# The pebbling threshold spectrum and paths

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May 31, 2019

**Abstract.** Given a distribution of pebbles on the vertices of a graph, say that we can *pebble* a vertex if a pebble is left on it after some sequence of moves, each of which takes two pebbles from some vertex and places one on an adjacent vertex. A distribution is *solvable* if all vertices are pebblable; the *pebbling threshold* of a sequence of graphs is, roughly speaking, the total number of pebbles for which random distributions with that number of pebbles on a graph in the sequence change from being almost never solvable to being almost always solvable. We show that any sequence of connected graphs with strictly increasing orders always has some pebbling threshold which is $\Omega(\sqrt{n})$ and $O(2^{\sqrt{2\log_2 n}} n/\sqrt{\log_2 n})$, and that it is possible to construct such a sequence of connected graphs which has any desired pebbling threshold between these bounds. (Here, $n$ is the order of a graph in the sequence.) It follows that the sequence of paths, which, improving earlier estimates, we show has pebbling threshold $\Theta(2^{\sqrt{\log_2 n}} n/\sqrt{\log_2 n})$, does not have the greatest possible pebbling threshold.

## Introduction

In the mathematical game of *pebbling*, one starts with a *distribution* on a graph assigning a nonnegative integral number of pebbles to each vertex of the graph. A *pebbling move* consists of taking two pebbles away from a vertex with at least two pebbles and adding one pebble to any adjacent vertex. A vertex is *pebblable* for a given distribution if there is some sequence of pebbling moves starting at the distribution and finishing with at least one pebble on that vertex, and a distribution is *solvable* if each vertex is pebblable for that distribution. In [5], Czygrinow et al. introduce the *pebbling threshold* for a sequence of graphs, which, roughly speaking, is the number of pebbles at which a random distribution with that number of pebbles on a graph in the sequence changes from being almost always unsolvable to being almost always solvable.

In [8, RP15], Hurlbert asks for the pebbling threshold of the sequence of paths. In this paper, we determine that it is $\Theta(2^{\sqrt{\log_2 n}} n/\sqrt{\log_2 n})$, where $n$ is the number of vertices of a path in the sequence. This makes more precise the estimates of [2], [5], [6], and [10]. We also prove some subsidiary results that may be of interest. To do this, we first (§1) define uniform and geometric probability distributions over multisets and the corresponding thresholds of sequences of families of multisets. In §2, we improve some estimates used in [2], and in §3, we use this to relate the uniform and geometric thresholds. In §4, we begin to compute the pebbling threshold of the sequence of paths, relating it to a certain hypoexponential distribution. In §5, we estimate asymptotically some probabilities of this distribution and finally complete the computation of the pebbling threshold of the sequence of paths in §6.

The pebbling threshold of the sequence of paths is not the largest possible pebbling threshold. The reason is that most vertices in the path can be moved onto from both directions; the ends are harder to reach since they can only be reached from one direction, but there are only two ends. A graph which contains a bouquet of paths joined at a point will then be harder to pebble since it has more ends (for an appropriate choice of path lengths and number of paths.) In §7 and §8, we analyze a construction of this type; in §9, we show that any sequence of connected graphs with strictly increasing orders has some pebbling threshold which is $\Omega(\sqrt{n})$ and $O(2^{\sqrt{2\log_2 n}} n/\sqrt{\log_2 n})$, and also conversely show that any positive function which is $\Omega(\sqrt{n})$ and $O(2^{\sqrt{2\log_2 n}} n/\sqrt{\log_2 n})$ is the pebbling threshold of a sequence of connected graphs with orders 1, 2, 3, .... Here, $n$ is the order of a graph in the sequence. This resolves the problem [8, RP17].

# 1 Definitions and notation

We use $\mathbb{Z}$, $\mathbb{Z}_{>0}$, $\omega$, $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{R}_{>0}$, $\mathbb{R}_{<0}$, $\mathbb{C}$, $\mathbb{P}$, $\mathbb{E}$, and Var to denote the integers, the positive integers, the nonnegative integers, the reals, the nonnegative reals, the positive reals, the negative reals, the complex numbers, probability, expectation, and variance. $\iota$ is the imaginary unit. For $x \in \mathbb{R}$, $\lfloor x \rfloor$ will be the largest integer no larger than $x$, $\lceil x \rceil$ the smallest integer no smaller than $x$, and $\{x\}$ the fractional part of $x$, $\{x\} := x - \lfloor x \rfloor$; for $x \in \mathbb{R}_{>0}$, $\log x$ will be the natural logarithm of $x$, and $\log_2 x$ will be the logarithm of $x$ to the base 2, $\log_2 x := \log x/(\log 2)$. The cardinality of a set $S$ is written $\#S$. For nonnegative integers $k \leq n$, $\binom{n}{k}$ denotes the binomial coefficient $n!/(k!\,(n-k)!)$. $A^B$ will be the set of functions from $B$ to $A$. If $f$, $g \in A^B$ and $A$ is ordered, we define $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in B$; similarly, if $A$ has an addition operation, we define $f + g \in A^B$ by $(f + g)(x) = f(x) + g(x)$ for all $x \in B$. We call the elements of $\omega^B$ *multisets* and write 0 for the empty multiset, i.e., the element of $\omega^B$ whose value is always 0.

For any $b \in B$, we take $e_b \in \omega^B$ to have $e_b(c) = 1$ if $b = c$, $e_b(c) = 0$ if $b \neq c$. For $S$ a subset of some $\omega^B$, we let $\partial S$ be $\{f \in \omega^B \mid f + e_b \in S \text{ for some } b \in B\}$, and if $B$ is finite and $T \in \omega$, we take $[S]_T$ to be $\{f \in S \mid \sum_{x\in B} f(x) = T\}$.

The geometric distribution on $\omega$ with parameter $0 < p \leq 1$ is the probability measure $\chi$ with $\chi(\{n\}) = p(1-p)^n$, where we take $0^0 = 1$. For $B$ finite and nonempty, if $T \in \omega$, we let $\mu_T$ be the probability measure on $\omega^B$ that is uniform on $[\omega^B]_T$ and zero elsewhere, and if $T \in \mathbb{R}_{\geq 0}$, we let $\nu_T$ be the probability measure on $\omega^B$ which is the product of $\#B$ copies of the geometric distribution with parameter $(1 + (T/\#B))^{-1}$.

If $(M_i)_{i\in\omega}$ is a sequence such that $\emptyset \neq M_i \subseteq \omega^{B_i}$ for each $i$, where each $B_i$ is a nonempty finite set, and each $M_i$ is an *upper set* ($x \in M_i$ and $x \leq y$ implies that $y \in M_i$), then we define the *uniform threshold* of $(M_i)_{i\in\omega}$ to be the sequence $(T_i)_{i\in\omega}$, where each $T_i \in \omega$ is minimal such that $\mu_{T_i}(M_i) \geq \frac{1}{2}$. If in addition $0 \notin M_i$ for each $i$, we define the *geometric threshold* of $(M_i)_{i\in\omega}$ to be the sequence $(T_i)_{i\in\omega}$, where each $T_i \in \mathbb{R}_{>0}$ is the unique $T_i$ satisfying $\nu_{T_i}(M_i) = \frac{1}{2}$. (We will see in §2 and §3 that these definitions are sensible.)

All graphs considered in this paper will be undirected and finite; also, we will take the graph with no vertices to be disconnected. For a given graph, $d(x, y)$ will mean the distance between vertices $x$ and $y$ in the graph, and, in a given distribution on the graph, $\mathcal{Z}(x)$ will mean the number of pebbles on vertex $x$.

# 2 Improved thresholds for multisets

The main result of this section is the following, which can be used to improve estimates like those used in the proof of [2, Theorem 1.5].

**Theorem 1.** *If $B$ is nonempty and finite, $T \in \omega$, $x \in \mathbb{R}_{\geq 0}$, $S \subseteq \omega^B$, and $\mu_{T+1}(S) \geq x/(T+1+x)$, then $\mu_T(\partial S) \geq x/(T+x)$. (Here we take $0/0 = 0$ in the case where $T = x = 0$.)*

**Lemma 2.** *Given $x \in \mathbb{R}_{\geq 0}$, $r \in \omega$ and positive integers $t$, $n$ and $d_0 \geq d_1 \geq \cdots \geq d_{r-1}$ with $t \geq r$ and (if $d_0$ exists) $n > d_0$, let*

$$p := \sum_{0 \leq i < r} \binom{t-i-1+d_i}{t-i} \Big/ \binom{t-1+n}{t},$$

$$q := \sum_{0 \leq i < r} \binom{t-i-2+d_i}{t-i-1} \Big/ \binom{t-2+n}{t-1}.$$

*Then $0 \leq p < 1$ and, if $p \geq x/(t+x)$, also $q \geq x/(t-1+x)$ (where we take $0/0 = 0$ in the case $t = 1$ and $x = 0$.)*

*Proof.* We first prove that $0 \leq p < 1$. This is clear if $r = 0$; otherwise

$$\begin{aligned} \sum_{0 \leq i < r} \binom{t-i-1+d_i}{t-i} &< \sum_{0 \leq j \leq t} \binom{j+d_0-1}{j} \\ &= \binom{t+d_0}{t} \\ &\leq \binom{t-1+n}{t}, \end{aligned}$$

so $p < 1$.

We can give an equivalent condition for $p \geq x/(t+x)$ implying that $q \geq x/(t-1+x)$ by observing that it will do to prove this for the maximal $x$ for which $p \geq x/(t+x)$, which is $x := pt/(1-p)$. In this case, $q \geq x/(t-1+x)$ reduces to $(t-1+p)q \geq pt$.

