SECOND COURSE IN ALGEBRA

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PREFACE

This book is designed to follow the authors' "First Course in Algebra" or any other text of similar scope and treatment. Experience shows that when a student returns to the study of algebra, after even a summer's vacation, a review is very necessary; and that it is absolutely indispensable if he comes back after a year spent on geometry. The review presented in the early chapters is brief, yet sufficiently thorough. Each review topic has been given a broader and more advanced treatment than is permissible in a first course. New material is used throughout and many new applications are given in order to make the entire review appeal to the student as fresh and inviting.

In the chapters which deal with the subjects not given in the "First Course," the aim has been to select those topics considered necessary for the best secondary schools and to treat each in a clear, practical, and attractive manner. It has been the purpose also to prepare a text that will lead the student to think clearly, as well as to acquire the necessary facility on the technical side of algebra. Lastly, it has been the desire to reduce the work of explanation and illustration on the teacher's part to a minimum. To accomplish these things every legitimate resource has been employed. The material has been carefully selected and graded, the explanations are unusually full, and the illustrative examples are especially numerous. Whenever graphs appeared to clarify a subject, they have been used; and if at any point an explanatory note or a bit of mathematical history seemed pertinent, it has been given. Along with the endeavor to accomplish these various ends a continuous effort has been made to produce a text that is modern, lucid, mathematically correct, and interesting.

We are under especial obligation for suggestions and criticisms, not only to those mentioned in the preface to our "First Course," but also to Mr. J. M: McPherron of Los Angeles, California; Mr. J. A. Foberg of Chicago, Illinois; and Mr. A. E. Booth of New Haven, Connecticut, who have read the manuscript critically. For a careful reading of the entire proof and for many helpful comments we are indebted to Mr. J. A. Avery of Somerville, Massachusetts, and Professor I. M. De Long of Boulder, Colorado.

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SECOND COURSE IN ALGEBRA

CHAPTER I

FUNDAMENTAL OPERATIONS

1. Order of fundamental operations. The numerical value of an arithmetical or an algebraic expression involving signs of addition, subtraction, multiplication, and division depends on the order in which the indicated operations are performed. It is understood that:

In a series of operations involving addition, subtraction, multiplication, and division, first the multiplications and divisions shall be performed in the order in which they occur. Then the additions and subtractions shall be performed in the order in which they occur or in any other order.

Within any parenthesis the preceding rule applies.

EXERCISES

Simplify:

1. $3 - 5 + 6 - 8.$ **3.** $24 \div 8 \cdot 4 - 4 + 6.$

2. $6 \div 2 + 1 - 4.$ **4.** $(7 - 6)(18 - 2 \cdot 4) \div (20 \div 4).$

5. $42 - 2(18 - 2 \cdot 3) \div 4 + 3 \cdot 5.$

6. $16 + 4 \div 8 - 10 + 51 \div 16 - 4 - 6 \cdot 3 \cdot 0 \cdot 2 + 18 \cdot 8 \div 48 - 2 \cdot 18 \div 12.$

7. $(16 \div 32 \times 48 \div 8 - 4 - 8 + 3) \times [12 \div 4 \div 3 - 1] + (42 \div 6 \cdot 7 - 42 - 6) \cdot 6.$

8. Does $a^4 = 4\,a$ when $a = 3$? when $a = 2$? when $a = 0$?

9. What name is given to each 4 in $a^4 = 4\,a$? Define each.

10. Define power. Distinguish between exponent and power.

Find the numerical value of:

11. $x^2 - 5x + 6$ when $x = 5$.

12. $x^3 - 3x^2 + 3x - 1$ when $x = 3$.

13. $x^3 - 3x^2y + 3xy^2 - y^3$ when $x = 4$ and $y = 2$.

14. $\frac{x^4 + x^2y^2 + y^4}{x^2 - xy + y^2} - \frac{x^3 + y^3}{x + y}$ when $x = 3$ and $y = 2$.

15. What is the absolute value of a number? Illustrate.

2. Addition. In algebra, **addition** involves the uniting of similar terms (see definitions below) which have the same or opposite signs into one term. For this we have the rule:

I. *To add two or more positive numbers, find the arithmetical sum of their absolute values and prefix to this sum the plus sign.*

II. *To add two or more negative numbers, find the arithmetical sum of their absolute values and prefix to this sum the minus sign.*

III. *To add a positive and a negative number, find the difference of their absolute values and prefix to this difference the sign of the number which has the greater absolute value.*

Obviously $2 + 4 + 7 = 2 + 7 + 4 = 7 + 2 + 4$, etc. Even if we have a series of positive and negative numbers, the order in which they occur does not affect the final result. This principle of addition is called the **Commutative Law for Addition.**

Similar terms are (a) integers and rational, numerical fractions; (b) like indicated roots, as $\sqrt{2}$ and $3\sqrt{2}$, or $\sqrt[3]{3}$ and $2\sqrt[3]{3}$; (c) terms having like literal parts, as $4a$ and $3a$, or $6xy^2$ and $\sqrt{2}\,xy^2$.

Dissimilar terms are unlike indicated roots or terms having unlike literal parts.

For the addition of polynomials we have the

RULE. *Write similar terms in the same column.*

Find the algebraic sum of the terms in each column and write the results in succession with their proper signs.

3. Subtraction. For the subtraction of polynomials we have the following

RULE. *Write the subtrahend under the minuend so that similar terms are in the same column.*

Change mentally the sign of each term of the subtrahend. Then find the algebraic sum of the terms in each column, and write the results in succession with their proper signs.

EXERCISES

Add:

1. 16, -3, $+2$, -8, -7, and 4.
2. $4a$, $-6a$, $-10a$, $+2a$, and $18a$.
3. $4x-3y+7$, $8x-10y-11$, and $10y-30+7x$.
4. $7x-4y-z$, $3x+z-8y$, and $18y-17x-14z$.
5. $4a^2-3a^2c-4ac^2$, $3a^2c-8ac^2-8a^2$, and $3a^2-6a^2c$.
6. If $x=1$, $y=2$, and $z=3$, find the numerical value of each of the three expressions and of the result obtained in Exercise 4. Compare the sum of the three numerical values with the numerical value of the result.
7. State a rule for checking work in addition of algebraic expressions.

Write with polynomial coefficients:

8. $ay+by+cy$.
9. $3ax-4bx+6x$.
10. $4x-abx-x$.
11. $7x-3ax-4a^2x$.
12. $3(a+b)-c(a+b)$.
13. $6a(x-2c)-3(x-2c)$.
14. $4b(3x-2)-8c(3x-2)$.
15. $4m(5a-3c)-6n(-3c+5a)$.

Subtract the first expression from the second in:

16. $4a$, $6a$.
17. $8a^3$, $5a^3$.
18. $4x+3$, $8x+6$.
19. $7x^2-10$, $5x^2+20$.
20. $x-3y^2+z-4ac+7ax$, $4x-y^2+8-5ax+9ac$.
21. $a^3-c+3x-a^2m-8ac$, $4a^3+m-8x-10ac+4a^2m$.

Find the expression which added to the first will give the second in:

22. $x^2 - 5x + 6$, $3x^2 - 5x + 2$.

23. $4x^2 - 3cx + c^2$, $8c^2 + 7cx - 10x^2 + 8$.

Find the expression which subtracted from the first will give the second in:

24. $4a^2 - 2ab + b^2$, $7a^2 - 10ab + 6b^2$.

25. $c^2 - 10cx + 8x^2$, $9x^2 - 10cx + 4 + c^2$.

26. State a rule for checking work in subtraction suggested by the directions preceding Exercises 22 and 24.

27. From the sum of $ax - ac - 3c^2$ and $4c^2 - 3ac$ take the sum of $4c^2 - 8ax + a^2$ and $4ac + 3ax - 5c^2$.

Remove parentheses and combine like terms:

28. $4x - 3 - (a - 2x) + (3x - a)$.

29. $6x + (3c - 8x + 2) - (c - x - 2)$.

30. $6x - [-(a - c) + (3c - 4a)]$.

31. $7c - [(3c - 4) - 6 - (4x - 3a - c)]$.

32. $4x - 2(x - 3) - 3[x - 3(4 - 2x) + 8]$.

33. $6x - 4(3 - 5x) - 4[2(x - 4) + 3(2x - 1) - (x - 7)]$.

34. $3x - 2[1 - 3(2x - 3 - a) - 5\{a - (3x - 2a) - 4\}]$.

35. State the rule for the removal of a parenthesis

(*a*) when it is preceded by the sign plus;
(*b*) when it is preceded by the sign minus.

Inclose in a parenthesis preceded by the sign plus those terms which contain x and y, and inclose all other terms in a parenthesis preceded by the sign minus.

36. $x^2 + 2xy + y^2 - a^2$. **37.** $x^2 + 14ab - 49a^2 - b^2$.

38. $y^2 + 6xy + 9x^2 - m^2 - 10m - 25$.

39. $x^4 + 10x^2y^3 - c^8 + 12c^4d - 36d^2 + 25y^6$.

40. State the rules for inclosing terms in a parenthesis preceded by (*a*) the sign plus; (*b*) the sign minus.

4. Multiplication. In multiplying one term by another the sign of the product, the coefficient of the product, and the exponent of any letter in the product are obtained as follows:

I. *The sign of the product is* **plus** *if the multiplier and the multiplicand have like signs, and* **minus** *if they have unlike signs.*

II. *The coefficient of the product is the product of the coefficients of the factors.*

III. *The exponent of each letter in the product is determined by the general law*

$$n^a \times n^b = n^{a+b}.$$

For the multiplication of polynomials we have the

Rule. *Multiply the multiplicand by each term of the multiplier in turn, and add the partial products.*

An extension of the law for exponents in multiplication is the **Law of Involution:**

$$(n^a)^b = n^{ab}.$$

This last law implies the more general forms:

$$(x^a y^b)^c = x^{ac} y^{bc},$$

and

$$((x^a)^b)^c = x^{abc}.$$

5. Division. In dividing one term by another the sign of the quotient, the coefficient of the quotient, and the exponent of any letter in the quotient are obtained as follows:

I. *The sign of the quotient is* **plus** *when the dividend and the divisor have like signs, and* **minus** *when they have unlike signs.*

II. *The coefficient of the quotient is the quotient of the coefficient of the dividend by that of the divisor.*

III. *The exponent of each letter in the quotient is determined by the law*

$$n^a \div n^b = n^{a-b}.$$

The method of dividing one polynomial by another is stated in the

Rule. *Arrange the dividend and the divisor according to the descending powers of some common letter, called the letter of arrangement.*

Divide the first term of the dividend by the first term of the divisor and write the result for the first term of the quotient.

Multiply the entire divisor by the first term of the quotient, write the result under the dividend, and subtract, being careful to write the terms of the remainder in the same order as those of the divisor.

Divide the first term of the remainder by the first term of the divisor for the second term of the quotient, and proceed as before until there is no remainder, or until the remainder is of lower degree (§ 8) in the letter of arrangement than the divisor.

6. Meaning of a zero exponent. The laws for exponents stated in the formulas of §§ 4 and 5 are assumed to hold for **all** values of a and b.

