# No more undulating squares

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A natural number is *undulating* if its decimal expansion is of the form $ab \dots abab$ or $bab \dots abab$ for some digits $a$ and $b$. We will prove that the only undulating squares are 0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 121, 484, 676, and 69696.

Examining the undulating numbers with 8 digits or fewer finds only the squares above. If we have an undulating square with 9 or more digits, however, $bababab ab$ will be a square mod $10^9$, which can happen only if $(a, b) \in \{(0, 0), (2, 1), (2, 9), (6, 1), (6, 9), (8, 4)\}$. If there is an undulating square with $(a, b) = (8, 4)$, there will be one of equal length with $(a, b) = (2, 1)$, so it will do to disprove the existence of undulating squares with 9 or more digits, $a$ 2 or 6, and $b$ 1 or 9.

For an undulating square $y^2$ with an even number, $2m > 0$, of digits, we have

$$99y^2 = (10a + b)(100^m - 1).$$

We can write $10a + b = qr$, where $q$ is 1 or 3 and $r \in \{7, 29, 61, 23\}$ is prime; then $r|y$, and if we write $y = rz$ we have

$$100^m - (99r/q)z^2 = 1,$$

or, if we set $v = 10^m$, $w = (3/q)z$, and $x = 11qr$,

$$v^2 - xw^2 = 1.$$

If we let $\alpha$ be $\sqrt{x}$, $K$ be the quadratic extension $\mathbf{Q}(\alpha)$, and $\mathcal{N}$ be the norm from $K$ to $\mathbf{Q}$, this will be the norm equation $\mathcal{N}(v + w\alpha) = 1$. Since for our $K$'s the norm of the fundamental unit is always 1, the solution to this will be $v + w\alpha = \pm f^N$, where $f$ is the fundamental unit of $K$, as shown below:

| $(a,b)$ | $q$ | $r$ | $\alpha$ | $f$ | $\mathcal{N}(f)$ |
|---|---|---|---|---|---|
| $(2,1)$ | 3 | 7 | $\sqrt{3\cdot 7\cdot 11}$ | $76+5\alpha$ | 1 |
| $(2,9)$ | 1 | 29 | $\sqrt{11\cdot 29}$ | $12901780+722361\alpha$ | 1 |
| $(6,1)$ | 1 | 61 | $\sqrt{11\cdot 61}$ | $58620+2263\alpha$ | 1 |
| $(6,9)$ | 3 | 23 | $\sqrt{3\cdot 11\cdot 23}$ | $551+20\alpha$ | 1 |

Let $R=\mathbf{Z}[\alpha]$. For $(a,b)=(2,1)$ and $(a,b)=(6,9)$, we observe that $f=1$ in $R/5R$, so all powers of $f$ are 1 in $R/5R$ and $v$ must be $\pm 1$ modulo 5, contradicting $v=10^m$. For $(a,b)=(6,1)$, $f=\alpha$ in $R/3R$, so $f^2=-1$ and the powers of $f$ are 1, $\alpha$, $-1$, $-\alpha$, 1, .... Recalling that $3|w$, we see that $N$ must be even; but since $f=-2\alpha$ in $R/5R$, the powers of $f$ in $R/5R$ are 1, $-2\alpha$, $-1$, $2\alpha$, 1, ..., so $v$ is $\pm 1$ modulo 5, contradicting $v=10^m$. Finally, if $(a,b)=(2,9)$, $f=\alpha$ in $R/5R$, with powers 1, $\alpha$, $-1$, $-\alpha$, 1, ..., so $N$ must be odd; but $f=4+\alpha$ in $R/8R$, making the powers of $f$ 1, $4+\alpha$, $-1$, $4-\alpha$, 1, ..., so $v$ is congruent to 4 modulo 8, contradicting $v=10^m$ (unless $m=2$, which we have already ruled out.)