We now induce on $t$ to prove this. If $t = 1$, we have two cases. If $r = 0$, we must have $p = 0$, making the result trivial; if $r = 1$, $q = 1$, making the result again trivial. Otherwise, let $t > 1$. If $r = 0$, we again have $p = 0$, making the result trivial. If $r > 0$, set

$$\begin{aligned}
\alpha &:= \sum_{1 \leq i < r} \binom{t-i-1+d_i}{t-i}, \\
\beta &:= \sum_{1 \leq i < r} \binom{t-i-2+d_i}{t-i-1}, \\
p' &:= \alpha \Big/ \binom{t-1+d_0}{t-1}, \qquad q' := \beta \Big/ \binom{t-2+d_0}{t-2}.
\end{aligned}$$

By the induction hypothesis, we can assume that $(t-2+p')q' \geq p'(t-1)$; since $0 \leq p' < 1$, this implies that $p' \leq q'$. We need to show that $(t-1+p)q \geq pt$, which, after clearing denominators, is equivalent to

$$\begin{aligned}
&\left((t-1)\binom{t-1+n}{t} + p'\binom{t-1+d_0}{t-1} + \binom{t-1+d_0}{t}\right) \cdot \\
&\left(q'\binom{t-2+d_0}{t-2} + \binom{t-2+d_0}{t-1}\right) \\
\geq\; &\binom{t-2+n}{t-1}\left(p'\binom{t-1+d_0}{t-1} + \binom{t-1+d_0}{t}\right)t. \qquad (1)
\end{aligned}$$

Taking a forward first difference of (1) with respect to $n$ gives

$$\begin{aligned}
&(t-1)\binom{t-1+n}{t-1}\left(q'\binom{t-2+d_0}{t-2} + \binom{t-2+d_0}{t-1}\right) \\
\geq\; &\binom{t-2+n}{t-2}\left(p'\binom{t-1+d_0}{t-1} + \binom{t-1+d_0}{t}\right)t,
\end{aligned}$$

which, using $\binom{t-1+n}{t-1} = \frac{t+n-1}{t-1}\binom{t-2+n}{t-2}$ and removing the common factor $\binom{t-2+n}{t-2}$, can be rewritten as

$$(t+n-1)\left(q'\binom{t-2+d_0}{t-2} + \binom{t-2+d_0}{t-1}\right) \geq \left(p'\binom{t-1+d_0}{t-1} + \binom{t-1+d_0}{t}\right)t.$$

Since we assume $n > d_0$, it's enough to prove this when $n = d_0 + 1$. Using $\binom{t-1+d_0}{t} = \frac{t-1+d_0}{t}\binom{t-2+d_0}{t-1}$, this simplifies to

$$q'(t+d_0)\binom{t-2+d_0}{t-2} + \binom{t-2+d_0}{t-1} \geq p'\binom{t-1+d_0}{t-1}t,$$

and since $p' \le q'$ and $p' \le 1$, it's enough to show that

$$(t+d_0)\binom{t-2+d_0}{t-2} + \binom{t-2+d_0}{t-1} = \binom{t-1+d_0}{t-1}t,$$

which is easy as both sides are multiples of $\binom{t-2+d_0}{t-2}$ by rational functions of $t$ and $d_0$.

It remains to prove (1) when $n = d_0 + 1$. In this case, looking at a portion of the left-hand side of (1),

$$\begin{aligned}
& q'\binom{t-2+d_0}{t-2}\left((t-1)\binom{t+d_0}{t} + p'\binom{t-1+d_0}{t-1}\right) \\
= \; & q'\binom{t-2+d_0}{t-2}\left((t-2+p')\binom{t-1+d_0}{t-1} + \binom{t+d_0}{t} + \right. \\
& \left. (t-2)\left(\binom{t+d_0}{t} - \binom{t-1+d_0}{t-1}\right)\right) \\
\ge \; & p'(t-1)\binom{t-2+d_0}{t-2}\binom{t-1+d_0}{t-1} + \\
& q'\binom{t-2+d_0}{t-2}\left(\binom{t+d_0}{t} + (t-2)\binom{t-1+d_0}{t}\right).
\end{aligned}$$

It's therefore enough to prove (1) with the left-hand side replaced by

$$\begin{aligned}
& p'(t-1)\binom{t-2+d_0}{t-2}\binom{t-1+d_0}{t-1} + \\
& q'\binom{t-2+d_0}{t-2}\left(\binom{t+d_0}{t} + (t-2)\binom{t-1+d_0}{t}\right) + \\
& q'\binom{t-2+d_0}{t-2}\binom{t-1+d_0}{t} + \\
& \left((t-1)\binom{t+d_0}{t} + p'\binom{t-1+d_0}{t-1} + \binom{t-1+d_0}{t}\right)\binom{t-2+d_0}{t-1}.
\end{aligned}$$

Using $p' \le q'$ and separating terms which involve and do not involve $p'$, it will do to show that

$$\begin{aligned}
& (t-1)\binom{t-2+d_0}{t-2}\binom{t-1+d_0}{t-1} + \\
& \binom{t-2+d_0}{t-2}\left(\binom{t+d_0}{t} + (t-2)\binom{t-1+d_0}{t}\right) + \\
& \binom{t-2+d_0}{t-2}\binom{t-1+d_0}{t} + \binom{t-1+d_0}{t-1}\binom{t-2+d_0}{t-1} \\
= \; & \binom{t-1+d_0}{t-1}^2 t
\end{aligned}$$

and

$$\left((t-1)\binom{t+d_0}{t}+\binom{t-1+d_0}{t}\right)\binom{t-2+d_0}{t-1}=\binom{t-1+d_0}{t-1}\binom{t-1+d_0}{t}t.$$

This can be done by expressing both sides of both of these equations as rational multiples of $\binom{t-2+d_0}{t-2}\binom{t-1+d_0}{t-1}$. $\square$

We now prove Theorem 1.

*Proof.* Set $n := \#B$, and intersect $S$ with $[\omega^B]_{T+1}$ if necessary so we can assume that $S \subseteq [\omega^B]_{T+1}$. If $S = [\omega^B]_{T+1}$, then $\partial S = [\omega^B]_T$, so $\mu_T(\partial S) = 1$ and the result is obvious; if $S$ is empty, then we must have $x = 0$ so the result is again obvious. Otherwise, set $t := T + 1$. Since we have $\emptyset \subsetneq S \subsetneq [\omega^B]_t$, we must have $0 < \#S < \#[\omega^B]_t = \binom{t+n-1}{t}$. By the theorem in [4], there is then a representation

$$\#S = \sum_{0\le i<r} \binom{t-i-1+d_i}{t-i}$$

with $t \ge r > 0$, $d_0 \ge d_1 \ge \cdots \ge d_{r-1} > 0$, and

$$\#(\partial S) \ge \sum_{0\le i<r} \binom{t-i-2+d_i}{t-i-1}.$$

Since $\#S < \binom{t+n-1}{t}$ we must then have $n > d_0$, so we can apply Lemma 2. $\square$

**Proposition 3.** *If $B$ is finite and nonempty and $M \subseteq \omega^B$ is an upper set, $T \ge U \in \omega$, $x \in \mathbb{R}_{\ge 0}$, and $\mu_T(M) \le T/(T+x)$, then $\mu_U(M) \le U/(U+x)$. (Here we take $0/0 = 1$ if $U = x = 0$.)*

*Proof.* If $x = 0$, the result is obvious. Otherwise, since $\mu_T(M) \le T/(T+x)$, $\mu_T(\omega^B \setminus M) \ge x/(T+x)$, so by repeated application of Theorem 1, $\mu_U(\partial^{T-U}(\omega^B \setminus M)) \ge x/(U+x)$. However, if $v \in \partial^{T-U}(\omega^B \setminus M)$, we have $v \le w$ for some $w \in \omega^B \setminus M$, so we cannot have $v \in M$ since then, as $M$ is an upper set, $w$ would also be in $M$. Therefore, $\partial^{T-U}(\omega^B \setminus M)$ is disjoint from $M$ so $\mu_U(M) \le \mu_U(\omega^B \setminus \partial^{T-U}(\omega^B \setminus M)) \le 1 - (x/(U+x)) = U/(U+x)$. $\square$

**Proposition 4.** *If $B$ is finite and nonempty and $M \subseteq \omega^B$ is an upper set, $T$ and $U$ are positive integers with $T \ge U$, $x \in \mathbb{R}_{\ge 0}$, and $\mu_U(M) \ge U/(U+x)$, then $\mu_T(M) \ge T/(T+x)$.*

*Proof.* Replace $x$ by $x+\epsilon$, apply the contrapositive of Proposition 3, and let $\epsilon \to 0$ from above. $\square$

**Proposition 5.** *If $B$ is finite and nonempty, $M \subseteq \omega^B$ is an upper set, $U \in \omega$, and $\mu_U(M) > 0$, then $\mu_U(M) \le \mu_{U+1}(M)$ and $\lim_{i\to\infty} \mu_i(M) = 1$. Also, if $0 < \mu_U(M) < 1$, then $\mu_U(M) < \mu_{U+1}(M)$.*

*Proof.* If $U = 0$, then we must have $0 \in M$ so $M = \omega^B$ and $\mu_i(M) = 1$ for all $i$. Otherwise, we can apply Proposition 4 with some value of $x$ to conclude that $\lim_{i\to\infty} \mu_i(M) = 1$. Applying it with the minimum possible value of $x$ allows us to conclude that $\mu_U(M) \leq \mu_{U+1}(M)$, or $\mu_U(M) < \mu_{U+1}(M)$ in the case where $x > 0$, i.e., $\mu_U(M) < 1$. □

**Theorem 6.** *If $B$ is finite and nonempty and $M \subseteq \omega^B$ is an upper set, then the sequence $(\mu_0(M), \mu_1(M), \ldots)$ is either:*

1. $(0, 0, \ldots)$*, if $M$ is empty.*

2. $(0, 0, \ldots, 0, r_0, \ldots, r_{N-1}, 1, 1, \ldots)$*, for some $N \in \omega$ and $0 < r_0 < \cdots < r_{N-1} < 1$.*

3. $(0, 0, \ldots, 0, r_0, r_1, \ldots)$*, for some strictly increasing sequence $(r_i)_{i\in\omega}$ of positive real numbers with $\lim_{i\to\infty} r_i = 1$.*

*(The initial sequence of zeroes may be empty in cases 2 and 3.)*

*Proof.* Apply Proposition 5 repeatedly. □

Theorem 6 shows that the definition of uniform threshold makes sense.

# 3 Uniform and geometric thresholds

In this section, we show that the definition of geometric threshold is sensible; also, if both uniform and geometric thresholds of a sequence of multiset families are defined, the geometric threshold approaches infinity, and the number of elements in the base sets of the multiset families approaches infinity, then the two thresholds have asymptotic ratio 1.

**Theorem 7.** *If $B$ is finite and nonempty and $M \subseteq \omega^B$ is an upper set, then the function $x \mapsto \nu_x(M)$ on $\mathbb{R}_{\geq 0}$ is either:*

1. *Identically 0, if $M$ is empty.*

2. *Identically 1, if $M = \omega^B$.*

3. *Strictly increasing and continuous with $\nu_0(M) = 0$ and $\lim_{x\to\infty} \nu_x(M) = 1$, otherwise.*

*Proof.* If $M$ is empty or $\omega^B$, this is obvious. Assume otherwise. Since $M \neq \omega^B$, $0 \notin M$ so $\nu_0(M) = 0$ and, since $\nu_x$ always assigns positive probability to 0, $\nu_x(M) < 1$ for all $x$.