Then $$x^a \div x^a = x^{a-a} = x^0.$$

But $$\frac{x^a}{x^a} = 1.$$

Hence $$\boldsymbol{x^0 = 1.}$$

That is, any number (except zero) whose exponent is zero is equal to 1.

7. Meaning of a negative exponent. If, in the formula of § 5, b is greater than a, we obtain a negative exponent for n. The meaning of such an exponent is illustrated as follows:

By § 5, $$\frac{x^a}{x^{a+b}} = x^{a-(a+b)} = x^{-b}.$$

But dividing both terms by x^a,

$$\frac{x^a}{x^{a+b}} \text{ or } \frac{x^a}{x^a x^b} = \frac{1}{x^b}.$$

Therefore $$\boldsymbol{x^{-b} = \frac{1}{x^b}.}$$

More generally, $\boldsymbol{cx^{-a} = \frac{c}{x^a}}$, and $\boldsymbol{\frac{c}{x^{-a}} = cx^a}$.

Hereafter it will be assumed that all the preceding exponential laws hold for positive, negative, zero, and fractional exponents.

EXERCISES

Perform the indicated operation:

1. $(4x^2 - 3x)(2x)$. **2.** $(2x + 3)(5x - 6)$.

3. Substitute 2 for x in each of the factors of Exercise 2, and in the product. Compare the numerical value of the product with the product of the numerical values of the factors. Then state a method of checking numerically work in multiplication.

4. $(3x^2 - 1)^2$.

5. $(7x^{2a} - 8x^a + 3)^2$.

6. $(x^{\frac{1}{2}} + x^{\frac{1}{3}})^2$.

7. $(x^{\frac{1}{2}} - x)^2$.

8. $(e^1 + e^{-1})^2$.

9. $(e^x + 2e^{-x})^2$.

10. $(e^x - e^{-x})^3$.

11. $(e^{2x} - 3e^{-x})^4$.

12. $(x^{\frac{1}{2}} + x^{\frac{1}{4}} + 1)(x^{\frac{1}{2}} - x^{\frac{1}{4}} + 1)$.

13. $\left(\frac{2x^2}{3} - \frac{3}{4}x - 2\right)\left(\frac{4x}{5} - \frac{x^2}{6} - \frac{1}{2}\right)$.

14. $(4x^{3e} - 6x^e + 3)(7x^{3e} - x^{2e} + 4)$.

15. $(x^2 - 2xy^2 + y^4)(x^2 + 2xy^2 + y^4)$.

16. $(x^{-1} - 3x - 2x^{-2})^2$.

17. $(x^{-\frac{1}{2}} + 2x^{\frac{1}{2}} - 3x^{\frac{3}{2}})^2$.

18. $\left(\frac{2a^2}{3} - \frac{a}{5} + \frac{2}{7}\right)\left(\frac{2a^3}{3} + \frac{a^2}{5} - \frac{2a}{7}\right)$.

19. $(5x^{2a} - 3x^{-2a} - 6x^{-a} + 8x^a)^2$.

20. $x^2 - x - 90 = ?$ if $x = -9$.

21. $x^2 - 4xy^2 + 4y^4 = ?$ if $x = 3$ and $y = 2$.

22. $x^3 - 3x^2y + 3xy^2 - y^3 = ?$ if $x = 2$ and $y = -3$.

23. $x^3 + 3x^2y + 3xy^2 + y^3 = ?$ if $x = -4$ and $y = -2$.

24. $(e^1 - e^{-1})^2 = ?$ if $e = 2$; if $e = -3$.

25. $e^{2x} - 2e^0 + e^{-2x} = ?$ if $e = 2$ and $x = 2$.

26. $(8x^4 - 6x^2 - 4x) \div (-2x)$.

27. $(x^2 - 7x + 12) \div (x - 3)$.

28. State the Associative Law of Multiplication. Illustrate.

29. State the Distributive Law of Multiplication. Illustrate.

30. $(x^3 - 64) \div (x - 4)(x^2 - 4x + 16)$.

31. $(x^4 - 8x^2 + 33x - 30) \div (x^2 + 3x - 5)$.

32. State a method of checking work in division similar to the check of multiplication.

Find the remainder in:

33. $(8x^3 - x^2 - 5) \div (2x - 3)$.

34. $(4x^4 - x^2 - 3) \div (2x^2 - x - 1)$.

Divide:

35. $x^3 + 8y^3 + 125 - 30xy$ by $x + 2y + 5$.

36. $x^3 + y^3 + z^3 - 3xyz$ by $x + y + z$.

37. $a^{\frac{3}{2}} - ab^{\frac{1}{2}} + a^{\frac{1}{2}}b - b^{\frac{3}{2}}$ by $a^{\frac{1}{2}} - b^{\frac{1}{2}}$.

38. $3x^{-10} + x^6 - 4x^{-6}$ by $2x^{-2} + x^2 + 3x^{-6}$.

39. $x^{\frac{3}{5}} - y^{\frac{3}{5}}$ by $[(x^{\frac{1}{5}} - y^{\frac{1}{5}}) \div (x^{\frac{1}{10}} + y^{\frac{1}{10}})]$.

40. $9m + 4m^{-1} - 13$ by $3m^{\frac{1}{2}} - 5 + 2m^{-\frac{1}{2}}$.

41. $x^{2a} + 4x^{-2a} - 29$ by $x^a - 2x^{-a} - 5$.

42. $9x^{2a} + 25x^{-4a} - 19x^{-a}$ by $5x^{-2a} + 3x^a - 7x^{-\frac{a}{2}}$.

43. $\left(\frac{x^3}{8} + \frac{27y^3}{64}\right) \div \left(\frac{3x}{2} + \frac{9y}{4}\right)(64x^2 + 96xy + 144y^2)$.

44. $\left(6a^3 + 6x^3 + \frac{35ax^2}{2} + \frac{35a^2x}{3}\right) \div \left(\frac{3a}{2} + \frac{2x}{3}\right)$.

45. $\left(\frac{9a^5}{5} + \frac{243a^3}{20} - 12 + \frac{59a}{4} - \frac{443a^2}{30} - \frac{43a^4}{8}\right)$
$\div \left(\frac{3a^3}{4} + a - \frac{3}{2} - \frac{5a^2}{6}\right)$.

8. Detached coefficients. A term is **rational** if it may be obtained from unity and the letters involved by means of the four fundamental operations without the extraction of any root.

A term is **integral** if it has no literal denominator and the exponent of each factor is a *positive* integer (or zero).

The **degree** of a rational integral term is the sum of the exponents of the letters in the term.

An algebraic expression is rational and integral if its terms are rational and integral.

An integral expression may not be rational. Nor is every rational expression integral. Thus, $\frac{x^2}{4} + \sqrt{x} + 8$ is integral but not rational, while $\frac{x^2}{4} + \frac{1}{x} + 8$ is rational but not integral.

A rational integral expression is **homogeneous** if its terms are all of the same degree.

In the multiplication or division of polynomials which involve but one letter or which are homogeneous in two letters much labor can be saved by using the coefficients only.

EXAMPLES

1. Multiply $3x^3 - 4x + 6$ by $2x^2 - 5x + 3$.

Solution: Since x^2 is missing in the first expression, its coefficient is zero. Inserting $0x^2$ and detaching coefficients, the multiplication is as follows:

$$\begin{array}{rrrrrr}
3 + & 0 - & 4 + & 6 & & \\
2 - & 5 + & 3 & & & \\
\hline
6 + & 0 - & 8 + & 12 & & \\
 & -15 + & 0 + & 20 - & 30 & \\
 & & +9 + & 0 - & 12 + & 18 \\
\hline
6 - & 15 + & 1 + & 32 - & 42 + & 18
\end{array}$$

Supplying the powers of x, we obtain as the product $6x^5 - 15x^4 + x^3 + 32x^2 - 42x + 18$.

2. Divide

$6x^4 - 11x^3y + 2x^2y^2 + 27xy^3 - 18y^4$ by $2x^2 - 5xy + 6y^2$.

Solution:

$$\begin{array}{rrrrr|l}
6 - & 11 + & 2 + & 27 - & 18 & 2 - 5 + 6 \\
\cline{6-6}
6 - & 15 + & 18 & & & 3 + 2 - 3 \\
\cline{1-3}
 & 4 - & 16 + & 27 & & \\
 & 4 - & 10 + & 12 & & \\
\cline{2-5}
 & & -6 + & 15 - & 18 & \\
 & & -6 + & 15 - & 18 & \\
\cline{3-5}
\end{array}$$

Therefore the quotient is $3x^2 + 2xy - 3y^2$.

In both multiplication and division by detached coefficients zero must be supplied for the coefficient of any missing term.

EXERCISES

Use detached coefficients and perform the indicated operation:

1. $(x^2 - 8x + 16)(2x - 3)$.
2. $(x^2 - 4x + 4)(x^2 + 4x + 4)$.
3. $(a^2 - ab + b^2)(a^2 + ab + b^2)$.
4. $(2x^2 + 5x + 2) \div (2x + 1)$.
5. $(x^3 + 4x - 16) \div (x - 2)$.
6. $(3xy - 6y^2 - 2x^2)(8x^2 - 6y^2 - 5xy)$.
7. $(9x^4 - 4x + 13x^2 + 4 - 6x^3) \div (3x^2 - x + 2)$.
8. $(x^4 + 4y^4) \div (x^2 - 2xy + 2y^2)$.
9. $(81a^4 - 171a^2b^2 + 25b^4) \div (9a^2 - 5b^2 + 9ab)$.
10. $(4a^3 - 2a^2 - 3a^{-2} - 5a^{-1} + 2a) \div (2a^2 - 2 - a^{-1})$.
11. $(8x - 12x^{\frac{2}{3}}y^{-1} + 6x^{\frac{1}{3}}y^{-2} - y^{-3}) \div (2x^{\frac{1}{3}} - y^{-1})$.
12. Which expressions in the preceding exercises are (*a*) not integral? (*b*) not rational?

Note. It is interesting to observe that our ordinary decimal notation really involves the use of detached coefficients. The number 649, for instance, is an abbreviated way of writing $6 \cdot 10^2 + 4 \cdot 10 + 9$. In fact, the various digits in any number in the decimal form are the detached coefficients of some power of the number 10.

