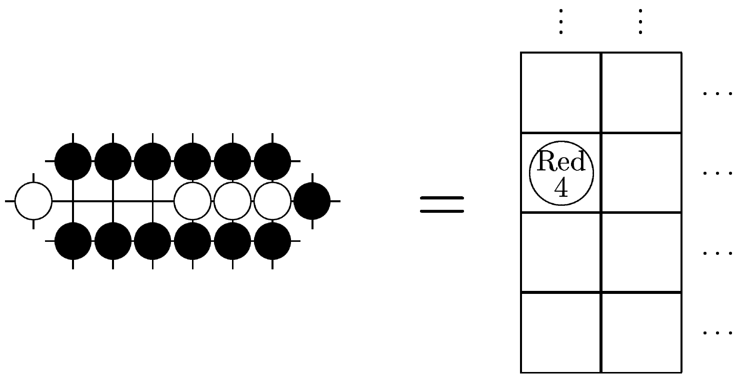


Figure 5.3: Coin-sliding and Go.

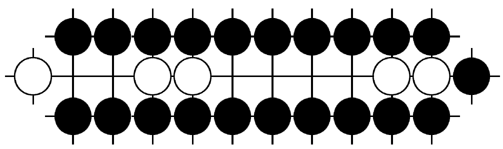


Figure 5.4: A more complicated corridor.

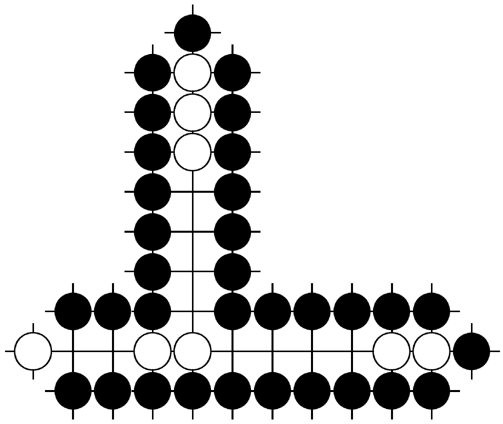


Figure 5.5: A still more complicated room.

If we have a Black corridor with White stones at intervals along it, the position may correspond to a coin in the leftmost column with a non-integral value. For example, the position in Figure 5.4 has value $13 \,||||||\, 12 \,|||||\, 7 \,||||\, 6 \,|||\, 5 \,||\, 4 \mid 0$, or, chilled, $12 + +_{4+\{-_2|0^2\}}$. This corresponds to a coin with value $4+\{-_2 \mid 0^2\}$. Theorem 22 lets us play multiples of this coin together with any number of coins of integral value. In fact, we can play arbitrary corridors of this form with each other, as long as the gaps between successive strings of White stones in all corridors are always at least 4 intersections long and always have the same parity (odd or even.)

Another application of Theorem 22 is for a hyper-Black room that, at some point, breaks up into two hyper-Black rooms, as shown in Figure 5.5. After two moves by White, the position will break up into two Black corridors, of values $7 \,||||\, 6 \,|||\, 5 \,||\, 4 \mid 0$ and $9 \,||||\, 8 \,|||\, 7 \,||\, 6 \mid 0$. The value of Figure 5.5 itself will then be $23 \,||\, 22 \mid (\{7 \,||||\, 6 \,|||\, 5 \,||\, 4 \mid 0\} + \{9 \,||||\, 8 \,|||\, 7 \,||\, 6 \mid 0\})$, or, after chilling, $22 + +_{6+\{-_4|0^2\}+\{-_2|0^2\}}$. This corresponds to a coin of value $6 + \{-_4 \mid 0^2\} + \{-_2 \mid 0^2\}$, which we could play together with coins of integral value, or even together with the coin of Figure 5.4. We could also handle corridors with more than one branching point by applying Theorem 22 recursively.

Chapter 6

Summary

If a Go endgame position, K, decomposes as $G + H$, where G contains loops and H does not contain loops—that is, H is an ender—then we have shown that the value of K, under Japanese ko-ban, is between $\phi_L^J(G) + H$ and $\phi_R^J(G) + H$. Here, $\phi_L^J(G)$ denotes the value of G in the limit where Left has very many ko-threats, and $\phi_R^J(G)$ denotes the value of G in the limit where Right has very many ko-threats.

There are two cases where we have shown how to compute the exact value of $\phi_L^x(G)$ and $\phi_R^x(G)$, where either $x = J$, for Japanese ko-ban, or $x = N$, for North American ko-ban. (Actually, we showed how to compute $\phi_L^x(G)$, but $\phi_R^x(G) = -\phi_L^x(-G)$.) One is where we are using Japanese scoring and G is a sum of kos with options infinitesimally shifted numbers; the other is when we are using Chinese scoring and G is a sum of kos with options integers, the Left option always exceeding -2 for $\phi_L^x(G)$, and the Right option never exceeding 1 for $\phi_R^x(G)$.

If G is not of the forms described above, we have shown how to adapt the method of sidling, described in [1, Chapter 11], to compute $\phi_L^x(G)$ and $\phi_R^x(G)$. This provides an algorithm that usually computes $\phi_L^x(G)$ and $\phi_R^x(G)$ in a finite length of time, the result being usually, but not always, an ender. Thus, in very many cases, all loopiness can be removed from an endgame, eliminating one of the difficulties for analysis.

In Chapter 5, we gave results relating to the ender component, H, of a Go endgame. We can determine who wins H for all H's which consist of sums and differences of positions in a set which includes almost all of the positions whose values are determined in [6] or [2]. This set also includes complicated corridors and rooms such as those in Figures 5.4 and 5.5.

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