If $(G_b)_{b\in B}$ is an i.i.d. family of geometric random variables, then, conditioned on $\sum_b G_b = T$, the function $b \mapsto G_b$ is uniform on $[\omega^B]_T$. Therefore, $\nu_x(M) = \mathbb{E}(\mu_{N_x}(M))$, where the random variable $N_x$ is the sum of $\#B$ i.i.d. geometric random variables with parameter $(1 + (x/\#B))^{-1}$. Given a geometric random variable $G_p$ with parameter $p$, we can realize $G_p$ as the smallest $i$ for which an i.i.d. sequence of random variables $U_0, U_1, \ldots$ uniform on $[0,1]$ has $U_i < p$. This lets us realize $G_p$ and $G_q$ ($p < q$) on the same probability space with $G_p = G_q + \Xi$, where $\Xi$ is a nonnegative integral random variable which, conditioned on $G_q$, always assigns positive probability to each nonnegative integer; in fact, $\mathbb{P}(\Xi = 0 \mid G_q)$ is always $p/q$. Summing $\#B$ independent copies of this, we can realize $N_x$ and $N_y$ ($x < y$) on the same probability space with $N_y = N_x + \Xi'$, where $\Xi'$ is a nonnegative integral random variable which, conditioned on $N_x$, always assigns positive probability to each nonnegative integer; also, $\mathbb{P}(\Xi' = 0 \mid N_x)$ is always $((x + \#B)/(y + \#B))^{\#B}$. But then

$$\begin{aligned} \nu_y(M) &= \mathbb{E}(\mu_{N_y}(M)) \\ &= \mathbb{E}(\mathbb{E}(\mu_{N_y}(M) \mid N_x)) \\ &= \mathbb{E}(\mathbb{E}(\mu_{N_x+\Xi'}(M) \mid N_x)). \end{aligned}$$

By Theorem 6, $\mathbb{E}(\mu_{N_x+\Xi'}(M) \mid N_x) \geq \mu_{N_x}(M)$. Also, by Theorem 6 and the above property of $\Xi'$, $\mathbb{E}(\mu_{N_x+\Xi'}(M) \mid N_x) > \mu_{N_x}(M)$ whenever $\mu_{N_x}(M) < 1$. Since $\nu_x(M) < 1$ we must have $\mathbb{P}(\mu_{N_x}(M) < 1) > 0$, so $\mathbb{E}(\mathbb{E}(\mu_{N_x+\Xi'}(M) \mid N_x)) > \mathbb{E}(\mu_{N_x}(M)) = \nu_x(M)$. This proves that $x \mapsto \nu_x(M)$ is strictly increasing. To show that it is continuous, observe that

$$\begin{aligned} \mathbb{E}(\mu_{N_x+\Xi'}(M) \mid N_x) &\leq \mathbb{P}(\Xi' = 0 \mid N_x)\mu_{N_x}(M) + 1 - \mathbb{P}(\Xi' = 0 \mid N_x) \\ &\leq \mu_{N_x}(M) + 1 - (\frac{x + \#B}{y + \#B})^{\#B}, \end{aligned}$$

and, taking expectations,

$$0 < \nu_y(M) - \nu_x(M) \leq 1 - (\frac{x + \#B}{y + \#B})^{\#B},$$

which implies that $x \mapsto \nu_x(M)$ is continuous.

Using Theorem 6 again, to prove that $\lim_{x\to\infty} \nu_x(M) = 1$, it will do to prove that $\lim_{x\to\infty} \mathbb{P}(N_x \leq j) = 0$ for each fixed $j \in \omega$. This is so because

$$\begin{aligned} \mathbb{P}(N_x \leq j) &\leq \mathbb{P}(G_{(1+(x/\#B))^{-1}} \leq j) \\ &\leq (j + 1)(1 + (x/\#B))^{-1} \\ &\to 0 \text{ as } x \to \infty. \end{aligned}$$

□

Theorem 7 shows that the definition of geometric threshold makes sense.

**Proposition 8.** *If $B$ is finite and nonempty, $M \subseteq \omega^B$ is an upper set, $T \geq U \in \omega$, and $\mu_U(M) \geq \frac{1}{2}$, then $\mu_T(M) \geq T/(T+U)$, where here we take $0/0 = 1$ if $T = U = 0$.*

*Proof.* If $U = 0$, $M$ must contain 0, so $M = \omega^B$ and the result is obvious. Otherwise, set $x := U$ and use Proposition 4. □

**Proposition 9.** *If $B$ is finite and nonempty, $\emptyset \neq M \subseteq \omega^B$ is an upper set, $U < T \in \omega$, and $T$ is minimal such that $\mu_T(M) \geq \frac{1}{2}$, then $\mu_U(M) \leq U/(T+U-1)$, where here we take $0/0 = 0$ if $U = 0$ and $T = 1$.*

*Proof.* Since $T \neq 0$, $0 \notin M$, so $\mu_0(M) = 0$; this proves the result if $U = 0$. Otherwise, set $x := T - 1$ and use Proposition 3 with $T$ decreased by 1. □

**Proposition 10.** *If $B$ is finite and nonempty, $\emptyset \neq M \subseteq \omega^B$ is an upper set with $0 \notin M$, $T \in \omega$ is minimal such that $\mu_T(M) \geq \frac{1}{2}$, $T'$ is the unique positive real number such that $\nu_{T'}(M) = \frac{1}{2}$, $S := \sqrt{T' + (T'^2/\#B)}$, and $\theta$ is a real number with $\sqrt{2} < \theta < T'/S$, then*

$$\lceil T' - \theta S \rceil (1 - \frac{2}{\theta^2}) \leq T \leq 1 + \lfloor T' + \theta S \rfloor (1 + \frac{2}{\theta^2 - 2}). \tag{2}$$

*Proof.* If $G_p$ is a geometric random variable with parameter $p$, then $\mathbb{E}G_p = p^{-1} - 1$ and $\mathrm{Var}\ G_p = p^{-2} - p^{-1}$. Therefore, if $N_{T'}$ is the sum of $\#B$ i.i.d. geometric random variables with parameter $(1 + (T'/\#B))^{-1}$, then $\mathbb{E}N_{T'} = T'$ and $\mathrm{Var}\ N_{T'} = T' + (T'^2/\#B) = S^2$. If $V := \lceil T' - \theta S \rceil$ and $W := \lfloor T' + \theta S \rfloor$, it follows from Chebyshev's inequality that

$$\mathbb{P}(V \leq N_{T'} \leq W) \geq 1 - \frac{1}{\theta^2},$$

so since, by Theorem 6, $\mu_i(M)$ is nondecreasing with $i$,

$$\frac{1}{2} = \nu_{T'}(M) = \mathbb{E}(\mu_{N_{T'}}(M)) \geq (1 - \frac{1}{\theta^2})\mu_V(M) \tag{3}$$

and

$$\frac{1}{2} = \nu_{T'}(M) = \mathbb{E}(\mu_{N_{T'}}(M)) \leq (1 - \frac{1}{\theta^2})\mu_W(M) + \frac{1}{\theta^2}. \tag{4}$$

If $V \leq T$, then the left-hand inequality of (2) is satisfied. If $T < V$, then by Proposition 8 and (3),

$$\frac{\frac{1}{2}}{1 - \theta^{-2}} \geq \mu_V(M) \geq \frac{V}{T + V},$$

which, after rearrangement, gives the left-hand inequality of (2). If $T \leq W$, then the right-hand inequality of (2) is satisfied. If $W < T$, then by Proposition 9 and (4),

$$\frac{\frac{1}{2} - \theta^{-2}}{1 - \theta^{-2}} \leq \mu_W(M) \leq \frac{W}{T + W - 1},$$

which, after rearrangement, gives the right-hand inequality of (2). □

**Theorem 11.** *If $(M_i)_{i \in \omega}$ is a sequence such that $\emptyset \neq M_i \subseteq \omega^{B_i}$ for each $i$, each $B_i$ is a nonempty finite set, each $M_i$ is an upper set, $0 \notin M_i$ for all $i$, $(T_i)_{i \in \omega}$ and $(T'_i)_{i \in \omega}$ are the uniform and geometric thresholds of $(M_i)_{i \in \omega}$, and $\#B_i$ and $T'_i$ both approach infinity as $i \to \infty$, then $T_i/T'_i \to 1$ as $i \to \infty$.*

*Proof.* For sufficiently large $i$, use Proposition 10 on each $B := B_i$, $M := M_i$, $T := T_i$, and $T' := T'_i$, setting

$$\theta := (\frac{T'}{S})^{1/3} = (\frac{T'}{\sqrt{T' + (T'^2/\#B)}})^{1/3}$$

and observing that since $T'/S \to \infty$ as $i \to \infty$, $\theta$ and $T'/(\theta S)$ both approach $\infty$ as $i \to \infty$. □

# 4 The threshold of the sequence of paths, I

We now begin to compute the pebbling threshold of the sequence of $n$-paths, where the $n$-path has $n$ vertices, $1, \ldots, n$, and edges between vertices $i$ and $i+1$ for all $i = 1, \ldots, n-1$. Because of Theorem 11 and [2, Theorem 1.3], it will do to find the geometric threshold of the sequence of families of solvable distributions of the $n$-paths. Therefore, fix some positive $n$, and suppose that, for some parameter $0 < p < 1$, we have i.i.d. geometric random variables $(Z_i)_{i \in \mathbb{Z}_{>0}}$ with parameter $p$, and that, for $i = 1, \ldots, n$, we place $Z_i$ pebbles on each vertex $i$. If $r := (2 \log n)/p$, then $\mathbb{P}(Z_i \geq r) = \mathbb{P}(Z_i \geq \lceil r \rceil) = (1-p)^{\lceil r \rceil} \leq e^{-pr} = n^{-2}$, so with probability at least $1 - n^{-1}$, $Z_i < r$ for all $i = 1, \ldots, n$. Let $L$ be a positive integer such that $n \geq 2L+1$. Now, for each $L+1 \leq i \leq n-L$, $i$ will be unpebblable iff $Z_i = 0$, $\sum_{1 \leq j < i} Z_j 2^{-(i-j)} < 1$, and $\sum_{i < j \leq n} Z_j 2^{-(j-i)} < 1$, so, given that $Z_j < r$ for all $j = 1, \ldots, n$, it is sufficient for unpebblability that $Z_i = 0$, $\sum_{1 \leq k \leq L} Z_{i-k} 2^{-k} < 1 - (r/2^L)$, and $\sum_{1 \leq k \leq L} Z_{i+k} 2^{-k} < 1 - (r/2^L)$. If we pick $i = L+1$, $L+1+(2L+1)$, $\ldots$, $L+1+(\lfloor \frac{n}{2L+1} \rfloor - 1)(2L+1)$, then, since the $Z_i$'s are i.i.d., the probability that the distribution is unsolvable will be at least

$$-\frac{1}{n} + 1 - (1-pq^2)^{\lfloor n/(2L+1) \rfloor}, \quad \text{where} \quad q := \mathbb{P}(\frac{Z_1}{2} + \cdots + \frac{Z_L}{2^L} < 1 - \frac{r}{2^L}),$$

so the probability that it is solvable will be no more than

$$\frac{1}{n} + (1-pq^2)^{\lfloor n/(2L+1) \rfloor} \leq \frac{1}{n} + \exp -pq^2 \lfloor \frac{n}{2L+1} \rfloor.$$

It is easy to see that $\sum_{i>0} Z_i/2^i$ converges a.s., and then

$$q \geq q' := \mathbb{P}(\sum_{i>0} \frac{Z_i}{2^i} < 1 - \frac{r}{2^L}).$$

If we let $(W_i)_{i \in \mathbb{Z}_{>0}}$ be an i.i.d. family of standard exponential random variables, with $\mathbb{P}(W_i \geq x) = e^{-x}$ for each $i$, then we can realize $Z_i$ by letting each $Z_i$ be $\lfloor W_i/\lambda \rfloor$, where $\lambda := -\log(1-p)$. Since $\sum_{i>0} W_i/2^i$ also converges a.s., we have

$$q' \quad = \quad \mathbb{P}(\sum_{i>0} \frac{\lfloor W_i/\lambda \rfloor}{2^i} < 1 - \frac{r}{2^L})$$

$$\geq \quad q'' := \mathbb{P}(\sum_{i>0} \frac{W_i/\lambda}{2^i} < 1 - \frac{r}{2^L})$$
$$= \mathbb{P}(Y_\infty < 2\lambda(1 - \frac{r}{2^L})),$$

where we have set

$$Y_\infty := W_1 + \frac{W_2}{2} + \frac{W_3}{4} + \cdots = \sum_{i\geq 0} \frac{W_{i+1}}{2^i}.$$

The probability that the distribution is solvable is then no more than

$$\frac{1}{n} + \exp -pq''^2 \lfloor \frac{n}{2L+1} \rfloor.$$

To estimate this, we must estimate the probability that $Y_\infty$ is below a small threshold.