9. Synthetic division. This method of division abbreviates the actual work, where the divisor is a binomial.

EXAMPLES

1. By long division we have

$$\begin{array}{r|l} 3x^3 - 8x^2 + 9x - 8 & x - 2 \\ \underline{3x^3 - 6x^2} \qquad\qquad & \overline{3x^2 - 2x + 5} \\ -2x^2 + 9x \qquad & \\ \underline{-2x^2 + 4x} \qquad & \\ 5x - 8 & \\ \underline{5x - 10} & \\ 2, \text{ Remainder} & \end{array}$$

The preceding division can be shortened by omitting the letters and arranging the work as was done in Example 2, page 9. But the usual process just given can be abbreviated still further. We may also omit the first number of each partial product, since it is merely a repetition of the number just above it. Thus we obtain

$$\begin{array}{l|l} 3 - 8 + 9 - 8 & 1 - 2 \\ \quad - 6 & \overline{3 - 2 + 5} \\ \hline \quad - 2 & \\ \qquad + 4 & \\ \hline \qquad\quad 5 & \\ \qquad\quad - 10 & \\ \hline \qquad\qquad 2, \text{ Remainder} & \end{array}$$

Since the sign minus before 2 in the divisor changes every sign in the partial products, if we replace -2 by $+2$, we may *add* the partial products thus formed to the dividend instead of subtracting them. (This change of sign is not an actual necessity, but it is a great convenience in practical work.) Then bringing the figures into horizontal lines and using only the second term of the divisor with its sign changed, we have a further abbreviation of the process:

$$\begin{array}{rrrr|l} 3 & -8 & +9 & -8 & 2 \\ & +6 & -4 & +10 & \\ \hline 3 & -2 & +5 & +2 & \end{array}$$

Here we have all the essential work for the complete division of $3x^3 - 8x^2 + 9x - 8$ by $x - 2$. For the figures on the lower line 3, -2, 5 up to the remainder, 2, are the coefficients of the partial quotient $3x^2 - 2x + 5$.

2. Divide $2x^4 - 14x^2 - 6x - 54$ by $x + 3$.

Solution:

$$\begin{array}{rrrrr|l} 2 & +0 & -14 & -6 & -54 & -3 \\ & -6 & +18 & -12 & +54 & \\ \hline 2 & -6 & +4 & -18 & 0 & \end{array}$$

Therefore the quotient is $2x^3 - 6x^2 + 4x - 18$.

Since the remainder is zero, $x + 3$ is a factor of the dividend. This method of obtaining factors will be used in Exercises 5–21, page 26.

EXERCISES

Divide by synthetic division:

1. $x^2 - x - 12$ by $x + 3$.
2. $x^3 - 2x - 4$ by $x - 2$.
3. $x^3 + 2x^2 + 96$ by $x + 4$.

Using synthetic division, find the remainder in Exercises 4, 6, 8, and 10:

4. $(x^2 - 5x + 6) \div (x - n)$.

5. Substitute n for x in $x^2 - 5x + 6$ and compare the result with the remainder obtained in Exercise 4.

6. $(x^2 + bx + c) \div (x - n)$.

7. Put n for x in $x^2 + bx + c$ and again compare results.

8. $(x^3 + ax^2 + bx + c) \div (x - n)$.

9. Compare the remainder in Exercise 8 with the result obtained by substituting n for x in $x^3 + ax^2 + bx + c$.

10. $(x^3 - 2x^2 + 6) \div (x - 4)$.

11. Substitute 4 for x in $x^3 - 2x^2 + 6$ and compare the result with the remainder obtained in Exercise 10.

12. Draw a general conclusion from Exercises 4 to 11.

Note. The approximate solution of equations of the third and higher degrees, having numerical coefficients, was a problem to which Newton devoted considerable attention. Little progress in this line was made from his time until 1819, when William George Horner (1786–1837), a teacher in Bath, England, published a method of solving equations by synthetic division. His procedure was in essence very similar to Newton's, and its element of originality lay in the very compact and elegant form in which he arranged the numerical work. In the ninety years which have intervened since its publication Horner's method has been improved but little, which is rather remarkable, as Horner did not have the advantage of a university training and was by no means a great mathematician.

10. Important special products. Certain products are of frequent occurrence. These should be memorized so that one can write or state the result without the labor of actual multiplication.

I. For the square of the sum of two terms we have the formula

$$(a+b)^2 = a^2 + 2\,ab + b^2.$$

II. For the square of the difference of two terms we have the formula

$$(a-b)^2 = a^2 - 2\,ab + b^2.$$

III. For the product of the sum and the difference of two terms we have the formula

$$(a+b)(a-b) = a^2 - b^2.$$

IV. For the product of two binomials having a common term we have the formula

$$(x+a)(x+b) = x^2 + (a+b)\,x + ab.$$

V. The square of the polynomial $(a+b-c)$ gives the formula

$$(a+b-c)^2 = a^2 + b^2 + c^2 + 2\,ab - 2\,ac - 2\,bc.$$

VI. The cube of the binomial $(a+b)$ gives the formula

$$(a+b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3.$$

Similarly, $(a-b)^3 = a^3 - 3\,a^2b + 3\,ab^2 - b^3.$

ORAL EXERCISES

1. Express in words each of the formulas I to VI which precede.

Perform the indicated operation:

2. $(x+3)^2$.

3. $(x-5)^2$.

4. $(2\,x+4)^2$.

5. $(4\,x-3)^2$.

6. $(x^2-x)^2$.

7. $(2\,x+c)^2$.

8. $(x-3\,c^2)^2$.

9. $(4\,x+xc)^2$.

10. $(3\,x^2-4\,xc)^2$.

11. $(7\,x+4\,ax^2)^2$.

12. $(x^2-x^{-2})^2$.

13. $(x^4-3\,x^{-4})^2$.

14. $(2\,a^x-a^{-x})(2\,a^x-a^{-x})(2\,a^x-a^{-x}) \div (2\,a^x-a^{-x})$.

15. $(16\,x^2-24\,x+9) \div (4\,x-3) =$? Why?

16. $(16\,x^2+8\,x^2c+x^2c^2) \div (4\,x+xc) =$? Why?

17. $(x - c)(x + c)$.

18. $(x - 3)(x + 3)$.

19. $(x + 6)(x - 6)$.

20. $(a - 3c)(a + 3c)$.

21. $(m - x)(x + m)$.

22. $(4a + c)(c - 4a)$.

23. $(4x - 3c)(3c + 4x)$.

24. $(a^2 + 4c)(a^2 - 4c)$.

25. $(x^3 - cx)(x^3 + cx)$.

26. $(4c^3 - a^5)(a^5 + 4c^3)$.

27. $\frac{m^2 - x^2}{x + m} = ?$ Why?

28. $\frac{a^4 - 16c^2}{a^2 - 4c} = ?$ Why?

29. $(x + 3)(x + 4)$.

30. $(x + 5)(x + 7)$.

31. $(a + 8)(a + 6)$.

32. $(b - 3)(b - 4)$.

33. $(m - 5)(m - 10)$.

34. $(c - 1)(c + 2)$.

35. $(x - 3)(x + 5)$.

36. $(S - 7)(S + 4)$.

37. $(x - 12)(x + 3)$.

38. $(a - 4c)(a + 2c)$.

39. $(a^2 - 4a)(a^2 + 6a)$.

40. $(a^x - 2a^{-x})(a^x + 5a^{-x})$.

41. $(ax - ac)(ax - 3ac)$.

42. $(cx - 4c^2)(cx + 8c^2)$.

43. $\frac{a^4 - 2a^3 - 24a^2}{a^2 - 6a} = ?$ Why?

44. $(a + b + c)^2$.

45. $(a + c + x)^2$.

46. $(a - c + x)^2$.

47. $(a - c - x)^2$.

48. $(a - c + 2)^2$.

49. $(x - c - 3a)^2$.

50. $\frac{a^2 + 9c^2 + x^2 - 6ac + 2ax - 6cx}{a - 3c + x} = ?$ Why?

51. $(2c - 2a - 4x)^2$.

52. $(3c - 5a + 2x)^2$.

53. $(x^{2a} + x^a - 5)^2$.

54. $(4a - 3c - 2x)^2$.

55. $(a + a^{-1} - 3)^2$.

56. $(a^x - a^{-x} + 4)^2$.

57. Can the expression in Exercise 44 be squared as a binomial? Explain.

58. $(x + c)^3$.

59. $(x - c)^3$.

60. $(x - 1)^3$.

61. $(x + 1)^3$.

62. $(x + 2)^3$.

63. $(x - 2)^3$.

64. $(x + 3)^3$.

65. $(x - 4)^3$.

66. $(x^2 + a)^3$.

67. $(x^2 - 2)^3$.

68. $(5a - 4c)^3$.

69. $(2x^5 - 7x^2)^3$.

70. $(x^3 - 6x^2 + 12x - 8) \div (x - 2) = ?$ Why?

71. $(8 - 12x + 6x^2 - x^3) \div (4 - 4x + x^2) = ?$ Why?

72. Square $(5 - 7)$ as a binomial and check the result by subtracting 7 from 5 and squaring the difference obtained.

73. Square $x + 9$ and $-x - 9$. Compare results and explain.

74. Find the product of $(9 - 4)(9 + 3)$ by the formula. Verify by simplifying each binomial and then multiplying.

75. Square: (*a*) 42, (*b*) 59, (*c*) 73, (*d*) 105, (*e*) 97, and (*f*) 1005.

76. Expand $(4 + 9 - 5)^2$ by the formula. Verify by simplifying and then squaring.

77. Expand $(3 - 2)^3$ by formula. Verify by simplifying the binomial and then cubing the result.

78. Expand $(a - 2b)^3$ and $(2b - a)^3$. Compare results and explain.

79. What must be added to $9x^2 + 6x$ to complete the trinomial square?

80. What must be added to or subtracted from $16a^2 + 9$ to complete the trinomial square? Why?

Form a perfect trinomial square of:

81. $x^2 - ? + 9$.

82. $4x^2 + ? + 1$.

83. $4x^2 - ? + 9a^2$.

84. $x^2 + 4x + ?$

85. $4x^2 + 4x + ?$

86. $9x^2 \pm 24x + ?$

87. $? \pm 12x + 9$.

88. $4x^2 - 18ax + ?$

89. $9x^2 - 4ax + ?$

90. $25x^2 - 12x + ?$

91. $a^2 + ? + a^{-2}$.

92. $a^{2x} - ? + a^{-2x}$.

93. $a^{2x} - ? + 16a^{-2x}$.

94. $a^{4x} + 10 + ?$

95. $a^{10} - ? + 49a^{-6}$.

96. $a^6 - 6a^2 + ?$

97. $a^{4x} - 12a^x + ?$

98. $4a^{6x} + ? + 25a^{-2x}$.

CHAPTER II

FACTORING

11. Definitions. Factoring is the process of finding the two or more expressions whose product is equal to a given expression.

In multiplication we have two factors given and are required to find their product. In division we have the product and one factor given and are required to find the other factor. In factoring, however, the problem is a little more difficult, for we have only the product given, and our experience is supposed to enable us to determine the factors.

In this chapter (except in § 17) only those expressions and factors which are rational will be considered.

An integral expression is here regarded as **prime** when no two rational integral expressions can be found (except the expression itself and 1) whose product is the given expression.

It must be remembered that to factor an integral expression means to resolve it into its *prime* factors.

The methods of this chapter enable one to factor all integral, rational expressions in one letter which are not prime, as well as some of the simpler expressions in two letters. No attempt is made even to define what is meant by prime factors of expressions which are not rational and integral.

There is no simple operation the performance of which makes us sure that we have found the *prime* factors of a given expression. Only insight and experience enable us to find prime factors with certainty.

A partial check, however, that may be applied to all the exercises in factoring, consists in actually multiplying together the factors that have been found. The result should be the original expression.