For an undulating square $y^2$ with an odd number, $2m+1$, of digits, we have

$$99y^2 = 10a(100^m-1)+b(100^{m+1}-1) = (10a+100b)100^m-(10a+b).$$

Multiplying by 99 and writing $v=99y$ and $z=3\cdot 10^m$, we get

$$v^2-11(10a+100b)z^2=-99(10a+b);$$

let $4^L$ be the maximal power of 4 dividing $10a+100b$, $w$ be $2^Lz$, $t$ be $-99(10a+b)$, and $x$ be $11(10a+100b)/4^L$; then

$$v^2-xw^2=t,$$

so letting $\alpha=\sqrt{x}$, $K=\mathbf{Q}(\alpha)$ and $\mathcal{N}$ be the norm from $K$ to $\mathbf{Q}$ as before, we get the norm equation $\mathcal{N}(v+w\alpha)=t$. $K$'s fundamental unit, as it turns out, still has norm 1, so the solution to this will be $v+w\alpha=\pm cf^N$, where $f$ is the fundamental unit, and $c$ is some integral element of $K$ with norm $t$. To find $c$ we may factor $-99(10a+b)$ as a product of prime ideals in $K$ and then attempt to split this product into two conjugate principal ideals, in the usual way. Note that the ring of integers $R$ in $K$ is $\mathbf{Z}[\alpha]$ except when $(a,b)=(6,9)$, when it is $\mathbf{Z}[\theta]$ for $\theta=\frac{1}{2}(1+\alpha)$ a root of $X^2-X-41=0$. We show $f$ and the possible values for $c$ below.

| $(a,b)$ | $2^L$ | $\alpha$ | $t$ | Possible $c$'s | $f$ | $\mathcal{N}(f)$ |
|---|---|---|---|---|---|---|
| $(2,1)$ | 2 | $\sqrt{2\cdot 3\cdot 5\cdot 11}$ | $-3^3\cdot 7\cdot 11$ | $99\pm 6\alpha$ | $109+6\alpha$ | 1 |
| $(2,9)$ | 2 | $\sqrt{2\cdot 5\cdot 11\cdot 23}$ | $-3^2\cdot 11\cdot 29$ | $18007\pm 358\alpha$<br>$297\pm 6\alpha$<br>$803\pm 16\alpha$ | $1259389+25038\alpha$ | 1 |
| $(6,1)$ | 4 | $\sqrt{2\cdot 5\cdot 11}$ | $-3^2\cdot 11\cdot 61$ | $99\pm 12\alpha$ | $21+2\alpha$ | 1 |
| $(6,9)$ | 8 | $\sqrt{3\cdot 5\cdot 11}$ | $-3^3\cdot 11\cdot 23$ | $15\theta+42$<br>$-15\theta+57$ | $6+\theta$ | 1 |

For $(a,b)=(6,9)$, we must impose the constraint that $v$ and $w$ are integral, i.e., that when we look in $R/2R$, the coefficient of $\theta$ in $cf^N$ vanishes. However, $c$ equals $\theta$ or $1+\theta$ in $R/2R$, depending on which value of $c$ we pick, and $f^N$ will equal 1, $\theta$, or $1+\theta$, according to whether $N$ is congruent to 0, 1, or 2 modulo 3; so $N\equiv -1$ modulo 3 if $c=15\theta+42$ and $N\equiv 1$ modulo 3 if $c=-15\theta+57$, meaning that we can replace $f$ by $f^3=1079+84\alpha$ and $c$ by $-(-15\theta+57)f=297+24\alpha$ or $-(15\theta+42)f^{-1}=297-24\alpha$.

For $(a,b)=(6,1)$, recall that $v=99y$, so $9|v$. We can write $(v+w\alpha)/3=\pm(33\pm 4\alpha)f^N$. In $R/3R$, we have $33\pm 4\alpha=\pm\alpha$, and $f=-\alpha$ has $f^2=-1$; therefore, $N$ must be even and we can replace $f$ by $f^2=881+84\alpha$. For $(a,b)=(2,9)$, similarly, $3|v$, so look at $cf^N$ in $R/3R$; we have $f=1$ in $R/3R$; but our three pairs of values for $c$ equal $1\pm\alpha$, 0, and $-1\pm\alpha$, respectively, in $R/3R$, so we must take the second pair of values for $c$, $c=297\pm 6\alpha$. Observe that, in all cases, we have forced $c$ to be the value of $c$ coming from a 1-digit undulating square, i.e., $c=99\sqrt{b}\pm 2^L\cdot 3\alpha$. The new values for $c$ and $f$ are shown below.

| $(a,b)$ | $c$ | $f$ | $\mathcal{N}(f)$ |
|---|---|---|---|
| $(2,1)$ | $99\pm 6\alpha$ | $109+6\alpha$ | 1 |
| $(2,9)$ | $297\pm 6\alpha$ | $1259389+25038\alpha$ | 1 |
| $(6,1)$ | $99\pm 12\alpha$ | $881+84\alpha$ | 1 |
| $(6,9)$ | $297\pm 24\alpha$ | $1079+84\alpha$ | 1 |