# 5 The asymptotics of $Y_\infty$

Let $n \in \mathbb{Z}_{>0}$, $Y_n := W_1 + \cdots + (W_n/2^{n-1})$, and, for $i = 0, \ldots, n-1$, let

$$R_{i,n}(x) := \prod_{0\leq j\leq n-1, j\neq i} \frac{2^j - x}{2^j - 2^i}$$

be the degree $n-1$ polynomial which is 0 at $2^j$, $j = 0, \ldots, n-1$, $j \neq i$, and 1 at $2^i$. Then for all $x \in \mathbb{R}_{\geq 0}$ [7, §I.13, ex. 12],

$$\mathbb{P}(Y_n \leq x) = \sum_{0\leq i\leq n-1} (1 - e^{-2^i x}) R_{i,n}(0).$$

For $0 \leq k < n$, $\sum_{0\leq i\leq n-1} 2^{ik} R_{i,n}(x)$ is a polynomial of degree at most $n-1$ which is $2^{ik}$ at $2^i$, $i = 0, \ldots, n-1$; it must then be $x^k$, so

$$\sum_{0\leq i\leq n-1} 2^{ik} R_{i,n}(0) = 0, \qquad 1 \leq k \leq n-1.$$

Therefore, if we write, for any $c \in \omega$,

$$e_c(x) := e^{-x} - \sum_{0\leq k\leq c} \frac{(-x)^k}{k!},$$

then

$$\mathbb{P}(Y_n \leq x) = -\sum_{0\leq i\leq n-1} e_c(2^i x) R_{i,n}(0), \qquad 0 \leq c \leq n-1.$$

Let

$$\mathcal{N} := \prod_{j\geq 1} \frac{2^j}{2^j - 1}, \qquad F_c(x) := \mathcal{N} \sum_{i\geq 0} \frac{(-1)^{i+1} e_c(2^i x)}{(2^1 - 1)\cdots(2^i - 1)}. \tag{5}$$

We have

$$\lim_{n\to\infty} \mathbb{P}(Y_n \le x) = \lim_{n\to\infty} \sum_{0\le i\le n-1} V_{c,i,n}(x),$$

where

$$\begin{aligned} V_{c,i,n}(x) &:= -e_c(2^i x) \prod_{0\le j\le n-1, j\ne i} \frac{2^j}{2^j - 2^i} \\ &= \left(\prod_{1\le j\le n-1-i} \frac{2^j}{2^j-1}\right) (-1)^{i+1} e_c(2^i x) \prod_{1\le k\le i} \frac{1}{2^k-1}. \end{aligned}$$

Then for all $n > i$,

$$|V_{c,i,n}(x)| \le U_{c,i}(x) := \left(\prod_{j\ge 1} \frac{2^j}{2^j-1}\right) |e_c(2^i x)| \prod_{1\le k\le i} \frac{1}{2^k-1},$$

and $\sum_{i\ge 0} U_{c,i}(x)$ converges, so by the dominated convergence theorem,

$$\begin{aligned} \lim_{n\to\infty} \mathbb{P}(Y_n \le x) &= \sum_{i\ge 0} \lim_{n\to\infty} V_{c,i,n}(x) \\ &= F_c(x). \end{aligned}$$

(If $x \in \mathbb{R}_{<0}$, we define $F_c(x) := 0 = \lim_{n\to\infty} \mathbb{P}(Y_n \le x)$.)

Since $Y_n \to Y_\infty$ a.s., $Y_n$ also converges to $Y_\infty$ in distribution, so $\mathbb{P}(Y_\infty \le x) = F_c(x)$ at every continuity point of $\mathbb{P}(Y_\infty \le x)$. Since $F_c(x)$ is continuous, it must equal $\mathbb{P}(Y_\infty \le x)$ everywhere.

Let $c$ and $x$ be positive. If, for complex $z$, we define $S_c(z)$ by

$$e^{cz} = \sum_{0\le k\le c} \frac{(cz)^k}{k!} + \frac{(cz)^c}{c!} S_c(z),$$

and $z$ has negative real part, then [3, Theorem]

$$S_c(z) = \frac{z}{1-z} + O(\frac{1}{c}). \tag{6}$$

In this section only, by $O(f(\cdots))$ for some function $f$, we mean any quantity for which there is an absolute constant $\mathcal{L}$ so that its absolute value is no larger than $\mathcal{L} f(\cdots)$ for all values of the parameters of $f$. Substituting (6) and

$$e_c(2^i x) = \frac{(-2^i x)^c}{c!} S_c(-\frac{2^i x}{c})$$

into (5) gives

$$\begin{aligned} F_c(x) &= \mathcal{N}^2 \sum_{i\ge 0} (-1)^{i+1} 2^{-i(i+1)/2} e_c(2^i x) \prod_{j>i} (1 - 2^{-j}) \\ &= \mathcal{N}^2 \sum_{i\ge 0} (-1)^i \frac{(-2^i x)^c}{c!} 2^{-i(i+1)/2} (\frac{2^i x}{c + 2^i x} + O(\frac{1}{c}))(1 + O(2^{-i})). \end{aligned}$$

Substituting $x := cy/2^c$, $i := j + c$, we get

$$\begin{aligned} F_c(x) &= \mathcal{N}^2 \frac{(cy)^c}{c!} 2^{-c(c+1)/2} \sum_{j \geq -c} (-1)^j 2^{-j(j+1)/2} \left(\frac{2^j y}{2^j y + 1} + O(\frac{1}{c})\right)(1 + O(2^{-(j+c)})) \\ &= \mathcal{N}^2 \frac{(cy)^c}{c!} 2^{-c(c+1)/2} \left(O(\frac{1}{c}) + \sum_{j \geq -c} (-1)^j 2^{-j(j+1)/2} \frac{2^j y}{2^j y + 1}\right), \end{aligned}$$

and since removing the lower limit at $-c$ changes the sum by only $O(2^{-c(c+1)/2})$, we have

$$\begin{aligned} \mathbb{P}(Y_\infty \leq x) &= \mathcal{N}^2 \frac{(cy)^c}{c!} 2^{-c(c+1)/2} (\mathcal{P}(y) + O(\frac{1}{c})), \quad \text{where} \\ \mathcal{P}(z) &:= \sum_{j \in \mathbb{Z}} (-1)^j 2^{-j(j+1)/2} \frac{2^j z}{2^j z + 1}, \qquad z \in \mathbb{C}. \end{aligned} \tag{7}$$

For small $\epsilon > 0$, the region $\mathcal{D}(\epsilon) := \{z \in \mathbb{C} \mid |z + 1| \leq \epsilon |z|\}$ is a small disk containing $-1$. On $\mathbb{C} \setminus \{0\}$ with $\mathcal{D}(\epsilon)$ and its scalings by $2^j$ ($j \in \mathbb{Z}$) removed, the sum defining $\mathcal{P}(z)$ converges uniformly, so it is analytic. Letting $\epsilon \to 0$, it follows that $\mathcal{P}(z)$ is analytic on $\mathbb{C} \setminus \{0, -2^j \mid j \in \mathbb{Z}\}$; similarly, it has simple poles at $-2^j$ ($j \in \mathbb{Z}$.) Where $\mathcal{P}$ is defined, we have

$$\begin{aligned} \mathcal{P}(z) &= z \sum_{j \in \mathbb{Z}} (-1)^j 2^{-j(j+1)/2} \frac{2^j}{2^j z + 1} \\ &= z \sum_{j \in \mathbb{Z}} (-1)^j 2^{-(j-1)j/2} \frac{1}{2^j z + 1} \\ &= z \sum_{j \in \mathbb{Z}} (-1)^j 2^{-(j-1)j/2} \left(\frac{1}{2^j z + 1} - 1\right), \\ &\qquad \text{since } \textstyle\sum_{j \in \mathbb{Z}} (-1)^j 2^{-(j-1)j/2} = 0 \\ &= z \sum_{j \in \mathbb{Z}} (-1)^{j-1} 2^{-(j-1)j/2} \frac{2^{j-1} \cdot 2z}{2^{j-1} \cdot 2z + 1} \\ &= z\mathcal{P}(2z). \end{aligned}$$

Now, set

$$\mathcal{Q}(z) := 2^{z(z-1)/2} \mathcal{P}(2^z); \tag{8}$$

then $\mathcal{Q}(z)$ is analytic on $\mathbb{C} \setminus (\mathbb{Z} + (2\pi\iota/\log 2)\mathbb{Z} + \pi\iota/\log 2)$, has simple poles at $\mathbb{Z} + (2\pi\iota/\log 2)\mathbb{Z} + \pi\iota/\log 2$, and, where $\mathcal{Q}$ is defined,

$$\begin{aligned} \mathcal{Q}(z+1) &= 2^{(z+1)z/2} \mathcal{P}(2^{z+1}) = 2^{z(z-1)/2} \cdot 2^z \mathcal{P}(2 \cdot 2^z) = \mathcal{Q}(z) \quad \text{and} \\ \mathcal{Q}(z + \frac{2\pi\iota}{\log 2}) &= 2^{(z+(2\pi\iota/\log 2))(z+(2\pi\iota/\log 2)-1)/2} \mathcal{P}(2^z) = -e^{2\pi\iota z} e^{-2\pi^2/\log 2} \mathcal{Q}(z). \end{aligned}$$

However, if $\theta_4(z, q)$ is the theta function [9, §21.1, §21.11, §21.12]

$$\theta_4(z, q) := 1 + 2 \sum_{i \geq 1} (-1)^i q^{i^2} \cos(2iz), \qquad q = e^{\pi\iota\tau}, \ |q| < 1, \ z, q, \tau \in \mathbb{C},$$

then, for fixed $q$, $\theta_4(z,q)$ is analytic on all of $\mathbb{C}$ and has zeroes at $\pi\mathbb{Z}+\pi\tau\mathbb{Z}+\frac{1}{2}\pi\tau$, and

$$\theta_4(z+\pi,q)=\theta_4(z,q), \qquad \theta_4(z+\pi\tau,q)=-\frac{e^{-2\iota z}}{q}\theta_4(z,q).$$

If we set $\tau := 2\pi\iota/\log 2$, $q := \exp -2\pi^2/\log 2$, then, $\mathcal{Q}(z)\theta_4(\pi z,q)$ will be analytic on $\mathbb{C}$ and doubly periodic, so it is constant and

$$\mathcal{Q}(z)=\frac{\mathcal{K}}{\theta_4(\pi z,q)}$$

for some $\mathcal{K}\in\mathbb{C}$, which is real and positive since both $\mathcal{Q}(0)=\mathcal{P}(1)$ and $\theta_4(0,q)$ are real and positive. Therefore, $\mathcal{Q}(r)$ cannot be zero for $r\in\mathbb{R}$, and since $\mathcal{Q}(r)$ is then real, it must always be real and positive for $r\in\mathbb{R}$, so $\mathcal{P}(r)$ is also always real and positive for $r\in\mathbb{R}_{>0}$. Also, since $\theta_4(\pi z,q)$ is even, $\mathcal{Q}(z)$ is even.