The types of factorable expressions previously considered will be reviewed in §§ 12, 13, 14, 15, 16, 18, and 20.

12. Polynomials with a common monomial factor. The type form is

$$ab + ac - ad.$$

Factoring, $ab + ac - ad = a(b + c - d)$.

EXERCISES

Factor:

1. $4x + 8$.
2. $5 - 10a^2$.
3. $ax - 7ay$.
4. $2cd - 4c^2 - 2$.
5. $a^2c - ac^2 - 4ac$.
6. $3xy + 21y^5 - 15y^3$.
7. $x^{2a} - 3x^a + 12x$.
8. $y^{2a} - 6y^a + 2y^{a-1}$.
9. $5xy + 30y(x^2 + xy)$.
10. $(7a^2 - 21ab + 7a) - 14ax$.
11. $14a^3x^5 - 21a^2x^5m - 49a^4x^8y^7$.
12. $(3c^2 - 3cd) - a(45c^2 - 15c^3x)$.
13. $2r^{2x+3} + 12r^{x-7} - 16sr^{x+2} + 8sr^{x+4}$.

13. Polynomials which may be factored by grouping terms. The type form is

$$ax + ay + bx + by.$$

Factoring,
$$\begin{aligned} ax + ay + bx + by &= (ax + ay) + (bx + by) \\ &= a(x + y) + b(x + y) \\ &= (x + y)(a + b). \end{aligned}$$

EXERCISES

Separate into polynomial factors:

1. $3(x + y) + a(x + y)$.
2. $a(x - 3) - b(x - 3)$.
3. $5x(a - b) + (b - a)$.
4. $2c(r - 2s) - 5d(2s - r)$.
5. $4x - 4y + bx - by$.

   HINT. $4x - 4y + bx - by = 4(x - y) + b(x - y)$, etc.
6. $3cx + 6ac + 8ax + 4x^2$.
7. $-6x^2 + 10x + 21xm - 35m$.
8. $x^5 - x^4 - x^3 + 2x^2 - 2x - 2$.
9. $rs - 2s + 3r - 6 - 5rx + 10x$.
10. $x^{3a} - 3x^{2a} - x^a + 3 - 6x^{4a} + 2x^{5a}$.
11. $x^{3a-2} + 2x^{a+1} - 15x^{2a-3} - 10 + 10x^{2a-8}$.

14. Trinomials which are perfect squares. The type form is

$$a^2 \pm 2ab + b^2.$$

Factoring, $a^2 \pm 2ab + b^2 = (a \pm b)^2.$

EXERCISES

Separate into binomial factors:

1. $x^2 + 6x + 9.$
2. $a^2 - 12a + 36.$
3. $x^2 + x + \frac{1}{4}.$
4. $9b^2 - 12b + 4.$
5. $28x + 49x^2 + 4.$
6. $9r^2 + 49 - 42r.$
7. $a^2 + 2 + a^{-2}.$
8. $a^2 - 2 + a^{-2}.$
9. $a^4 - 6 + 9a^{-4}.$
10. $\dfrac{x^2}{9a^2} - \dfrac{8x}{a} + 144.$
11. $81a^4x^2 - 36a^2bx + 4b^2.$
12. $1 + 4a^4b^8c^{12} - 4a^2b^4c^6.$
13. $x^{4a} - 10ax^{2a} + 25a^2.$
14. $r^{4x+2} + 4 - 4r^{2x+1}.$
15. $a^{2x} + 4a^{-2x} - 4.$
16. $a^{4x} - 2a^x + a^{-2x}.$
17. $(x-2)^2 + 14(x-2) + 49.$
18. $4(a+5)^2 - 12b(a+5) + 9b^2.$
19. $(a-b)^{2x} - 18x(a-b)^x + 81x^2.$

15. The quadratic trinomial. The type form is

$$x^2 + bx + c.$$

For factoring expressions of this type we have the

RULE. *Find two numbers whose algebraic product is* **c** *and whose algebraic sum is* **b**.

Write for the factors two binomials which have **x** *for their common term and the numbers just obtained for the other terms.*

EXERCISES

Separate into binomial factors:

1. $x^2 - 8x + 12.$

Solution: The two numbers whose sum is -8 and whose product is $+12$ are -2 and -6. Therefore $x^2 - 8x + 12 = (x-2)(x-6)$.

2. $x^2 - 9x + 18.$
3. $x^2 + 2x - 24.$
4. $x^2 + x - \frac{3}{4}.$
5. $a^2 - 12 - 11a.$
6. $a^2 + .3a - .1.$
7. $c^2 - ac - 90a^2.$
8. $x^2 - 8x - 9.$
9. $r^2s^2 + 6rs - 40.$
10. $a^2 + 5 + 6a^{-2}.$

11. $1 - 9xy + 8x^2y^2$.

12. $11d^4 - 12d^2c^3 + c^6$.

13. $15m^2 - 14mx - x^2$.

HINT. This may be written as $-1(x^2 + 14xm - 15m^2)$, etc.

14. $90 + x - x^2$.

15. $12a^4 - a^2x - x^2$.

16. $a^{2x} - 20 + 19a^x$.

17. $a^2 + 12a^{-2} + 7$.

18. $a^{2x} - 8a^{-2x} - 2$.

19. $120 + 7m^n - m^{2n}$.

20. $a^{4x} - a^x - 6a^{-2x}$.

21. $(m+n)^{4e} - 9(m+n)^{2e} - 22$.

16. The general quadratic trinomial. The type form is

$$ax^2 + bx + c.$$

This important type really includes the two preceding types.

If a trinomial of this type has two rational factors, they have the forms $dx + e$ and $fx + g$.

Now $$(dx+e)(fx+g) = dfx^2 + fex + dgx + ge \quad (1)$$
$$= dfx^2 + (fe + dg)x + ge. \quad (2)$$

In (2) the product of the coefficient of x^2, df, and the constant term, ge, is $dfge$. But $dfge$ equals fe times dg, and fe plus dg equals the coefficient of x. Therefore, if $ax^2 + bx + c$ has rational factors, it can be written in the form (1) and factored by grouping terms. Hence the

RULE. *Find two numbers whose algebraic product is* **ac** *and whose algebraic sum is* **b**.

Replace **bx** *by two terms in* **x** *whose respective coefficients are the numbers just found, and factor by grouping terms.*

EXERCISES

Separate into binomial factors:

1. $4x^2 - 7x - 15$.

Solution: Here $ac = 4 \cdot (-15)$, or -60, and $b = -7$. The numbers whose product is -60 and whose sum is -7 are $+5$ and -12.

Hence $$4x^2 - 7x - 15 = 4x^2 - 12x + 5x - 15$$
$$= 4x(x-3) + 5(x-3) = (x-3)(4x+5).$$

2. $2a^2 - 3a - 2$.

3. $3a^2 + 8a - 3$.

4. $4a^2 + a - 5$.

5. $9c^2 - 71c - 8$.

6. $5r^2 - 22r + 8$.

7. $7x^2 + 62x - 9$.

8. $6x^2+19x-7$.
9. $6x^2+13x-5$.
10. $2x^2+7x-15$.
11. $3x^2-ax-2a^2$.
12. $4a^4-12a^2+9$.
13. $25+4c^2d^2-20cd$.
14. $-8n^4+3n^8-3$.
15. $6x^{2y}-13x^y+6$.
16. $20x^2-9xy^3-20y^6$.
17. $2x^2-(a+2b)x+ab$.
18. $5m^{2n-4}+9am^{n-2}-2a^2$.
19. $6x^{4a}+(3-2y^b)x^{2a}-y^b$.
20. $20a^2b^{4-2x}-9a-20b^{2x-4}$.
21. $6a^{2x+6}-25a^{x+3}b^{y-1}+4b^{2y-2}$.

17. The factors of ax^2+bx+c by formula.* The factors of any trinomial of this type, whether they are rational or irrational, can **always** be obtained as follows:

Let $$ax^2+bx+c=0. \quad (1)$$

Then $$x^2+\frac{b}{a}x+\frac{c}{a}=0. \quad (2)$$

If (2) be solved by completing the square (see Exercise 1, page 271, "First Course in Algebra"), we obtain

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}. \quad (3)$$

From (3), $$x-\frac{-b+\sqrt{b^2-4ac}}{2a}=0, \quad (4)$$

and $$x-\frac{-b-\sqrt{b^2-4ac}}{2a}=0. \quad (5)$$

From (4) and (5),

$$\left(x-\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(x-\frac{-b-\sqrt{b^2-4ac}}{2a}\right)=0. \quad (6)$$

Now (6) and (2) are equivalent, that is, they have the same roots. Further, if both members of (2) be multiplied by a, the result is (1). If both members of (6) be multiplied by a, we obtain an equation which is identical with (1), as can be verified by performing the indicated multiplication. That is,

$$\boldsymbol{ax^2+bx+c=a\left(x-\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(x-\frac{-b-\sqrt{b^2-4ac}}{2a}\right)}. \quad (7)$$

* See also page 240.

EXAMPLE

Factor $10x^2 - 7x - 12$.

Here $a = 10$, $b = -7$, and $c = -12$. Using (7) as a formula gives:

$$10x^2 - 7x - 12$$
$$= 10\left(x - \frac{-(-7) + \sqrt{(-7)^2 - 4 \cdot 10(-12)}}{20}\right)\left(x - \frac{7 - \sqrt{49 + 480}}{20}\right)$$
$$= 10\left(x - \frac{7 + 23}{20}\right)\left(x - \frac{7 - 23}{20}\right)$$
$$= 2(x - \tfrac{3}{2})5(x + \tfrac{4}{5}) = (2x - 3)(5x + 4).$$

We may infer from the preceding work that a trinomial of the form $ax^2 + bx + c$ has *rational* factors only when $b^2 - 4ac$ is a *perfect square.*

EXERCISES

Factor by use of the formula:

1. $3x^2 + 5x + 2$.

2. $6x^2 + x - 2$.

3. $x^2 + 4x + 1$.

4. $3x^2 - 5x - 12$.

5. $5x^2 - 19x - 4$.

6. $7x^2 - 20x - 3$.

7. $15x^2 - 11x - 14$.

8. $3x^2 + 5mx - 2m^2$.

9. $4x^2 - 5nx - 6n^2$.

10. $x^2 - 6x + 1$.

11. $2x^2 + 5x + 1$.

12. $3x^2 + 6x - 1$.

13. $4x^2 - 8x + 3$.

14. $5x^2 + 7x - 3$.

15. $x^2 - 2x + 1 - n$.

16. $x^2 + 6x - 4n + 9$.

17. $x^2 - (n + 5)x + (2n + 6)$.

18. $x^2 - nx - 2x + 3n - 3$.