Let the conjugate of $\beta$ in $K$ be $\bar{\beta}$. Shifting to our revised values of $f$ and $c$, we have the equation

$$\pm(cf^N - \bar{c}\bar{f}^N) = 2w\alpha = 2^{L+1} \cdot 3 \cdot 10^m \alpha. \tag{1}$$

Observe that, for each $(a, b)$, our two values of $c$ are conjugates of each other. Since $\mathcal{N}(f) = 1$ implies that $\bar{f} = f^{-1}$, $cf^N - \bar{c}\bar{f}^N = -(\bar{c}f^{-N} - c\bar{f}^{-N})$, so we may negate the left-hand side of (1) if necessary to make $c = 99\sqrt{b} + 2^L \cdot 3\alpha$. We will then have $c > 0$, so since $\mathcal{N}(c) = t < 0$, $\bar{c} < 0$; similarly, $f > 0$, and since $\mathcal{N}(f) = 1$, $\bar{f} > 0$. The parenthesis in (1) is then positive, and so is the right-hand side, so we may take the positive sign in (1):

$$cf^N - \bar{c}\bar{f}^N = 2^{L+1} \cdot 3 \cdot 10^m \alpha = (c - \bar{c})10^m.$$

If $N = 0$, then $m = 0$, so assume that $N > 0$. We can then rearrange terms to get

$$1 - \frac{10^m(c - \bar{c})}{cf^N} = \frac{\bar{c}}{c}(\frac{\bar{f}}{f})^N,$$

so

$$|1 - \frac{10^m(c - \bar{c})}{cf^N}| \leq |\frac{\bar{c}}{c}|f^{-2N}.$$

For $|\epsilon| < \frac{1}{2}$, we have $|\log(1+\epsilon)| \leq 2|\epsilon|$; but $|\bar{c}/c| < 1$ and $f > 100$, so $|\bar{c}/c|f^{-2N} < \frac{1}{2}$ and

$$|\log \frac{10^m(c - \bar{c})}{cf^N}| \leq 2|\frac{\bar{c}}{c}|f^{-2N},$$

i.e.,

$$|\log(1 - \frac{\bar{c}}{c}) + m \log 10 - N \log f| \leq 2|\frac{\bar{c}}{c}|f^{-2N}. \tag{2}$$

If $N < 0$, negate $N$; since $\bar{f} = f^{-1}$, we get

$$c\bar{f}^N - \bar{c}f^N = (c - \bar{c})10^m;$$

we can then proceed as above to get

$$|1 - \frac{10^m(\bar{c} - c)}{\bar{c}f^N}| \leq |\frac{c}{\bar{c}}|f^{-2N}.$$

We have

$$\begin{aligned} -\frac{c}{\bar{c}} \leq 1 - \frac{c}{\bar{c}} &\leq (1 - \frac{c}{\bar{c}})(1 - \frac{\bar{c}}{c}) \\ &= \mathcal{N}(\frac{c - \bar{c}}{c}) \\ &= \mathcal{N}(\frac{2^{L+1} \cdot 3\alpha}{c}) \\ &= \frac{-4^{L+1} \cdot 9x}{t} \\ &= \frac{4(10a + 100b)}{10a + b} \\ &\leq 400, \end{aligned}$$

but $f > 100$, so $|c/\bar{c}|f^{-2N} < \frac{1}{2}$ and, in the same way as before, we get

$$|\log(1 - \frac{c}{\bar{c}}) + m\log 10 - N\log f| \le 2|\frac{c}{\bar{c}}|f^{-2N}. \qquad (3)$$

For an algebraic number $\delta$, let the *minimal polynomial* of $\delta$ be the unique polynomial in $\mathbf{Z}[X]$ with $\delta$ as a root that has minimal degree, unit content, and positive leading coefficient, and let the *absolute logarithmic Weil height* [1] of $\delta$ be

$$h_0(\delta) = \frac{1}{d}(\log A + \sum_{1\le i\le d} \log\max(1, |\delta^{(i)}|)),$$

where $d$ is the degree of the minimal polynomial of $\delta$, $A$ is its leading coefficient, and $\delta^{(1)}, \ldots, \delta^{(d)}$ are its roots. In [1] we find the following theorem:

Let $\log\gamma_1, \ldots, \log\gamma_n$ be logarithms of algebraic numbers, and let $d = [\mathbf{Q}(\gamma_1, \ldots, \gamma_n) : \mathbf{Q}]$. Define

$$h'(\gamma_i) = \max(h_0(\gamma_i), \frac{1}{d}|\log\gamma_i|, \frac{1}{d}).$$

If $k_1, \ldots, k_n \in \mathbf{Z}$ have $\sum_i k_i \log\gamma_i \neq 0$, and if $B \ge \max(e, k_1, \ldots, k_n)$, then

$$\log|\sum_i k_i \log\gamma_i| \ge$$

$$-18(n+1)!\, n^{n+1}(32d)^{n+2}\log B\log(2nd)\prod_i h'(\gamma_i). \qquad (4)$$

We wish to apply this theorem with $n = 3$, $\gamma_1 = 10$, $\gamma_2 = f$, and $\gamma_3$ equalling $1 - \bar{c}/c = 2^{L+1}\cdot 3\alpha/c$, or its conjugate, $1 - c/\bar{c} = -2^{L+1}\cdot 3\alpha/\bar{c}$. In this case, we clearly cannot have $\sum_i k_i \log\gamma_i = 0$ for $k_i \in \mathbf{Z}$ not all zero, since then $\gamma_1^{k_1}\gamma_2^{k_2}\gamma_3^{k_3} = 1$, and the sets of primes dividing $\gamma_1$ and $\gamma_3$ would have to be the same, which they are not. Evidently, $\mathbf{Q}(\gamma_1, \gamma_2, \gamma_3) = K$, so $d = 2$. It is obvious that $h_0(10) = \log 10$, and since $10 > \sqrt{e}$, $h'(10) = \log 10$ as well. Since $f$ is an algebraic integer, it has monic minimal polynomial, and since $f > 1$ and $f\bar{f} = 1$, $0 < \bar{f} < 1$. This implies that $h_0(f) = \frac{1}{2}\log f$, and since $f > e$, $h'(f) = \frac{1}{2}\log f$ as well.

We have already seen that $\mathcal{N}(1 - \bar{c}/c) = 4(10a + 100b)/(10a + b)$, and if $\mathcal{T}$ is the trace from $K$ to $\mathbf{Q}$,

$$\mathcal{T}(1 - \frac{\bar{c}}{c}) = 2 - \frac{\bar{c}}{c} - \frac{c}{\bar{c}} = (1 - \frac{\bar{c}}{c})(1 - \frac{c}{\bar{c}}) = \mathcal{N}(1 - \frac{\bar{c}}{c}).$$

Write, as before, $10a + b = qr$, where $q \in \{1, 3\}$ and $r \in \{7, 23, 29, 61\}$. Then $r$ will be minimal with $r\mathcal{T}(1 - \bar{c}/c)$ and $r\mathcal{N}(1 - \bar{c}/c)$ integral, so $r$ will be the leading coefficient of the minimal polynomial of $1 - \bar{c}/c$, and

$$h_0(1 - \frac{\bar{c}}{c}) \quad = \quad \frac{1}{2}(\log r + \log(1 - \frac{\bar{c}}{c}) + \log(1 - \frac{c}{\bar{c}}))$$

$$\begin{aligned} &= \frac{1}{2}\log r\mathcal{N}(1-\frac{\bar{c}}{c}) \\ &= \frac{1}{2}\log\frac{4(10a+100b)}{q}, \end{aligned}$$

and this is clearly also equal to the height of the conjugate, $h_0(1-c/\bar{c})$. This height is a sum of nonnegative terms, one of which is $\frac{1}{2}\log\gamma_3$, so it is at least $\frac{1}{2}\log\gamma_3$, and since $4(10a+100b)\geq qe$, it is at least $\frac{1}{2}$. Therefore,

$$h'(\gamma_3) = \frac{1}{2}\log\frac{4(10a+100b)}{q}.$$

It follows from (4), then, that if $B \geq \max(e, k_1, k_2, k_3)$ and the $k_i$'s are not all zero, then

$$\log|\sum_i k_i \log\gamma_i| \geq$$

$$-18\cdot 4!\cdot 3^4\cdot 64^5\cdot\log B\cdot\log 12\cdot\log 10\cdot\frac{1}{2}\log f\cdot\frac{1}{2}\log\frac{4(10a+100b)}{q},$$