Supposing now that $x=c'/2^{c'}$ for some real $c'\geq 1$, we may let $c:=\lfloor c'\rfloor\geq 1$. Then

$$y=\frac{c'/c}{2^{c'-c}}=2^{-\{c'\}}(1+\frac{\{c'\}}{c}) \tag{9}$$

so $\frac{1}{2}<y<2$, and, from (7) and (8),

$$\begin{aligned}\mathbb{P}(Y_\infty\leq x) &= \mathcal{N}^2\frac{(cy)^c}{c!}2^{-c(c+1)/2}(\mathcal{P}(y)+O(\frac{1}{c}))\\ &= \mathcal{N}^2\frac{(cy)^c}{c!}2^{-c(c+1)/2}\mathcal{P}(y)(1+O(\frac{1}{c}))\\ &= \mathcal{N}^2\frac{(cy)^c}{c!}2^{-c(c+1)/2}y^{(1-\log_2 y)/2}\mathcal{Q}(\log_2 y)(1+O(\frac{1}{c})).\end{aligned}$$

After some simplification, using Stirling's approximation, (9), $\mathcal{Q}(\log_2 y)=\mathcal{Q}(-\{c'\})(1+O(1/c))$, $O(1/c)=O(1/c')$, and the periodicity and evenness of $\mathcal{Q}$, we get

**Theorem 12.** *For real $c'\geq 1$,*

$$\mathbb{P}(Y_\infty\leq\frac{c'}{2^{c'}})=\frac{\mathcal{N}^2}{\sqrt{2\pi c'}}e^{c'}2^{-c'(c'+1)/2}\mathcal{Q}(c')(1+O(\frac{1}{c'})).$$

It will be convenient later to find $\mathbb{P}(Y_\infty\leq c''y/2^{c''})$, where $c''$ and $y$ are positive real and $(\log_2 y)^4\leq c''$. We need then to find $c'$ with

$$\frac{c'}{2^{c'}}\Big/\frac{c''y}{2^{c''}}=1, \tag{10}$$

and if we set

$$c':=c''-\log_2 y-\frac{\log_2 y}{c''\log 2}+\frac{K}{c''^{3/2}}, \tag{11}$$

we can verify that, if $c''\geq 6$, the left-hand side of (10) is less than 1 if $K=4$ and bigger than 1 if $K=-7$, so (10) must be satisfied with some $-7\leq K\leq 4$. Substituting (11) into Theorem 12, we get

**Proposition 13.** *For real* $c'' \geq 6$ *and* $2^{-c''^{1/4}} \leq y \leq 2^{c''^{1/4}}$,

$$\mathbb{P}(Y_\infty \leq \frac{c''y}{2^{c''}}) = \frac{\mathcal{N}^2}{\sqrt{2\pi c''}}(ey)^{c''}2^{-c''(c''+1)/2}y^{(1-\log_2 y)/2}\mathcal{Q}(c''-\log_2 y)(1+O(\frac{1}{\sqrt{c''}})).$$

Finally, we observe that

$$\begin{aligned}
\mathbb{P}(Y_\infty > x) &= 1 - F_0(x) \\
&= 1 - \mathcal{N}\sum_{i\geq 0}\frac{(-1)^{i+1}(e^{-2^i x}-1)}{(2^1-1)\cdots(2^i-1)} \\
&= 1 + \mathcal{N}\sum_{i\geq 0}\frac{(-1)^{i+1}}{(2^1-1)\cdots(2^i-1)} + \mathcal{N}\sum_{i\geq 0}\frac{(-1)^i e^{-2^i x}}{(2^1-1)\cdots(2^i-1)}.
\end{aligned}$$

The last term is an alternating series whose terms decrease in magnitude, so its value is between 0 and the first term of the series, which is $\mathcal{N}e^{-x}$. Since $\mathbb{P}(Y_\infty > x)$ must approach 0 as $x \to \infty$, the first two terms must cancel, so we have

**Proposition 14.** *For* $x \in \mathbb{R}_{>0}$,

$$\mathbb{P}(Y_\infty > x) \leq \mathcal{N}e^{-x}.$$

# 6 The threshold of the sequence of paths, II

Returning to the situation of §4, we now let $n$ be large and set $L := \lfloor \log_2 n \rfloor$, $c'' := \sqrt{\log_2 n}$, $p := (c'' + \sqrt{c''})/(e2^{c''})$. If we use Proposition 13 to estimate $q''$, we will take $y := 2(1-(r/2^L))\lambda(1+c''^{-1/2})/(ep)$. Recalling that $r = (2\log n)/p$ and, since $\lambda = -\log(1-p)$, $|(\lambda/p) - 1| \leq p$ for all $0 \leq p \leq \frac{1}{2}$, we find that $\frac{1}{2} \leq y \leq 1$ for all sufficiently large $n$ and so

$$q'' = \Theta(\frac{1}{\sqrt{c''}n}2^{c''/2}\Delta^{c''}), \qquad \text{where } \Delta := (1-\frac{r}{2^L})\frac{\lambda}{p}(1+\frac{1}{\sqrt{c''}}).$$

Then

$$pq''^2\lfloor\frac{n}{2L+1}\rfloor = \Theta(\frac{\Delta^{2c''}}{L}) = \Theta(\frac{e^{2\sqrt{c''}}}{L})$$

will approach infinity as $n \to \infty$, so our random distribution is solvable with a probability that approaches 0 as $n \to \infty$. This means that, for this choice of $p$,

$$n(\frac{1}{p}-1) = e\frac{2^{\sqrt{\log_2 n}}}{\sqrt{\log_2 n}}n(1+O(\frac{1}{(\log n)^{1/4}}))$$

is eventually below the geometric threshold of the sequence of families of solvable distributions of the $n$-paths.

We now need to find an upper bound for the geometric threshold. We first prove a preliminary lemma.

**Lemma 15.** *If $L \in \omega$, $0 < p < 1$ and $r \in \mathbb{R}_{\geq 0}$, define*

$$\mathcal{X}(L,p,r) := \mathbb{P}(Z_1 + \frac{Z_2}{2} + \cdots + \frac{Z_L}{2^{L-1}} < r),$$

*where $Z_1, \ldots, Z_L$ are i.i.d. geometric random variables with parameter $p$. Then*

$$\mathcal{X}(L,p,r) \leq \mathbb{P}(Y_\infty < (r+3)\lambda) + \mathcal{N} \exp -2^L \lambda, \tag{12}$$

*where $\lambda := -\log(1-p)$.*

*Proof.* As in §4, let $W_1, W_2, \ldots$ be i.i.d. standard exponential random variables; then we can realize each $Z_i$ as $\lfloor W_i/\lambda \rfloor$, so

$$\begin{aligned}
\mathcal{X}(L,p,r) &= \mathbb{P}(\lfloor W_1/\lambda \rfloor + \cdots + \frac{\lfloor W_L/\lambda \rfloor}{2^{L-1}} < r) \\
&\leq \mathbb{P}((W_1/\lambda) + \cdots + \frac{W_L/\lambda}{2^{L-1}} < r+2) \\
&\leq \mathbb{P}(\sum_{i\geq 1} \frac{W_i}{\lambda 2^{i-1}} < r+3) + \mathbb{P}(\sum_{i \geq L+1} \frac{W_i}{\lambda 2^{i-1}} > 1) \\
&= \mathbb{P}(Y_\infty < (r+3)\lambda) + \mathbb{P}(Y_\infty > 2^L \lambda) \\
&\leq \mathbb{P}(Y_\infty < (r+3)\lambda) + \mathcal{N} \exp -2^L \lambda,
\end{aligned}$$

by Proposition 14. □

Suppose now as before that we place $Z_i$ pebbles on each vertex $i$, where the $Z_i$'s are i.i.d. geometric random variables with parameter $p$, and let $M \leq L$ be positive integers with $n \geq 2(L+M)+1$. If $i \leq L+M$, if $i$ is unpebblable, then $\sum_{0 \leq k \leq L-1} Z_{i+k} 2^{-k}$ must be less than 1, and if $i \geq n-(L+M)+1$, if $i$ is unpebblable, then $\sum_{0 \leq k \leq L-1} Z_{i-k} 2^{-k}$ must be less than 1. In both cases, then, since the $Z_i$'s are i.i.d.,

$$\mathbb{P}(i \text{ unpebblable}) \leq \mathcal{X}(L,p,1).$$

If $L+M+1 \leq i \leq n-(L+M)$, we observe that for $i$ to be unpebblable, $Z_{i-M}, \ldots, Z_{i+M}$ must all be less than $2^M$, and $\sum_{0 \leq k \leq L-1} Z_{i+M+1+k} 2^{-k}$ and $\sum_{0 \leq k \leq L-1} Z_{i-M-1-k} 2^{-k}$ must both be less than $2^{M+1}$. In this case, then,

$$\mathbb{P}(i \text{ unpebblable}) \leq \mathbb{P}(Z_1 < 2^M)^{2M+1} \mathcal{X}(L,p,2^{M+1})^2.$$

Summing these probabilities, and using

$$\mathbb{P}(Z_1 < 2^M) = \sum_{0 \leq j < 2^M} \mathbb{P}(Z_1 = j) \leq 2^M p,$$

we find that the probability that our random distribution is unsolvable is no more than

$$\begin{aligned}
& 2(L+M)\mathcal{X}(L,p,1) + (n-2(L+M))(2^M p)^{2M+1} \mathcal{X}(L,p,2^{M+1})^2 \\
\leq\ & 4L\mathcal{X}(L,p,1) + n(2^M p)^{2M+1} \mathcal{X}(L,p,2^{M+1})^2. \qquad (13)
\end{aligned}$$

We now let $n$ be large, set $L := \lfloor \log_2 n \rfloor$, $M := \lfloor (\log_2 n)^{1/16} \rfloor$, $c'' := \sqrt{\log_2 n}$, and $p := (c'' - \sqrt{c''})/(e2^{c''})$, and estimate $\mathcal{X}(L, p, r)$ using (12). To estimate the first term in (12), we use Proposition 13, setting $y := (r+3)\lambda(1 - c''^{-1/2})/(ep)$. In the case $r = 1$ we have $1 \leq y \leq 2$ for all sufficiently large $n$ so

$$\mathcal{X}(L, p, 1) = \Theta(\frac{1}{\sqrt{c''}n} 2^{3c''/2} \Delta'^{c''}) + O(e^{-\sqrt{n}}), \quad \text{where } \Delta' := \frac{\lambda}{p}(1 - \frac{1}{\sqrt{c''}}). \tag{14}$$