18. A binomial the difference of two squares. The type form is

$$a^2 - b^2.$$

Factoring, $\quad a^2 - b^2 = (a + b)(a - b).$

More generally,

$$a^2 + 2ab + b^2 - c^2 + 2cd - d^2$$
$$= a^2 + 2ab + b^2 - (c^2 - 2cd + d^2)$$
$$= (a + b)^2 - (c - d)^2$$
$$= (a + b + c - d)(a + b - c + d).$$

EXERCISES

Factor:

1. $m^2 - n^2$.
2. $a^2 - c^2$.
3. $x^2 - 4$.
4. $x^2 - \frac{1}{9}$.
5. $81 - x^4$.
6. $25x^2 - 49b^2$.
7. $a^2 - a^{-2}$.
8. $1 - 9a^2c^6$.
9. $a^4 - \dfrac{4c^2}{25}$.
10. $16a^6 - 25c^4d^{10}$.
11. $a^{2x} - a^{-2x}$.
12. $a^4 - a^{-4}$.
13. $(a + c)^2 - 1$.
14. $(a - x)^4 - 4$.
15. $9 - (2 + x)^2$.
16. $16 - (x - a)^{6a}$.
17. $5^{2a} - (m - 1)^{10a}$.
18. $(a + c)^2 - (m + n)^2$.
19. $(a + c)^2 - (m - n)^2$.
20. $(a - x)^{2r} - (c - 5)^2$.
21. $a^2 + 2ax + x^2 - 9$.

    Hint. $a^2 + 2ax + x^2 - 9 = (a + x)^2 - 9$.
22. $a^2 - 4ac + 4c^2 - x^2$.
23. $25 - 10x + x^2 - 16m^2$.
24. $9 - 12a + 4a^2 - b^{2m}$.
25. $m^2 - a^2 - 6a - 9$.

    Hint. $m^2 - a^2 - 6a - 9 = m^2 - (a^2 + 6a + 9) = m^2 - (a + 3)^2$.
26. $x^2 - 4y^2 + 20y - 25$.
27. $16 - 9a^2 + 12ax - 4x^2$.
28. $4m^4 + 30n^2x - 9x^2 - 25n^4$.
29. $-28c^3d^2 - 49c^6 + 1 - 4d^4$.
30. $12r^s - 36 + 5^{2a} - r^{2s}$.
31. $(m - 2)^2 - 4n^2 + 28n - 49$.
32. $x^2 - 6x + 9 - y^2 + 8ay - 16a^2$.
33. $4bd + 4c^2 - 4d^2 - 4c - b^2 + 1$.
34. $6c + h^2 - 1 - 9c^2 - 4hk + 4k^2$.
35. $h^2 - 4y^2 - 10h + 8xy + 25 - 4x^2$.
36. $25x^2 - 20x + 4 - 4y^2 - 9a^2 - 12ay$.

19. Expressions reducible to the difference of two squares. The type form is

$$a^4 + ka^2b^2 + b^4.$$

If k has such a value that the trinomial is not a perfect square, a trinomial of this type can often be written as the **difference** of two squares. Thus, if $k = 1$, the adding and subtracting of a^2b^2 transforms the expression into the difference of two squares.

EXAMPLES

1. Factor $a^4 + a^2b^2 + b^4$.

Solution: $a^4 + a^2b^2 + b^4 = a^4 + 2a^2b^2 + b^4 - a^2b^2$
$= (a^2 + b^2)^2 - (ab)^2$
$= (a^2 + b^2 + ab)(a^2 + b^2 - ab)$.

2. Factor $49h^4 + 34h^2k^2 + 25k^4$.

Solution: If $36h^2k^2$ is added, the expression becomes a perfect trinomial square. Adding and subtracting $36h^2k^2$, we have

$$49h^4 + 34h^2k^2 + 25k^4 = 49h^4 + 70h^2k^2 + 25k^4 - 36h^2k^2$$
$$= (7h^2 + 5k^2)^2 - (6hk)^2$$
$$= (7h^2 + 5k^2 + 6hk)(7h^2 + 5k^2 - 6hk).$$

EXERCISES

Factor:

1. $x^4 + x^2 + 1$.
2. $x^4 + x^2y^2 + y^4$.
3. $x^4 + 4x^2 + 16$.
4. $16y^4 + 4y^2 + 1$.
5. $c^4 + c^2d^2 + 25d^4$.
6. $1 - 19y^6 + 25y^{12}$.
7. $4x^4 + 3x^2y^2 + 9y^4$.
8. $4x^4 - 28x^2y^6 + 9y^{12}$.
9. $9c^4 - 55c^2d^2 + 25d^4$.
10. $9a^8 - 19a^4b^2 + 25b^4$.
11. $49h^4 - 44h^2k^4 + 4k^8$.
12. $x^8 + x^4 + 1$.
13. $c^8 - 6c^4 + 1$.
14. $16 + 4x^4 + x^8$.
15. $25y^4 - 11y^2 + 1$.
16. $y^8 + 16y^4 + 256$.
17. $3x^9y + 3x^5y^5 + 3xy^9$.
18. $16h^4 - 33h^2k^2 + 36k^4$.
19. $25c^4 - 51c^2d^2 + 49d^4$.
20. $49a^4 - 32a^2b^2 + 64b^4$.
21. $64x^4 + 119x^2y^2 + 81y^4$.
22. $81a^4 - 171a^2b^2 + 25b^4$.
23. $1 + 4x^4$. Hint. $1 + 4x^4 = 1 + 4x^2 + 4x^4 - 4x^2$.
24. $64c^4 + 1$.
25. $x^4 + 4y^4$.
26. $x^8 + 4y^8$.
27. $x^8 + 64$.
28. $x^{4a} + 4y^{8a}$.
29. $a^{4b}c^{12d} + 64e^{4x+4}$.

20. The sum or difference of two cubes. The type form is

$$a^3 \pm b^3.$$

Factoring, $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$,
$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

EXERCISES

Factor:

1. $x^3 + 64$.

Solution: $x^3 + 64 = x^3 + 4^3 = (x + 4)(x^2 - x \cdot 4 + 4^2)$
$= (x + 4)(x^2 - 4x + 16)$.

2. $x^3 + 27$.
3. $a^3 - 64$.
4. $8 + m^3$.
5. $27 - m^6$.
6. $x^3 - \frac{a^3}{8}$.
7. $x^3 - y^{-3}$.
8. $8a^3 + 27b^3$.
9. $1 - 125x^6$.
10. $\frac{x^3}{27} - y^6$.
11. $x^6 - y^6$.
12. $x^6 + y^6$.
13. $x^9 - a^3$.
14. $x^6 + a^9$.
15. $(x + y)^3 - 8$.
16. $x^3 - 9x^2 + 27x - 28$.
17. $a^3b^6c^9 - 8d^{12}$.
18. $x^{3e} - y^{-3e}$.
19. $c^{6e} + 27d^{9x}$.

21. The Remainder Theorem. If any rational integral expression in x be divided by $x - n$, the remainder is the same as the original expression with n substituted for x.

EXAMPLE

$$
\begin{array}{l|l}
x^2 - 5x \quad + 6 & x - n \\
\underline{x^2 - nx \qquad\quad} & \overline{x + (n - 5)} \\
(n - 5)x + 6 & \\
\underline{(n - 5)x \qquad - n^2 + 5n} & \\
\qquad\qquad n^2 - 5n + 6, \text{ Remainder} &
\end{array}
$$

Here the remainder $n^2 - 5n + 6$ is the same as $x^2 - 5x + 6$ when n is substituted for x.

Now if n is a letter or a number such that the remainder $n^2 - 5n + 6$ is zero, the division is exact; and the value of n, if substituted for x, will make $x^2 - 5x + 6$ zero also.

Hence, if by trial we can discover a number n which, when put for x, makes $x^2 - 5x + 6$ zero, $x - n$ will be an exact divisor of $x^2 - 5x + 6$. If 2 is put for x in $x^2 - 5x + 6$, we get $4 - 10 + 6$, or zero. Therefore $x - 2$ is a factor of $x^2 - 5x + 6$.

The last paragraph illustrates the following theorem:

22. Factor Theorem. *If any rational integral expression in* x *becomes zero when any number* n *is put for* x, $x - n$ *is a factor of the expression.*

The Factor Theorem may be used to factor many of the preceding exercises. Moreover, many expressions which, by previous methods, are very difficult to factor, may be readily factored by the aid of this theorem.

Note. By means of the Factor Theorem we are able to solve cubic equations when the roots are integers. The solution of the general cubic equation is one of the famous problems of mathematics, and one which is accompanied by many interesting applications. This problem was first solved by the Italian, Tartaglia, about 1530, but was published by Cardan, to whom Tartaglia explained his solution on the pledge that he would not divulge it. For many years the credit for the discovery was given to Cardan, and to this day it is usually called Cardan's solution.

EXAMPLES

Factor:

1. $x^3 + x - 2$.

Solution: If $x - n$ is a factor of $x^3 + x - 2$, then n must be an integral divisor of 2. Now the integral divisors of 2 are $+1$, -1, $+2$, and -2. If 1 be put for x, $x^3 + x - 2 = 1 + 1 - 2 = 0$. Therefore $x - 1$ is a factor of $x^3 + x - 2$. Dividing $x^3 + x - 2$ by $x - 1$, the quotient is $x^2 + x + 2$. None of the integral divisors of 2, when put for x, make $x^2 + x + 2$ zero; hence $x^2 + x + 2$ is prime.

Therefore $x^3 + x - 2 = (x - 1)(x^2 + x + 2)$.

2. $x^3 + 2x^2 - 5x - 6$.

Solution: The integral divisors of 6 are $+1$, -1, $+2$, -2, $+3$, -3, $+6$, and -6. If we put 1 for x, $x^3 + 2x^2 - 5x - 6 = 1 + 2 - 5 - 6 = -8$. If we put -1 for x, $x^3 + 2x^2 - 5x - 6 = -1 + 2 + 5 - 6 = 0$. Therefore $x - (-1)$ or $x + 1$ is a factor. Dividing $x^3 + 2x^2 - 5x - 6$ by $x + 1$, the quotient is $x^2 + x - 6$, which equals $(x + 3)(x - 2)$.

Therefore $x^3 + 2x^2 - 5x - 6 = (x + 1)(x - 2)(x + 3)$.

In the following exercises, when searching for the values of x which will make the given expression zero, only integral divisors of the last term of the expression (arranged according to the descending powers of x) need be tried.

EXERCISES

1. Divide $x^2 + bx + c$ by $x - n$ and show that the remainder is $n^2 + bn + c$.

2. Find the remainder in $(x^3 + ax^2 + bx + c) \div (x - n)$.

Find, by the Remainder Theorem, the remainder when:

3. $x^3 - x - 8$ is divided by $x - 3$.

4. $x^4 - x + 6$ is divided by $x + 2$ by synthetic division.

Factor:

(Instead of ordinary substitution, the student should use synthetic division to find whether the remainder is or is not zero.)

5. $x^3 - 3x + 2$.
6. $x^3 - 4x + 3$.
7. $x^3 + 2x + 3$.
8. $x^3 - x - 6$.
9. $x^3 - x + 6$.
10. $x^3 - 11x + 6$.
11. $x^3 - 11x - 6$.
12. $x^3 - 14x - 8$.
13. $x^3 - 27x - 10$.
14. $x^3 + 3x^2 + 3x + 2$.
15. $x^3 + 4x^2 + 5x + 2$.
16. $x^3 - 6x^2 + 11x - 6$.
17. $x^4 - 11x^2 + 2x + 12$.
18. $4x^4 - 3x - 1$.
19. $x^3 - 5a^2x + 2a^3$.
20. $x^3 - 7m^2x - 6m^3$.
21. $x^3 - 2nx^2 - 5n^2x + 6n^3$.