and using $f \leq f+\bar{f} = \mathcal{T}(f) \leq 3\cdot 10^6$ and $4(10a+100b)/q \leq 4000$, we get

$$|\sum_i k_i\log\gamma_i| \geq \exp(-10^{17}\log B). \tag{5}$$

Since $\log\gamma_3 \geq 0$, and since $|\log\gamma_3 + m\log 10 - N\log f| \leq 2\cdot\frac{1}{2} = 1$, we have $m\log 10 - N\log f \leq 1$, so $m \leq (1+N\log f)/\log 10$, and we can set $B = 3 + N\log f/\log 10$. Recalling that $|\bar{c}/c| \leq 1$ and $|c/\bar{c}| \leq 400$, we see that for (2) or (3) to be compatible with (5), we need

$$\begin{aligned} 10^{17}\log B &\geq 2N\log f - \log 800 \\ &= 2(B-3)\log 10 - \log 800, \end{aligned}$$

and therefore $B < 10^{18}$.

We need to verify that (2) and (3) are not satisfied with $B < 10^{18}$. Let $\gamma$ be the 3-tuple $(\gamma_1,\gamma_2,\gamma_3)$. Let $j_1$, $j_2$, $j_3 \in \mathbf{Z}^3$ satisfy $\mathbf{Z}j_1+\mathbf{Z}j_2+\mathbf{Z}j_3 = \mathbf{Z}^3$, and point in the same approximate direction as $\gamma$.

If $0 \neq l \in \mathbf{Z}^3$ has $|l\cdot\gamma| \leq \epsilon$, then there must be some $i$ with $l\cdot j_i \neq 0$, so $|l\cdot j_i| \geq 1$ and

$$\begin{aligned} l\cdot j_i &= l\cdot(j_i - \frac{j_i\cdot\gamma}{||\gamma||^2}\gamma + \frac{j_i\cdot\gamma}{||\gamma||^2}\gamma) \\ &= l\cdot(j_i - \frac{j_i\cdot\gamma}{||\gamma||^2}\gamma) + \frac{(j_i\cdot\gamma)(l\cdot\gamma)}{||\gamma||^2}, \end{aligned}$$

so

$$\begin{aligned} |l \cdot j_i| &\leq \frac{||l||}{||\gamma||^2}\, ||\, ||\gamma||^2 j_i - (j_i \cdot \gamma)\gamma|| + \frac{||j_i||\, ||\gamma||\, |l \cdot \gamma|}{||\gamma||^2} \\ &\leq \frac{||l||}{||\gamma||^2}\, ||\, ||\gamma||^2 j_i - (j_i \cdot \gamma)\gamma|| + \frac{\epsilon ||j_i||}{||\gamma||}, \end{aligned}$$

so if

$$\epsilon ||j_i|| < \frac{1}{2}||\gamma||, \qquad i = 1,\, 2,\, 3, \tag{6}$$

and

$$||l||\, ||\, ||\gamma||^2 j_i - (j_i \cdot \gamma)\gamma|| < \frac{1}{2}||\gamma||^2, \qquad i = 1,\, 2,\, 3, \tag{7}$$

we will have a contradiction.

3-tuples of vectors $(j_1, j_2, j_3)$ satisfying (6) and (7) are listed in the Appendix, for all of our four values of $(a, b)$ and two values of $\gamma_3$, and for $\epsilon = 10^{-58}$ and $||l|| \leq \sqrt{3} \cdot 10^{18}$. It follows that any solution to (2) must have $2|\bar{c}/c| f^{-2N} > 10^{-58}$, and any solution to (3) must have $2|c/\bar{c}| f^{-2N} > 10^{-58}$. Keeping in mind that $f > 100$, this implies that $N \leq 30$, but since $f \leq 3 \cdot 10^6 \leq 10^{13/2}$, we have $m \leq 1 + N \log f / \log 10 \leq 1 + 15 \cdot 13 = 196$. Therefore, any undulating square other than those already seen must have no more than $1 + 2 \cdot 196 = 393$ digits. A computer search for such undulating numbers which are squares finds no squares other than those already seen, so there are no undulating squares other than those we have mentioned.

## References


[1] A. Baker and G. Wüstholz, Logarithmic forms and group varieties, *J. reine angew. Math.* **442** (1993), pp. 19–62.