In the case $r = 2^{M+1}$, $\log_2 y = O(M)$ so

$$\mathcal{X}(L, p, 2^{M+1}) = \Theta(\frac{1}{\sqrt{c''}n} (2^{M+1} + 3)^{c''} 2^{-c''/2} 2^{O(M^2)} \Delta'^{c''}) + O(e^{-\sqrt{n}}). \tag{15}$$

Combining (13), (14), (15), and $\Delta'^{c''} = \Theta(e^{-\sqrt{c''}})$ shows that our random distribution is unsolvable with a probability that approaches 0 at $n \to \infty$, so, for this choice of $p$,

$$n(\frac{1}{p} - 1) = e\frac{2^{\sqrt{\log_2 n}}}{\sqrt{\log_2 n}} n(1 + O(\frac{1}{(\log n)^{1/4}}))$$

is eventually above the geometric threshold of the sequence of families of solvable distributions of the $n$-paths. Together with our previous lower bound on the geometric threshold, this proves

**Theorem 16.** *For all positive integers $n$, let the $n$-path have $n$ vertices, $1, \ldots, n$, and edges between vertices $i$ and $i+1$ for $i = 1, \ldots, n-1$. Then the geometric threshold of the sequence of families of solvable distributions of the $n$-paths is*

$$e\frac{2^{\sqrt{\log_2 n}}}{\sqrt{\log_2 n}} n(1 + O(\frac{1}{(\log n)^{1/4}})).$$

**Corollary 17.** *The pebbling threshold of the sequence of $n$-paths is*

$$\Theta(\frac{2^{\sqrt{\log_2 n}}}{\sqrt{\log_2 n}} n).$$

*Proof.* Use Theorem 16, Theorem 11 and [2, Theorem 1.3]. □

# 7 One-ended path estimates

**Lemma 18.** *Let $H$ be a graph which contains $L \geq 1$ vertices, $v_1, \ldots, v_L$, such that $d(v_1, v_i) = i - 1$ for all $i = 2, \ldots, L$. There exists some absolute constant $G_- \geq 3$ such that, if $g \geq G_-$ and an independent, geometrically distributed number of pebbles with parameter $p := (1 + (2^{\sqrt{2 \log_2 g}} e/(2(1 + (\log_2 g)^{-1/4})\sqrt{\log_2 g})))^{-1}$ is placed on each of $v_1, \ldots, v_L$, then, with probability at least $2/g$, $v_1$ is unpebblable, provided that, for the set of vertices $V$ of $H$ apart from $v_1, \ldots, v_L$,*

$$\sum_{x \in V} \mathcal{Z}(x) 2^{-d(x, v_1)} \leq \frac{2e}{\sqrt{\log_2 g}}. \tag{16}$$

*Proof.* This is similar to the first half of the proof of Theorem 16. Let $Z_i$ be the number of pebbles on $v_i$, $i = 1, \ldots, L$. The quantity

$$Q := \sum_{x \text{ a vertex of } H} \mathcal{Z}(x) 2^{-d(x,v_1)}$$

is at least 1 if there is a pebble on $v_1$, and it cannot be increased by pebbling moves. It follows that $v_1$ will be unpebblable provided that $Q < 1$, which, by (16), will certainly be true if $\sum_{1 \le i \le L} Z_i 2^{-(i-1)} < 1 - (2e/\sqrt{\log_2 g})$; so, if we let $Z_{L+1}$, $Z_{L+2}$, ... be additional independent geometric random variables with parameter $p$, $v_1$ will be be unpebblable if $\sum_{i \ge 1} Z_i 2^{-(i-1)} < 1 - (2e/\sqrt{\log_2 g})$. As in §4, we can now set $Z_i := \lfloor W_i/\lambda \rfloor$, where $\lambda := -\log(1-p)$ and $W_1$, $W_2$, ... are i.i.d. standard exponential random variables, so it suffices for unpebblability that

$$Y_\infty < \lambda\left(1 - \frac{2e}{\sqrt{\log_2 g}}\right).$$

We can compute the probability $q$ of this event using Proposition 13, setting

$$c'' := \sqrt{2 \log_2 g}, \quad p' := (p^{-1} - 1)^{-1} = \frac{2\sqrt{\log_2 g}}{e 2^{\sqrt{2 \log_2 g}}}(1 + (\log_2 g)^{-1/4}),$$

$$y := \frac{\sqrt{2}}{e}\Delta, \quad \Delta := \frac{\lambda}{p}\frac{1}{p'+1}(1 + (\log_2 g)^{-1/4})\left(1 - \frac{2e}{\sqrt{\log_2 g}}\right).$$

For large $g$, $\Delta$ will be close to 1, so after choosing $G_-$ appropriately, $y$ will be between $\frac{1}{2}$ and 1. According then to the proposition, if we choose $G_-$ so as to make $c''$ sufficiently large, there is some positive constant $C$ such that

$$q \ge C(ey)^{c''} 2^{-c''(c''+1)/2}/\sqrt{c''},$$

or such that $q \ge C\Delta^{c''}/(g\sqrt{c''})$. Now $\Delta$ is a function only of $g$ and, for large $g$, $\log \Delta = 2^{1/4} c''^{-1/2} + O(c''^{-1})$, so choose $G_-$ large enough to ensure that $\Delta^{c''}/\sqrt{c''} \ge 2/C$. ☐

**Lemma 19.** *Let $H$ be a graph which contains a path with $L \ge 1$ vertices, $v_1$, ..., $v_L$. Then there exists some absolute constant $G_+ \ge 3$ such that, if $g \ge G_+$ and an independent, geometrically distributed number of pebbles with parameter $p := (1 + (2^{\sqrt{2\log_2 g}} e/(2(1 - (\log_2 g)^{-1/4})\sqrt{\log_2 g})))^{-1}$ is placed on each of $v_1$, ..., $v_L$, then (A) if $L \ge 1.1\sqrt{2 \log_2 g}$, with probability at least $1 - 1/(4g)$, $v_1$ is pebblable, and (B) if*

$$2.2\sqrt{2\log_2 g} \le L \le \exp(2 \log_2 g)^{1/4},$$

*with probability at least $1 - 1/(4g)$, all of $v_1$, ..., $v_L$ are pebblable.*

*Proof.* This is similar to the second half of the proof of Theorem 16. We start with (A). Let $Z_j$ be the number of pebbles on $v_j$, $j = 1, \ldots, L$. Set $M := \lfloor (\log_2 g)^{1/16} \rfloor$. Since $L \ge 1.1\sqrt{2 \log_2 g}$, we can choose $G_+$ large enough to ensure that $L \geq M+1$. For $v_1$ to be unpebblable, we must have $Z_j < 2^M$, $j = 1, \ldots, M+1$, and $\sum_{0 \leq j \leq L-M-2} Z_{M+2+j} 2^{-j} < 2^{M+1}$; since the $Z_j$'s are independent and geometrically distributed with parameter $p$, this will have probability no more than $(2^M p)^{M+1} X$, where $X := \mathcal{X}(L-M-1, p, 2^{M+1})$, and by Lemma 15, $X \leq q + X'$, where

$$q := \mathbb{P}(Y_\infty < (2^{M+1}+3)\lambda), \quad X' := \mathcal{N} \exp -2^{L-M-1}\lambda,$$

$$\lambda := -\log(1-p).$$

For an appropriate choice of $G_+$, we will have $X' \leq \mathcal{N} \exp -2^{0.09\sqrt{2\log_2 g}}$, so by choosing $G_+$ large enough, we can force $X'$ to be less than $\frac{1}{8}$ when multiplied by $(2^M p)^{M+1} g$. To estimate $q$, use Proposition 13, setting

$$c'' := \sqrt{2\log_2 g}, \quad p' := (p^{-1}-1)^{-1} = \frac{2\sqrt{\log_2 g}}{e 2^{\sqrt{2\log_2 g}}}(1-(\log_2 g)^{-1/4}),$$

$$y := 2^{M+1}\frac{\sqrt{2}}{e}\Delta', \quad \Delta' := \frac{\lambda}{p}\frac{1}{p'+1}(1-(\log_2 g)^{-1/4})(1+\frac{3}{2^{M+1}}).$$

For large $g$, $\Delta'$ will be close to 1, so $2^M \leq y \leq 2^{M+1}$, and by the proposition, if we choose $G_+$ appropriately, we will have, for some constant $C > 0$,

$$q \leq \frac{C}{g\sqrt{c''}} 2^{(M+1)c''} 2^{-(M-1)M/2} \Delta'^{c''},$$

so

$$(2^M p)^{M+1} q g \leq \frac{C}{\sqrt{c''}} (\frac{\sqrt{2}}{e} c'')^{M+1} 2^{M(M+3)/2} \Delta''^{M+1} \Delta'^{c''}, \tag{17}$$

where

$$\Delta'' := \frac{1}{p'+1}(1-(\log_2 g)^{-1/4}).$$

The logarithm of the right-hand side of (17) is

$$-2^{1/4} c''^{1/2} + \frac{M(M+3)\log 2}{2} + O(M \log\log g),$$

so we can choose $G_+$ so that the right-hand side of (17) is less than $\frac{1}{8}$.

For (B), it will suffice to show that for each $i = 1, \ldots, L$, $v_i$ is unpebblable with probability no more than $1/(4gL)$. We fix some $i$ and let $\delta := 1$ if $i \leq L/2$, $\delta := -1$ if $i > L/2$; we now try to move pebbles onto $v_i$ from $v_{i+\delta}$, $v_{i+2\delta}$, $\ldots$, $v_{i+\lfloor L/2 \rfloor \delta}$. The proof is then similar to (A), except that $L-M-1$ is replaced by $\lfloor L/2 \rfloor - M$; also, since we have assumed that $L \leq e^{\sqrt{c''}}$, we must bound $e^{\sqrt{c''}}(2^M p)^{M+1} X$ instead of $(2^M p)^{M+1} X$. $\square$

## 8 The bouquet of paths

For positive $n$ and $L$ and nonnegative $g$ such that $g(L-1)+1 \leq n$, let the graph $\mathcal{B}_{n,g,L}$ be the graph which has $n$ vertices and is made by taking $g$ paths with $L$ vertices each and a complete graph, choosing one vertex from the complete graph and one end-vertex from each of the paths, and identifying these $g+1$ vertices into a single vertex. Also, for a graph $H$ with $n > 0$ vertices, we define the *geometric pebbling threshold* of $H$ to be the unique positive real $x$ for which, if an independent, geometrically distributed number of pebbles with parameter $(1 + x/n)^{-1}$ is placed on each of the vertices of $H$, the probability of the distribution being solvable is $\frac{1}{2}$. (See Theorem 7 for a proof that this probability is strictly increasing with $x$ and that this definition is sensible.)