23. The sum or difference of two like powers. The type form is

$$a^n \pm b^n.$$

The cases in which $a^n \pm b^n$ is divisible by $a + b$ or $a - b$ can be determined by the Factor Theorem.

Thus in $a^n - b^n$, n being either an odd or an even integer, let $a = b$. Then $a^n - b^n$ becomes $b^n - b^n = 0$. Therefore $a - b$ is a factor of $a^n - b^n$.

In $a^n - b^n$, n being even, let $a = +b$ or $-b$. Then $a^n - b^n$ becomes $b^n - b^n = 0$, since both $(+b)^n$ and $(-b)^n$ are *positive* when n is *even*. Therefore when n is even, both $a - b$ and $a + b$ are exact divisors of $a^n - b^n$.

In $a^n + b^n$, n being even, let a equal either $+b$ or $-b$. Then $a^n + b^n$ becomes $b^n + b^n$, which is not zero. Therefore $a^n + b^n$ is never divisible by $a + b$ or $a - b$ when n is even.

In $a^n + b^n$, n being odd, let $a = -b$. Then $a^n + b^n$ becomes $(-b)^n + b^n = 0$, since $(-b)^n$ is *negative* when n is *odd*. Therefore when n is odd, $a + b$ is a divisor of $a^n + b^n$.

Summing up:

I. $a^n - b^n$ is *always* divisible by $a - b$.

II. $a^n - b^n$, when n is *even*, is always divisible by $a - b$ and $a + b$.

III. $a^n + b^n$ is *never* divisible by $a - b$.

IV. $a^n + b^n$, when n is *odd*, is always divisible by $a + b$.

It is worth noting that $a^n + b^n$ is usually prime when n is a power of 2. (See, however, Exercises 23–29, page 23.)

Thus $a^2 + b^2$, $a^4 + b^4$, $a^8 + b^8$, etc., are prime.

In every other case $a^n + b^n$ is not prime.

Thus
$$a^6 + b^6 = (a^2)^3 + (b^2)^3,$$
$$a^{10} + b^{10} = (a^2)^5 + (b^2)^5,$$
$$a^{12} + b^{12} = (a^4)^3 + (b^4)^3, \text{ etc.}$$

EXAMPLES

1. Factor $a^5 - b^5$.

Solution: $a^5 - b^5 = (a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4)$.

2. Factor $a^6 - b^6$.

Solution: $a^6 - b^6$ is divisible by $a + b$ and $a - b$. It is better, however, to regard all such binomials with even exponents as the difference of two squares. Thus

$$a^6 - b^6 = (a^3 - b^3)(a^3 + b^3), \text{ etc.}$$

3. Factor $a^5 + b^5$.

Solution: $a^5 + b^5 = (a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4)$.

Note the signs of the second factor in Examples 1 and 3, — all plus in one case, alternately plus and minus in the other.

4. Factor $x^5 - y^{10}$.

Solution:
$$x^5 - y^{10} = x^5 - (y^2)^5$$
$$= (x - y^2)[x^4 + x^3(y^2)^1 + x^2(y^2)^2 + x(y^2)^3 + (y^2)^4]$$
$$= (x - y^2)(x^4 + x^3y^2 + x^2y^4 + xy^6 + y^8).$$

5. Factor $32x^{15} + y^{15}$.

Solution: $32x^{15} + y^{15}$

$$= (2x^3)^5 + (y^3)^5$$
$$= (2x^3 + y^3)[(2x^3)^4 - (2x^3)^3(y^3) + (2x^3)^2(y^3)^2 - (2x^3)(y^3)^3 + (y^3)^4]$$
$$= (2x^3 + y^3)(16x^{12} - 8x^9y^3 + 4x^6y^6 - 2x^3y^9 + y^{12}).$$

EXERCISES

Factor:

1. $x^5 + 1$.

2. $x^5 + y^5$.

3. $x^7 - 1$.

4. $x^7 - y^7$.

5. $x^7 + 1$.

6. $x^5 - 32$.

7. $243 + a^5$.

8. $3125 - c^5$.

9. $c^7 - 128$.

10. $c^7 - x^{14}$.

11. $x^{10} + y^5$.

12. $1 + c^7d^{14}$.

13. $x^{14} + y^7$.

14. $c^{14} + 128$.

15. $x^{10} - a^5$.

16. $x^{10} + a^{15}$.

17. $32x^5 - 1$.

18. $x^5 - 32y^{10}$.

19. $128x^7 - 1$.

20. $243x^5 + 1$.

21. $1024 - 243x^5$.

24. General directions for factoring. The following suggestions will prove helpful:

I. *First look for a common monomial factor, and if there is one (other than 1), separate the expression into its greatest monomial factor and the corresponding polynomial factor.*

II. *Then determine, by the form of the polynomial factor, with which of the following types it should be classed, and use the methods of factoring applicable to that type.*

1. $ax + ay + bx + by$.

2. $a^2 + 2ab + b^2$.

3. $x^2 + bx + c$.

4. $ax^2 + bx + c$.

5. $a^n \pm b^n$.

6. $\begin{cases} a^2 - b^2. \\ a^2 + 2ab + b^2 - c^2. \\ a^2 + 2ab + b^2 - c^2 - 2cd - d^2. \end{cases}$

7. $a^4 + ka^2b^2 + b^4$.

III. *Proceed again as in II with each polynomial factor obtained until the original expression has been separated into its prime factors.*

IV. *If the preceding steps fail, try the Factor Theorem.*

REVIEW EXERCISES

Factor:

1. $x^3 - x$.
2. $x^5 - x$.
3. $x^{10} - x^2$.
4. $x^4 - 2x^2 + 1$.
5. $x^8 - 2x^4 + 1$.
6. $x^5 - 8x^3 + 16x$.
7. $x^4 - 10x^2 + 9$.
8. $x^4 - 13x^2 + 36$.
9. $3a^2x^4 - 12a^3x^2 + 12a^4$.
10. $18a^2x^2 - 24a^2x - 10a^2$.
11. $x^2 - \frac{x}{6} - \frac{1}{6}$.
12. $16x^4 + 8x^2 - 3$.
13. $a^3 - a + a^2b - b$.
14. $3x^4 - 15x^2 + 12$.
15. $x^3 - x^2 - 4x + 4$.
16. $4n^6 + 48n^2 - 28n^4$.
17. $12a - 39ay - 51ay^2$.
18. $x^4 - 3x^3 + 4x^2 - 12x$.
19. $3a^3 + 3a^2 - 27a - 27$.
20. $2a^3b + 3a^2b - 8ab - 12b$.
21. $4a^2 - a^4 + 81 + 10a^2x - 36a - 25x^2$.
22. $12cd^3 - 6a^3x - a^6 + 4c^2 + 9d^6 - 9x^2$.
23. $x^6 + 1$.
24. $x^5 - \frac{a^4x}{16}$.
25. $x^{10} - y^{10}$.
26. $x^8 - y^8$.
27. $x^8 - x^2$.
28. $x^{12} + y^{12}$.
29. $x^4 - 64x^8$.
30. $x^{12} - y^{12}$.
31. $x^{12} - 64$.
32. $x^{10} + xy^9$.
33. $x^{12} - 8$.
34. $x^4 - y^{16}$.
35. $64x^{12} - 4x^4$.
36. $x^{12} + 64y^{12}$.
37. $32x^{10} + y^{10}$.
38. $x^{16} - y^{16}$.
39. $16x^{16} - 1$.
40. $x^5 + y^{15}$.
41. $243 - h^5$.
42. $5^{-4x} - 1$.
43. $20 - x - x^2$.
44. $10 - 10c^{14}d^4$.
45. $2cd - c^2 - d^2$.
46. $x^5 - x^4 - x^3 + x^2$.
47. $x^4 - 9x^2 - x + 3$.
48. $x^4 - 7x^2y^2 + 81y^4$.
49. $4c^4 + 20c^3d - 11c^2d^2$.
50. $a^4 - a^2 + a + 1$.
51. $a^5 - a^3 + a - 1$.
52. $a^5 - a^4 - a^3 + a$.
53. $5d^2 - 5cd - 10c^2$.
54. $x^3 - 3x^2 + 8x - 12$.
55. $121x^4 - 476x^2y^2 + 100y^4$.
56. $x^4 - x^2 + 12xy - 36y^2$.
57. $y^4 - 18y^2 + 81 - 16x^4 - 24x^2y^3 - 9y^6$.

58. $4h^5 + 32h^2k^3$.

59. $h^5k^5 - 1024k^5$.

60. $x^3 - 83x^5 + 289x^7$.

61. $14 + 5x^5 - x^{10}$.

62. $1 - x + x^6 - x^7$.

63. $x^2 - y^2 - x - y$.

64. $x^2 - y^2 + x - y$.

65. $289 - 100a^2 - b^2 - 20ab$.

66. $625a^8 - 169d^4 + 78cd^2 - 9c^2$.

67. $c^7 - 2187$.

68. $x^{3n} - 125y^{6n}$.

69. $5 - 8x - 4x^2$.

70. $4x^4 - 37x^2 + 9$.

71. $256 - 16k^4 + 8h^2k^2 - h^4$.

72. $x^4 + 4$.

73. $x^{12} - 729$.

74. $a^{4x+8} + 64$.

75. $a^4 + 225a^{-4} - 39$.

76. $x^3 - 6x^2 + 12x - 8$.

77. $a^2 - 9d^2 - 8ab + 6cd - c^2 + 16b^2$.

78. $4h^{-6} - 20h^{-3}k + 25k^2 - 6ab^{-2} - 9a^2 - b^{-4}$.

79. $4x^4 - 9x^2 - 9$.

80. $x^{2a} - 2x^a - 15$.

81. $a^3 + a + b^3 + b$.

82. $x^{2a} - 12x^a + 36$.

83. $25x^{2c} + 50x^c - 39$.

84. $64 + c^8$.

85. $128 - x^{28}$.

86. $3x^2 + 10x - 8$.

87. $a^5 - a^3 - a^2 + 1$.

88. $2c^2 + 3cd - 27d^2$.

89. $x^3 + 3x^2 + 9x + 27$.

90. $x^3 - 6x^2 + 12x - 7$.

91. $4a^4b^6 - 40a^2b^4c^2 + 100b^2c^4$.

92. $4x^4 - 25y^6 + 10y^3 - 12x^2 + 8$.

93. $x^2 - k^2 + 6xy + 6h^2k + 9y^2 - 9h^4$.

94. $e^{2x} - 2 + e^{-2x}$.

95. $e^{2x} - 5 + 6e^{-2x}$.

96. $3x^{2a} + 5x^a - 28$.

97. $e^{3x} - 2e^x - 24e^{-x}$.