[2] J. Hastad, B. Just, J. C. Lagarias, and C. P. Schnorr, Polynomial time algorithms for finding integer relations among real numbers, *SIAM J. Comput.* **18** (1989), pp. 859–881.


## Appendix

These $j_i$'s were found by the algorithm of Hastad, Just, Lagarias, and Schnorr [2]. For each choice of $(a, b)$ and $\gamma_3$, we list $j_1$, $j_2$, and $j_3$ in order.

$(a, b) = (2, 1)$:
$\gamma_3 = 1 - \bar{c}/c$ :
(394811817334324410568258359276989856480158 8, 92324664992545180208612846693964653848557 28, 804802534371895870648105852276543147409 03)

(415866494788507540768102237705345174929262 1, 972481904726285100450108067274224648567278 3,
847721355520454831093680063889391003249 92)

(147083238303038086664040715205344843400399 4, 343946409558650893505025229719188120838599 1,
299821225588273125298907964074072438418 29)

$\gamma_3 = 1 - c/\bar{c}:$

(146564309781183936019859373837731496833672 87, 342732922529343174357196634411393859720104 62,
196196478822108043510451715094804364654041 84)

(787475664479412501746148442584090358104809 9, 184147038464349762254568134980583376085682 25,
105414444184688790757111439793005299469105 79)

(164647407714390634723340734820737212495007, 385019295059498672212302290773264214757090,
220403191534004352998224336986633657016439)

$(a, b) = (2, 9)$:

$\gamma_3 = 1 - \bar{c}/c:$

(2220632315394692021163872091924467029073734639947943283 83,
1421468912970367647036122762476742177348738777064373804880,
769140016609597630987243664588874711252968957440120464)

(2643907667349593089827812283076262192368458154412973826 11,
1692415503389386450275636134778713053914201095022061707427,
915746012107398749115991993481158331559709836545941558)

(1907889394067827761052262807926270125828196096227327623 16,
1221276230311572557698305351313201322409113501512643011490,
660818123770203966944359980963683613686222412714915369)

$\gamma_3 = 1 - c/\bar{c}:$

(760972676921373190485867617213543510597992 1, 487113060710059731848004800597008903249738 00,
159803212972562521383842603068009906572052 36)

(163442459433604573311980083880717317819077 5, 104622621914333255907588903350760672758811 94,
343226911369465709464675935497798460767283 9)

(283744727442523819015960804958403106146 52, 181630388102818144431502035750421101979435,
595860015536725984315449874114839332451 41)

$(a, b) = (6, 1)$:

$\gamma_3 = 1 - \bar{c}/c:$

(358842098738994946724802666613558555886785, 116480352094100303676213781064901871111354 5,
175836406542611462161015812195979137901 82)

(826673259383218232843925100119133694493 6, 268338616506009520082504024414151854990 63,
405078600937903940303079368746695358932)

(101587244231780424318763326496512506500910, 329752780344589895606077149330164237424887,
497788192607674700286879380121720889302 5)

$\gamma_3 = 1 - c/\bar{c}:$

(1373888988363038021143329105883018025547243, 4459651575584310223922374777416636137304179,
1335212757740216676754635643863749333092071)

(241202372140900123689637608530813823598180, 782944290305789847984692019915460623302175,
234412304929714369056012762447403094035602)

(3795167286035620301946862038227844615157, 12319134886538933581581919590240254453370,
3688329858521342429307975174564386798308)

$(a,b) = (6,9):$

$\gamma_3 = 1 - \bar{c}/c:$

(4116736281246703786084754764070761112107677, 13725410144104289757206788143567657980812602,
33028083200032285507591556974390092072476)

(1965748780966300628904098728373993764416980, 6553907371220952512377224157729949875519849,
15770967546275588441309217904538260226626)

(400038325001276561820747842753419733855606, 1333748316358078919416218870860783428163691,
3209459673560258347199843102236447039041)

$\gamma_3 = 1 - c/\bar{c}:$

(195241826410046637693552320988234780575093, 650946274350701367349745450924255654763638,
339224717895823151503590522039512677805407)

(366296257469152206257891128655079841493330, 1221250530649002550702889547560020228766996,
636424821929831278932518317441568656538223)

(38477574409878730099994608407626305606502, 128286208794006656800340159128213307583223,
66853217696774250907713557322046036908953)