**Lemma 20.** *For all $\delta > 0$ there is some $m_0 = m_0(\delta)$ such that, if $\alpha \in \mathbb{R}_{\geq 0}$, $N$ is a sum of $m$ independent geometric random variables with parameter $(1+\alpha)^{-1}$, and both $m$ and $\alpha m$ exceed $m_0$, then*

$$\mathbb{P}(|N - \alpha m| \leq \delta \alpha m) \geq \frac{8}{9}.$$

*Proof.* A geometric random variable with parameter $(1+\alpha)^{-1}$ has mean $\alpha$ and variance $\alpha(1+\alpha)$, so $N$ has mean $m\alpha$ and variance $m\alpha(1+\alpha)$. Then, use Chebyshev's inequality. □

**Proposition 21.** *There is some integer $G_0 \geq 3$ such that if $g \geq G_0$, $2gL \leq n$, and*

$$\sqrt{2\log_2 g} \leq L - \log_2 n \leq \exp(2\log_2 g)^{1/4},$$

*then the geometric pebbling threshold of $\mathcal{B}_{n,g,L}$ is $\alpha n$, where*

$$\alpha := \beta(1+\eta)^{-1}, \quad \beta := \frac{2^{\sqrt{2\log_2 g}}e}{2\sqrt{\log_2 g}}, \qquad |\eta| \leq (\log_2 g)^{-1/4}.$$

*Proof.* Let $H := \mathcal{B}_{n,g,L}$ and let $\alpha := \beta(1+\eta)^{-1}$, where $\eta$ is now arbitrary but satisfies $|\eta| \leq (\log_2 g)^{-1/4}$. We have $L \geq 2$ and by taking $G_0$ large enough we can ensure that $g \geq G_-$, $g \geq G_+$, $\beta \geq 12$, $\alpha \geq 1$, $\log_2 n \geq 1.2\sqrt{2\log_2 g}$, and $\exp(2\log_2 g)^{1/4} \geq \lceil 2.2\sqrt{2\log_2 g}\rceil$. Suppose that an independent, geometrically distributed number of pebbles with parameter $(1+\alpha)^{-1}$ is placed on each vertex of $H$.

Set $\eta := (\log_2 g)^{-1/4}$, consider one of the paths $v_1, \ldots, v_L$ which was identified to make $H$, and let its unidentified end-vertex be $v_1$. Let the total number of pebbles on $H$ be $N$, assume that $N \leq 2\alpha n$, and let $V$ be the set of vertices of $H$ apart from $v_1, \ldots, v_{L-1}$. Then

$$\sum_{x \in V} \mathcal{Z}(x) 2^{-d(x,v_1)} \leq 2^{-(L-1)} \sum_{x \in V} \mathcal{Z}(x) \leq N 2^{-(L-1)} \leq 2\alpha n 2^{-(L-1)}$$

and

$$2\alpha n 2^{-(L-1)} \leq 4\beta 2^{-\sqrt{2\log_2 g}} = \frac{2e}{\sqrt{\log_2 g}},$$

so we can apply Lemma 18 to this path (with $L$ decreased by 1) to show that $v_1$ is unpebblable with probability at least $2/g$. After doing this to each of the paths in $H$ in turn we can conclude that the probability that the distribution is solvable is no more than

$$\mathbb{P}(N > 2\alpha n) + (1 - \frac{2}{g})^g \leq \mathbb{P}(N > 2\alpha n) + e^{-2}. \tag{18}$$

If we apply Lemma 20 (with $m := n$), then, since $\alpha \geq 1$ and $n \geq 2gL$, we can choose $G_0$ so that $n$ and $\alpha n$ are forced to be so large that $\mathbb{P}(|N - \alpha n| \leq \alpha n) \geq \frac{8}{9}$. Since $\frac{1}{9} + e^{-2} < \frac{1}{2}$, (18) then implies that the geometric pebbling threshold of $\mathcal{B}_{n,g,L}$ is at least $\beta n(1 + (\log_2 g)^{-1/4})^{-1}$.

Set $\eta := -(\log_2 g)^{-1/4}$. Let $N'$ be the total number of pebbles on the vertices of the complete graph which was identified to make $H$, and let there be $m$ of these vertices. If $N' \geq \alpha m/2$, then, since $m = n - g(L-1) \geq n - gL \geq n/2$,

$$N' \geq \frac{\alpha m}{2} \geq \frac{\alpha n}{4} \geq \frac{\beta n}{4} \geq 3n.$$

By moving from vertices in the complete graph to any vertex $w$ of the complete graph, we can then place at least $(3n - m)/2 \geq (3n - n)/2 = n$ pebbles on $w$. Now, again consider one of the paths $v_1, \ldots, v_L$ which was identified to make $H$, letting its unidentified end-vertex be $v_1$. By moving from vertices in the complete graph to $v_L$, we can place at least $n$ pebbles on $v_L$; by moving along the path, we can then place at least one pebble on $v_{L-j}$, for any $j = 1, \ldots, \lceil \log_2 n \rceil - 1$. If $L - \lceil \log_2 n \rceil \geq 2.2\sqrt{2 \log_2 g}$, we can apply Lemma 19 to the path with $L$ decreased by $\lceil \log_2 n \rceil$ and conclude that, with probability at least $1 - 1/(4g)$, each of $v_1, \ldots, v_{L - \lceil \log_2 n \rceil}$ are pebblable; otherwise, we can apply Lemma 19 to the path with $L$ replaced by $\lceil 2.2\sqrt{2 \log_2 g} \rceil$ and conclude that, with probability at least $1 - 1/(4g)$, each of $v_1, \ldots, v_{\lceil 2.2\sqrt{2 \log_2 g} \rceil}$ are pebblable. Applying this reasoning to each path, then, the probability that $H$ is solvable is at least

$$\mathbb{P}(N' \geq \frac{\alpha m}{2}) - g\frac{1}{4g} = \mathbb{P}(N' \geq \frac{\alpha m}{2}) - \frac{1}{4}.$$

If we apply Lemma 20, since $m \geq n/2$, we can choose $G_0$ so that $\mathbb{P}(|N' - \alpha m| \leq \frac{1}{2}\alpha m) \geq \frac{8}{9}$. Since $\frac{8}{9} - \frac{1}{4} > \frac{1}{2}$, this means that the geometric pebbling threshold of $\mathcal{B}_{n,g,L}$ is no more than $\beta n(1 - (\log_2 g)^{-1/4})^{-1}$, completing the proof. □

**Lemma 22.** *If we define* $\Phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *by*

$$\Phi(\alpha) := \frac{\alpha^2}{2\alpha + 1},$$

*then* $\Phi$ *is a strictly increasing bijection, and if* $\alpha_2 > \alpha_1 > 0$,

$$(\frac{\alpha_2}{\alpha_1})^2 \geq \frac{\Phi(\alpha_2)}{\Phi(\alpha_1)} \geq \frac{\alpha_2}{\alpha_1}.$$

*Proof.* Easy. □

The following result is similar to [6, Theorem 4].

**Proposition 23.** *For any $0 < \epsilon < 1$, there is some integer $L_0 = L_0(\epsilon) \geq 2$ such that if $g \geq 1$, $2gL \leq \epsilon n$, and $L_0 \leq L \leq (\log_2 n) - L_0$, then the geometric pebbling threshold of $\mathcal{B}_{n,g,L}$ is $\alpha n$, where*

$$\alpha := \beta(1+\eta), \quad \beta > 0 \text{ and } \Phi(\beta) = \frac{2^{L-1}}{n}, \qquad |\eta| \leq \epsilon.$$

*Proof.* Fix $\epsilon$, let $H := \mathcal{B}_{n,g,L}$ and $\alpha := \beta(1+\eta)$, where $\eta$ is arbitrary such that $|\eta| \leq \epsilon$, and suppose that an independent, geometrically distributed number of pebbles with parameter $(1+\alpha)^{-1}$ is placed on each vertex of $H$.

Set $\eta := -\epsilon$ and let $v_1, \ldots, v_L$ be one of the paths which was identified to make $H$, with $v_1$ being the unidentified end vertex. If $V$ is the set of vertices in $H$ other than $v_1, \ldots, v_L$, then

$$\mathcal{Z}(v_1) + \frac{\mathcal{Z}(v_2)}{2} + \cdots + \frac{\mathcal{Z}(v_L)}{2^{L-1}} + 2^{-(L-1)} \sum_{x \in V} \lfloor \frac{\mathcal{Z}(x)}{2} \rfloor$$

is at least 1 if there is a pebble on $v_1$, and it cannot be increased by pebbling moves. Arguing as in Lemma 18, then, the probability that $v_1$ is unpebblable conditioned on the distribution on $V$ is at least

$$\mathbb{P}(Y_\infty < \lambda(1 - 2^{-(L-1)}N)),$$

where

$$N := \sum_{x \in V} \lfloor \frac{\mathcal{Z}(x)}{2} \rfloor, \quad \lambda := \log(1 + \alpha^{-1}).$$

Now, for each $x$, $\lfloor \mathcal{Z}(x)/2 \rfloor$ is independently geometrically distributed with parameter $1 - (1 - (1+\alpha)^{-1})^2 = (1 + \Phi(\alpha))^{-1}$. Also, $n \geq 2gL \geq 2L_0$, so if $m$ is the number of vertices in $V$, then

$$m \geq n - gL \geq n(1 - \frac{\epsilon}{2}) \geq L_0,$$

and, by Lemma 22,

$$\Phi(\alpha)m \leq \Phi(\beta)(1-\epsilon)m = \frac{2^{L-1}m}{n}(1-\epsilon) \leq 2^{L-1}(1-\epsilon)$$

and

$$\Phi(\alpha)m \geq \Phi(\beta)(1-\epsilon)^2 m = \frac{2^{L-1}m}{n}(1-\epsilon)^2 \geq 2^{L_0-1}(1-\epsilon)^2(1-\frac{\epsilon}{2}).$$

Using Lemma 20, then, we can choose $L_0$ large enough so that $N$ is no more than $2^{L-1}(1 - (\epsilon/2))$ with probability at least $\frac{8}{9}$, so the probability that $v_1$ is unpebblable is at least

$$\frac{8}{9}\mathbb{P}(Y_\infty < \frac{\lambda\epsilon}{2}). \tag{19}$$

Since $L \le (\log_2 n) - L_0$, we have $\Phi(\beta) \le 2^{-L_0-1}$, so we may choose $L_0$ large enough so that $\beta$ is forced to be small enough, and $\lambda\epsilon/2$ large enough, so that (19) is greater than $\frac{1}{2}$. This proves that, for an appropriate choice of $L_0$, the geometric pebbling threshold of $\mathcal{B}_{n,g,L}$ is at least $\beta n(1-\epsilon)$.