98. $6e^{2x} - 5e^{-2x} - 13$.

99. $x^3 - 7xy^2 + 6y^3$.

100. $e^{3x} + e^{2x} + e^x + 1$.

101. $a^{4x-2} - 10 + 25a^{2-4x}$.

102. $e^{3x} - e^{-3x} + 3e^{-x} - 3e^x$.

103. $e^{x+3} + e^{x+2} - e^{3-x} - e^{2-x}$

104. $xy^2 + xz^2 + x^2y + x^2z + yz^2 + y^2z + 2xyz$.

105. $ab^3 - a^3b + ac^3 - a^3c + bc^3 - b^3c$.

25. Solution of equations by factoring. For solving an equation by factoring we have the

RULE. *Transpose the terms so that the right-hand member is zero. Then factor the member on the left, set each factor which contains an unknown equal to zero, and solve the resulting equations.*

One should not divide each member of an equation by an expression containing the unknown, for by this process one or more roots are usually lost (see Example 3, page 44).

EXERCISES

Solve by factoring and check:

1. $x^2 - 25 = 0$.
2. $x^2 + 10 = 7x$.
3. $y^2 - 9y = 0$.
4. $r^2 - ra = 30a^2$.
5. $4z^3 - 36z = 0$.
6. $x^4 + 4 = 5x^2$.
7. $t^6 = 13t^4 - 36t^2$.
8. $3x^2 - xb - 2b^2 = 0$.
9. $x^3 - 2x^2 = x - 2$.
10. $x^{2x} - 2x^x + 1 = 0$.
11. $x^{2x} - 8x^x + 16 = 0$.
12. $x^3 + 5x^2 - 18x - 72 = 0$.

    HINT. Apply the Factor Theorem.
13. $x^3 - 6x^2 + 12x = 8$.
14. $x^4 + 3x^3 - 8x^2 + 16 = 12x$.
15. $x^3 + 6a^3 = 2ax^2 + 5a^2x$.
16. $x^5 + 9ax^2 = 9x^4a + x^3$.
17. $x^3 + 5x^2c - 16xc^2 - 80c^3 = 0$.

26. Highest common factor. One method of finding the H.C.F. of two or more rational, integral expressions, which can be readily factored, is stated in the

RULE. *Separate each expression into its prime factors. Then find the product of such factors as occur in each expression, using each factor the least number of times it occurs in any one expression.*

EXERCISES

Find the H.C.F. of the following:

1. 28, 56, 84, and 35.
2. 225, 120, 210, and 135.
3. 198, 495, 693, and 1155.
4. 816, 1224, 1360, and 4080.
5. $91x^4y^3$, $133x^2y^6$, and $343x^5y^2$.
6. x^2-9, x^2+2x-3, and x^2+3x.
7. $a^2-9a+14$, a^2-4, and $5a^2-10a$.
8. x^3+27, $2x^2+3x-9$, and $5x^3+15x^2$.
9. $4x^2+20x$, x^3+4x^2-5x, and $4ax^2+20ax$.
10. $2x^5+8x$, $3ax^3+6ax+6ax^2$, and $3ax^7+12ax^3$.
11. $32-x^5$, x^2y-4y, $3xy-6y$, and x^2-4x+4.
12. $[(x+y)(x-y)]^3$, $x^4-2x^2y^2+y^4$, and $(x^6-y^6)^2$.
13. x^7+a^{7e}, x^2-a^{2e}, $x^2-3xa^e-4a^{2e}$, and x^8+a^{8e}.
14. $a^{5e}+4a^eb^8$, $a^{2e+1}b^2+2ab^6-2a^{e+1}b^4$, and $a^{8e}-16b^{16}$.

27. Lowest common multiple. The **lowest common multiple** of two or more rational, integral expressions is the expression of lowest degree which will exactly contain each.

For such expressions as can be readily factored the method of finding the L.C.M. is stated in the

RULE. *Separate each expression into its prime factors. Then find the product of all the different prime factors, using each factor the greatest number of times it occurs in any one expression.*

EXAMPLE

Find the L.C.M. of

$$12x^4y+12xy^4,\ 4x^3-4xy^2, \text{ and } 8x^4y-8x^3y^2+8x^2y^3.$$

Solution: $12x^4y+12xy^4=2^2\cdot 3\cdot xy(x+y)(x^2-xy+y^2).$

$$4x^3-4xy^2=2^2\cdot x(x+y)(x-y).$$

$$8x^4y-8x^3y^2+8x^2y^3=2^3x^2y(x^2-xy+y^2).$$

Therefore the L.C.M. is $2^3\cdot 3x^2y(x+y)(x-y)(x^2-xy+y^2)$.

EXERCISES

Find the L.C.M. of:

1. 24, 30, and 54.

2. 105, 140, and 245.

3. 15, 30, 35, 70, and 105.

4. 174, 485, 4611, and 5141.

5. $30\,ax^2$, $225\,a^5xy^2$, and $75\,a^4x^3y$.

6. $12\,x^2 + 6\,x$, $12\,x^3 - 3\,x$, and $16\,x^4 + 2\,x$.

7. $a^3 - 8\,b^3$, $4\,b^2 - a^2$, and $a^3b + 4\,ab^3 + 2\,a^2b^2$.

8. $x^4 + x^3y + x^2y^2$, $4\,x^3 - 4\,x^2y$, and $3\,x^3 - 3\,y^3$.

9. $x^3 - 2\,x^2 - 5\,x + 6$, $4 - x^2$, and $a - ax^2$.

10. $a^4 + 4\,a^2 + 16$, $a^2 - 4$, $a^3 + 8$, and $a^3 - 8$.

11. $x^3 - 2\,a^2x + ax^2$, $2\,a^3 + 3\,a^2x + ax^2$, and $4\,a^4x^2 - a^2x^4$.

12. $m^4 - 3\,m^2n^2 + 9\,n^4$, $m^3 + 3\,mn^2 + 3\,m^2n$, and $3\,n^3 + nm^2 - 3\,mn^2$.

13. $e^{2x} + e^{-2x} - 2$, $e^{2x} - e^{-2x}$, and $e^{2x} - 3 + 2\,e^{-2x}$.

14. $x^3 - 2\,x^2 - 2\,x - 3$, $x^3 - 27$, and $x^2 + x - 12$.

CHAPTER III

FRACTIONS

28. Operations on fractions. The four fundamental operations on fractions and the reduction of fractions to lower or to higher terms depend on the

PRINCIPLE. *The numerator and the denominator of a fraction may be multiplied by the same expression or divided by the same expression without changing the value of the fraction.*

29. Changes of sign in a fraction. In the various operations on fractions three signs must be considered, — the sign of the numerator, the sign of the denominator, and the sign before the fraction. Since a fraction is an indicated quotient, the law of signs in division gives us the

PRINCIPLE. *In a fraction the signs of both numerator and denominator, or the sign of the numerator and the sign before the fraction, or the sign of the denominator and the sign before the fraction, may be changed without altering the value of the fraction.*

30. Equivalent fractions. Two fractions are equivalent when one can be obtained from the other by multiplying or by dividing both of its terms by the same expression.

Two fractions having unlike denominators cannot be added, nor can one be subtracted from the other, until they have been reduced to respectively equivalent fractions having like denominators.

To change two or more fractions (in their lowest terms) to respectively equivalent fractions having the L.C.D., we have the

RULE. *Rewrite the fractions with their denominators in factored form.*

Find the L.C.M. of the denominators of the fractions.

Multiply the numerator and the denominator of each fraction by those factors of this L.C.M. which are not found in the denominator of the fraction.

If the L.C.D. is not easily obtained, any integral multiple of the denominators or the product of the denominators may be used. The result, however, will not be in its lowest term unless one uses the L.C.D.

31. Addition and subtraction of fractions. To find the algebraic sum of two or more fractions in their lowest terms, we have the

RULE. *Reduce the fractions to respectively equivalent fractions having the lowest common denominator. Write in succession over the lowest common denominator the numerators of the equivalent fractions, inclosing each numerator in a parenthesis preceded by the sign of the corresponding fraction.*

Rewrite the fraction just obtained, removing the parentheses in the numerator.

Then combine like terms in the numerator, and, if necessary, reduce the resulting fraction to its lowest terms.

EXERCISES

Reduce to lowest terms:

1. $\dfrac{18\,a^3c^2}{24\,a^2c^3}$.

2. $\dfrac{3\,x-6}{x^2-4}$.

3. $\dfrac{7\,a^2-14\,b^2}{a^4+a^2b^2-6\,b^4}$.

4. $\dfrac{x^3-8}{(x-2)^3}$.

5. $\dfrac{x^{2a}-c^{2b}}{(x^a+c^b)^2}$.

6. $\dfrac{(x^2-c^2)^2}{x^4-c^4}$.

7. $\dfrac{2\,x^{6e}-128}{x^{2e}-4}$.

8. $\dfrac{x^6-1}{x^4+x^2+1}$.

9. $\dfrac{3\,x^2+2\,x-21}{27\,x^4-147\,x^2}$.

10. $\dfrac{x^4-x^2+x-1}{x^4-1}$.

By the use of § 29 write in three other ways:

11. $\dfrac{-a}{c}$.

12. $-\dfrac{c}{-2\,a}$.

13. $\dfrac{c}{a-2\,c}$.

14. $\dfrac{2\,x-3\,y}{3\,x^2-6\,y^2-xy}$.

Change to equivalent fractions, writing the letters in the denominators in alphabetical order and making the first term in each factor positive:

15. $\dfrac{-3}{(c-a)(b-a)}$.

16. $\dfrac{x-y}{(y-x)(z-x)(z-y)}$.

17. Does $\dfrac{x-3}{5-x-x^2}=\dfrac{x+3}{x^2+x-5}$? Why?

Change to respectively equivalent fractions having the lowest common denominator:

18. $\frac{3}{4}$, $\frac{2}{5}$, $\frac{1}{6}$.

19. $\frac{7}{24}$, $\frac{9}{56}$.

20. $\dfrac{4}{ab}$, $\dfrac{3}{a^2c}$.

21. $\dfrac{3a+b}{6a}$, $\dfrac{a-2b}{4ab}$.

22. $\dfrac{2}{6x-12}$, $\dfrac{3}{x^2-2x}$.

23. $\dfrac{x-1}{x^2-5x+6}$, $\dfrac{x}{x^2-9}$.

24. Does $\dfrac{2x+5}{3a+5}$ equal $\dfrac{2x}{3a}$? Explain.

25. Define cancellation. Illustrate.

Find the algebraic sum of:

26. $\dfrac{3a}{6}-\dfrac{a-x}{9}-\dfrac{3x}{4}$.

27. $\dfrac{a-3}{a^2c}-\dfrac{2c-5}{ac^2}$.

28. $\dfrac{x-3}{4x^2}-\dfrac{3x^2-2}{14x^3}-\dfrac{x^3-8}{3x^4}$.

29. $\dfrac{5-x}{x-4}-\dfrac{5}{7}$.

30. $\dfrac{4m^2}{x^2-9m^2}-\dfrac{3m-x}{x+3m}$.