Now, set $\eta := \epsilon$, let $V'$ be the set of vertices on the complete graph which was identified to make $H$, let $V'$ have size $m'$, and let $N' := \sum_{x \in V'} \lfloor \mathcal{Z}(x)/2 \rfloor$. If $N' \ge 2^{L-1}$, we can place $2^{L-1}$ pebbles on the identified vertex in $H$, and from there place at least one pebble on any vertex in $H$. We need then to show that $\mathbb{P}(N' \ge 2^{L-1}) > \frac{1}{2}$. As before, for each $x$, $\lfloor \mathcal{Z}(x)/2 \rfloor$ is independently geometrically distributed with parameter $(1+\Phi(\alpha))^{-1}$, so since

$$m' \ge n - gL \ge n(1 - \frac{\epsilon}{2}) \ge L_0$$

and, by Lemma 22,

$$\begin{aligned} \Phi(\alpha)m' \ge \Phi(\beta)(1+\epsilon)m' \ge 2^{L-1}(1+\epsilon)(1-\frac{\epsilon}{2}) &= 2^{L-1}(1+\frac{\epsilon(1-\epsilon)}{2}) \\ &\ge 2^{L_0-1}, \end{aligned}$$

we can choose $L_0$ large enough so that, by Lemma 20, $\mathbb{P}(N' \ge 2^{L-1}) \ge \frac{8}{9}$. This proves that the geometric pebbling threshold of $\mathcal{B}_{n,g,L}$ is no more than $\beta n(1+\epsilon)$, completing the proof. □

# 9 The pebbling threshold spectrum

**Lemma 24.** *Let $H$ be a connected graph with $n \ge 2$ vertices, let $v$ be a vertex of $H$ such that all vertices of $H$ are within distance $d \in \mathbb{Z}$ of $v$, $d \ge 2$, and let each vertex of $H$ have a number of pebbles which is independently geometrically distributed with parameter $(1+\alpha)^{-1}$, $\alpha \in \mathbb{R}_{>0}$. Then $v$ is unpebblable with probability at most*

$$\left( \frac{e(2^{d-1} + \lceil n^{1/d} - 1 \rceil - 1)}{\lceil n^{1/d} - 1 \rceil (1+\Phi(\alpha))} \right)^{\lceil n^{1/d} - 1 \rceil}.$$

*Proof.* For $i = 0, 1, \ldots,$ let $D_i$ be the number of vertices at distance $i$ from $v$ and let $D'_i := \sum_{0 \le j \le i} D_j$. Since $\log D'_0 = 0$ and $\log D'_d = \log n$, there must be some $0 \le i \le d-1$ for which $(\log D'_{i+1}) - (\log D'_i) \ge (\log n)/d$, and then $D_{i+1}/D'_i = (D'_{i+1}/D'_i) - 1 \ge n^{1/d} - 1$. Since $D'_i \ge D_i$, $D_{i+1}/D_i$ is also at least $n^{1/d} - 1$. This means that there must be some vertex $w$ at distance $i$ from $v$ which has at least $\lceil n^{1/d} - 1 \rceil$ neighbors at distance $i+1$. Letting a set of $\lceil n^{1/d} - 1 \rceil$ of these neighbors be $V$, $v$ will be pebblable if

$$\sum_{x \in V} \lfloor \mathcal{Z}(x)/2 \rfloor \ge 2^{d-1}, \tag{20}$$

since if so we can move $2^{d-1}$ pebbles to $w$ and then place a pebble on $v$. Each $\lfloor \mathcal{Z}(x)/2 \rfloor$ is independently geometrically distributed with parameter $p := (1 +$

$\Phi(\alpha))^{-1}$, so the probability of (20) is the probability that, if we flip a coin with success probability $p$, it takes at least $2^{d-1} + \lceil n^{1/d} - 1\rceil$ flips to get $\lceil n^{1/d} - 1\rceil$ successes. Another way of saying this is that there are no more than $\lceil n^{1/d}-1\rceil-1$ successes in $2^{d-1} + \lceil n^{1/d} - 1\rceil - 1$ flips, so if (20) is false, there must be at least $\lceil n^{1/d} - 1\rceil$ successes in this number of flips. Set

$$p' := \lceil n^{1/d} - 1\rceil/(2^{d-1} + \lceil n^{1/d} - 1\rceil - 1).$$

If $\lceil n^{1/d} - 1\rceil(1 + \Phi(\alpha)) \leq 2^{d-1} + \lceil n^{1/d} - 1\rceil - 1$, then the claimed bound on the probability of unpebblability is 1 or greater and there is nothing to prove. We can assume then that $\lceil n^{1/d} - 1\rceil(1 + \Phi(\alpha)) > 2^{d-1} + \lceil n^{1/d} - 1\rceil - 1$; now $p < p' < 1$, so by [1, Theorem 1], the probability that (20) is false is no more than

$$\exp -(2^{d-1} + \lceil n^{1/d} - 1\rceil - 1)\Omega, \quad \text{where}$$
$$\Omega := p' \log \frac{p'}{p} + (1 - p') \log \frac{1 - p'}{1 - p}. \tag{21}$$

Since $y \log y \geq y - 1$ for all $0 < y < 1$, $\Omega \geq p'(-1 + \log(p'/p))$. Together with (21), this gives the claimed bound. □

**Theorem 25.** *There is some $n_0 \geq 3$ such that if $H$ is a connected graph with $n \geq n_0$ vertices, then the geometric pebbling threshold of $H$ is no more than*

$$\frac{2^{\sqrt{2\log_2 n}}e}{2\sqrt{\log_2 n}}(1 - (\log_2 n)^{-1/4})^{-1}n.$$

*Proof.* Choose $n_0$ such that $n_0 \geq G_+$. Let $p := (1 + (2^{\sqrt{2\log_2 n}}e/(2(1 - (\log_2 n)^{-1/4})\sqrt{\log_2 n})))^{-1}$, and suppose that an independent, geometrically distributed number of pebbles with parameter $p$ is placed on each vertex of $H$. Pick some vertex $v$ of $H$. It will do to show that $v$ is pebblable with probability at least $1 - 1/(4n)$. If there is some vertex $x$ of $H$ with $d(v, x) \geq 1.1\sqrt{2\log_2 n}$, then this follows immediately from Lemma 19. Otherwise, apply Lemma 24 with $d := \lceil 1.1\sqrt{2\log_2 n}\rceil$. For an appropriate choice of $n_0$, this will always show that $v$ is unpebblable with probability at most $1/(4n)$. □

**Theorem 26.** *There is some $n_1 \geq 1$ such that, if $H$ is a graph with $n \geq n_1$ vertices, then the geometric pebbling threshold of $H$ is at least $\sqrt{n \log 2}$.*

*Proof.* A distribution on any graph with $n \geq 1$ vertices will not be solvable if no vertex has two or more pebbles and some vertex has no pebbles. If the number of pebbles on each vertex is independently geometrically distributed with parameter $p$, the probability of this event is $q := (1-(1-p)^2)^n - (p(1-p))^n$. For large $n$, if $p := (1+\sqrt{(\log 2)/n})^{-1} = 1-\sqrt{(\log 2)/n}+((\log 2)/n)+O(n^{-3/2})$, then $q = \frac{1}{2} + (\log 2)^{3/2}n^{-1/2} + O(n^{-1})$, which eventually exceeds $\frac{1}{2}$. □

**Corollary 27.** *If $(H_i)_{i\in\mathbb{Z}_{>0}}$ is any sequence of connected graphs such that the number of vertices in $H_i$ is strictly increasing with $i$, then the sequence has some pebbling threshold $t(n)$ which is $\Omega(\sqrt{n})$ and $O(2^{\sqrt{2\log_2 n}}n/\sqrt{\log_2 n})$, where $n$ is the number of vertices in a graph in the sequence.*

*Proof.* By Theorems 25 and 26, for sufficiently large $i$, the geometric pebbling threshold $T_i$ of $H_i$ satisfies

$$\sqrt{n_i \log 2} \leq T_i \leq \frac{2^{\sqrt{2\log_2 n_i}} e}{2\sqrt{\log_2 n_i}} (1 - (\log_2 n_i)^{-1/4})^{-1} n_i,$$

where $n_i$ is the number of vertices in $H_i$. Define the function $t(n)$ by $t(n_i) := T_i$ for each $i \in \mathbb{Z}_{>0}$ and $t(n) := n$ if $n$ is not equal to any $n_i$. Now apply Theorem 11 and argue as in the proof of [2, Theorem 1.3] to prove that $t(n)$ is a pebbling threshold for $(H_i)_{i\in\mathbb{Z}_{>0}}$. □

**Theorem 28.** *There is some constant $K > 1$ such that, if $n \geq 2$ is an integer and*

$$\sqrt{n} \leq t \leq \frac{2^{\sqrt{2\log_2 n}}}{\sqrt{\log_2 n}} n,$$

*then there is some connected graph $H$ with $n$ vertices whose geometric pebbling threshold is between $t/K$ and $Kt$.*

*Proof.* Set $L_0 := L_0(\frac{1}{2})$. We are free to choose an arbitrary connected graph for $H$ for a finite number of values of $n$, at the cost of worsening $K$, so we can assume that $n \geq 2^{2L_0}$ and $n/(4\log_2 n) \geq G_0$. Then, we will always choose $H$ to be some $\mathcal{B}_{n,g,L}$. Set $\beta := t/n$ and $\beta_c := 2^{\sqrt{2\log_2 G_0}} e/(2\sqrt{\log_2 G_0})$.

1. If $\beta < \beta_c$, $g$ will always be 1. Let $\hat{L} := 1 + \log_2(\Phi(\beta)n)$. Then we let $L$ be $L_0$ if $\hat{L} < L_0$, $\lfloor \hat{L} \rfloor$ if $L_0 \leq \hat{L} \leq (\log_2 n) - L_0$, and $\lfloor \log_2 n \rfloor - L_0$ if $\hat{L} > (\log_2 n) - L_0$.
2. If $\beta \geq \beta_c$, let $g$ be the maximal integer in $G_0, G_0 + 1, \ldots, \lfloor n/(4\log_2 n) \rfloor$ with $2^{\sqrt{2\log_2 g}} e/(2\sqrt{\log_2 g}) \leq \beta$. Let $L$ be $\lceil (\log_2 n) + \sqrt{2\log_2 g} \rceil$.

It is straightforward to verify that, regardless of $t$ or $n$, the geometric pebbling threshold $t'$ of $H$ can then be computed with Proposition 21 or Proposition 23, and that there is some absolute constant $K > 1$ such that $t'/t$ is always in $[1/K, K]$. □

**Corollary 29.** *If $t(n)$ is any positive function of integral $n \geq 1$ which is $\Omega(\sqrt{n})$ and $O(2^{\sqrt{2\log_2 n}} n/\sqrt{\log_2 n})$, then there is some sequence of connected graphs $(H_n)_{n\in\mathbb{Z}_{>0}}$ with pebbling threshold $t(n)$ such that $H_n$ has $n$ vertices for each $n$.*

*Proof.* Let $t'(1) := 1$ and for all integral $n \geq 2$, let

$$t'(n) := \min(\max(t(n), \sqrt{n}), 2^{\sqrt{2\log_2 n}} n/\sqrt{\log_2 n}).$$

Let $H_1$ be the 1-vertex graph, which has geometric pebbling threshold $t''(1) := 1$, and, for each $n \geq 2$, let $H_n$ be the connected graph given by Theorem 28 which has $n$ vertices and geometric pebbling threshold $t''(n)$ between $t'(n)/K$ and $Kt'(n)$. Then, apply Theorem 11 and [2, Theorem 1.3] to prove that $t''(n)$ is a pebbling threshold of $(H_n)_{n\in\mathbb{Z}_{>0}}$. It follows that $t'(n)$ and $t(n)$ are also pebbling thresholds for $(H_n)_{n\in\mathbb{Z}_{>0}}$. □

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