31. $\dfrac{x-3}{x^2-9x+14}-\dfrac{3}{x^2-4}$.

32. $\dfrac{x}{x^2-2x}-\dfrac{1}{x}-\dfrac{2+x}{x^2-4x+4}$.

33. $2-\dfrac{1}{v-1}+\dfrac{1}{1-v}$.

34. $x^2+x+1-\dfrac{x^3}{x-1}$.

35. $x^2-\dfrac{x^4-2x^2}{x^2-x+1}+x+1$.

36. $\dfrac{2x^2-3a^2}{27x^3-64a^3}-\dfrac{2x+a}{9x^2-16a^2}$.

37. $\dfrac{2}{(a-c)(x-c)}-\dfrac{3}{(c-a)(c-x)}$.

38. $\dfrac{2x-4}{9-x^2}-\dfrac{3x+1}{x-3}-\dfrac{x}{9-6x+x^2}$.

39. $\dfrac{2x+12c}{6x^2-13cx-5c^2}-\dfrac{3x-7c}{4x^2+4cx-35c^2}$.

40. $\dfrac{(a-s)(c-s)}{s(s-m)}+\dfrac{ac}{ms}+\dfrac{(a-m)(c-m)}{m(m-s)}$.

41. $\dfrac{2x^3-x^2}{x^2-x+1}-\dfrac{x^2-3}{x^4+x^2+1}+\dfrac{3x-5}{x^3+1}$.

Show that:

42. $c(c-y)\div d-(y+d)d\div c=y$, when $y=c-d$.

43. $\dfrac{a}{b}-\dfrac{c}{d}=\dfrac{16}{x^4+4x^2+16}$, when $a=x+2$, $b=x^2+2x+4$, $c=x-2$, and $d=x^2-2x+4$.

44. $\dfrac{e^{3x}-e^{-3x}}{e^x-e^{-x}}=5\frac{1}{4}$, when $e^{2x}=4$.

32. Multiplication of fractions. To find the product of two or more fractions or mixed expressions we have the

RULE. *If there are integral or mixed expressions, reduce them to fractional form.*

Separate each numerator and each denominator into its prime factors.

Cancel the factors (factor for factor) common to any numerator and any denominator.

Write the product of the factors remaining in the numerator over the product of the factors remaining in the denominator.

33. Division of fractions. For division of fractions we have the

RULE. *Reduce all integral or mixed expressions to fractional form.*

Then invert the divisor, or divisors, and proceed as in multiplication of fractions.

The **reciprocal** of a number is 1 divided by the number.

Thus the reciprocal of 2 is $\frac{1}{2}$; of $\frac{3}{5}$ is $\frac{5}{3}$; and of $3\frac{2}{7}$ is $\frac{7}{23}$.

Therefore the quotient of one fraction by another is the product of the first and the reciprocal of the second. Also the quotient of a fraction by an integral expression is the product of the fraction and the reciprocal of the integral expression.

34. Complex fractions. A **complex** fraction is a fraction containing a fractional expression either in its numerator or in its denominator or in both.

To simplify a complex fraction we have the

RULE. *Reduce both the numerator and the denominator to simple fractions, then perform the indicated division.*

EXERCISES

Perform the indicated operations:

1. $\frac{4x^2c^2}{9a^3}\cdot\frac{27a^2}{10x^3c}.$

2. $\left(\frac{2a}{cx}\right)^2\cdot\left(\frac{3c^2x}{4a^2}\right)^3.$

3. $\left(\frac{a}{6c^2}\right)\cdot\left(\frac{3a}{c}\right)^2\cdot\left(\frac{2c^2}{3a^4}\right)^2.$

4. $\left(\frac{-a}{x}\right)^3\cdot\left(\frac{-2x}{a}\right)^2\div\left(\frac{-c}{2a}\right)^4.$

5. $\left(\frac{3c^2}{2a}\right)^2\div\left(\frac{9c^2x^3}{4a^2}\right)^2\div\left(\frac{-2a^3c^2}{5x^3}\right)^3\frac{(3a^2c)^4}{(10x^3)^2}.$

6. $(3c-ac)\div(4m)\cdot(8m^2x^2)\div(9m-a^2m).$

7. $\frac{4x^e-2y^e}{(2x^e-y^e)^2}\cdot\frac{4x^{2e}-y^{2e}}{(2x^e+y^e)^2}\cdot\frac{2x^e+y^e}{3}.$

8. $\frac{x^2-1}{x^4-1}\left(1-\frac{2}{x^6+1}\right)\frac{x^4-x^2+1}{x^2+x+1}.$

9. $\left(x^2-2x+4-\frac{16}{x+2}\right)\frac{x^2-4}{3}.$

10. In Exercise 9 multiply each term of the first expression by $\frac{x^2-4}{3}$. Then add the partial products, and compare this sum with the product obtained in Exercise 9. Explain. Name and state the law of multiplication here illustrated.

11. $\dfrac{a^2 - a - 90}{a^2 - 100} \div \dfrac{a^3 + 9a^2}{a^2 + 10a} \div \dfrac{4a + 6}{2a^2x + 3ax}.$

12. $\left(a - \dfrac{13a}{a + 6}\right) \div \dfrac{a^2 - a - 42}{a^2 - 36} \div \dfrac{2a^3 - 12a}{ax + 6x}.$

13. $\left(\dfrac{3a^4 - 75a^2}{3a - 7}\right)\left(6a - \dfrac{7}{a} - 11\right) \div \left(2 + \dfrac{5}{a^2} + \dfrac{11}{a}\right).$

14. $\left(\dfrac{c^2 + a^2 + 2ac}{a^2 - 3ac - 4c^2}\right)\left(2c + a - \dfrac{5a + 10c}{c + a}\right) \div (a + 2c).$

15. $\left(\dfrac{4c}{a} - \dfrac{15c^2}{a^2} + 4\right)\left(3 - \dfrac{4a + 20c}{2a + 5c}\right) \div \left(4 - \dfrac{16c}{a} + \dfrac{15c^2}{a^2}\right).$

16. $\dfrac{1}{mx}\left(\dfrac{m}{x} + \dfrac{x}{m}\right) \div \left(\dfrac{m^6 + x^6}{m^3x^3}\right)\left(m^2 - x^2 + \dfrac{x^4}{m^2}\right)\left(\dfrac{am + mx}{cx - ax}\right).$

17. Multiply $\dfrac{x^3 - 8}{x^4 + 4x^2 + 16} \cdot \dfrac{x^2 - 4}{(x-2)^2}$ by the reciprocal of $\dfrac{(x+2)^2}{x^3 + 8}.$

18. $\dfrac{2x^2 + 5x + 3}{6x^2 + x - 1} \cdot \dfrac{3x^2 + 3ax - x - a}{2x^2 - 2cx + 3x - 3c}$

$$\left(\frac{c + 2a + (2a + 1 + 2c)x}{x^2 + x + ax + a} - 2\right).$$

Simplify each term in:

19. $\left(\dfrac{2x^3}{c^2}\right)^5 - 5\left(\dfrac{2x^3}{c^2}\right)^4\left(\dfrac{c^3}{15x^4}\right)$

$$+ 10\left(\frac{2x^3}{c^2}\right)^3\left(\frac{c^3}{15x^4}\right)^2 - 10\left(\frac{2x^3}{c^2}\right)^2\left(\frac{c^3}{15x^4}\right)^3.$$

20. $\left(\dfrac{a^2}{b^3}\right)^{10} - 10\left(\dfrac{a^2}{b^3}\right)^9\left(\dfrac{b^2}{6a^4}\right)$

$$+ \frac{10 \cdot 9}{2}\left(\frac{a^2}{b^3}\right)^8\left(\frac{b^2}{6a^4}\right)^2 - \frac{10 \cdot 9 \cdot 8}{2 \cdot 3}\left(\frac{a^2}{b^3}\right)^7\left(\frac{b^2}{6a^4}\right)^3.$$

21. Explain why the divisor is inverted in division of fractions.

22. Show how the rule for the division of fractions is based on the first principle of this chapter.

Simplify:

23. $\dfrac{8-\frac{1}{27}}{2-\frac{1}{3}}$.

24. $\dfrac{\frac{1}{16}+\frac{1}{4}+1}{\frac{1}{4}-\frac{1}{2}+1}$.

25. $\dfrac{7\frac{3}{7}}{40\frac{5}{9}}\cdot\dfrac{146}{17\frac{1}{3}}$.

26. $\dfrac{3\frac{1}{2}\cdot 3\frac{1}{2}\cdot 3\frac{1}{2}-1}{3\frac{1}{2}\cdot 3\frac{1}{2}-1}$.

27. $\dfrac{\frac{a}{x}+3}{\frac{a^2}{x^2}-9}$.

28. $\dfrac{1-\frac{m}{x}}{\frac{4m^4}{x^4}-4}$.

29. $\dfrac{2a-\frac{a^2+c^2}{c}}{\frac{a}{c}-1}$.

30. $\dfrac{3-\frac{a}{c}}{\frac{(a+3c)^2}{2ac}-6}$.

31. $\dfrac{c-\frac{4}{c}}{\frac{1}{c^2}+\frac{2}{c^3}-\frac{8}{c^4}}$.

32. $\dfrac{\left(\frac{4c+3a}{3a}\right)^2-\frac{6c}{a}}{\frac{(3a+2c)^2}{3a}-8c}$.

33. $1+\dfrac{1}{1+\frac{1}{1+2}}$.

34. $1-\dfrac{2}{3+\frac{4}{5-\frac{6}{7}}}$.

35. $3-\dfrac{1}{1-\frac{4}{6+\frac{3}{7-\frac{2}{5}}}}$.

36. $\dfrac{1-\frac{a^2b^2}{(a^2+b^2)^2}}{\frac{ab}{a^2+b^2}+1}$.

37. $\dfrac{1+\frac{y}{x}+\frac{x}{y}}{\frac{x^2}{y^2}+1+\frac{y^2}{x^2}}$.

38. $\dfrac{\frac{m^2}{n^2}+\frac{mn}{mn}+\frac{n^2}{m^2}}{\frac{m^2+n^2}{mn}-1}$.

39. $\dfrac{6x-11-\frac{7}{x}}{2+\frac{11}{x}+\frac{5}{x^2}}\div\dfrac{\frac{5-x}{3x}}{x^2-25}$.

40. $\dfrac{1}{1-\frac{1}{1-\frac{1}{x}}}$.

41. $\dfrac{1}{1+\frac{a}{a+1+\frac{2a^2}{1-a}}}$.

42. $\dfrac{\frac{xy+1}{y}}{x+\frac{1}{\frac{yz+1}{z}}}-\dfrac{1}{y(x+xyz+z)}$.

43. $\dfrac{\frac{1}{x}-\frac{x-a}{x^2-a^2}}{\frac{1}{a}-\frac{a-x}{a^2+x^2}}$.

44. What value has $\dfrac{a^3 - b^3}{a^2 + b^2}$ when $a = e^x - \dfrac{1}{e^x}$ and $b = e^x + \dfrac{1}{e^x}$?

45. Brouncker (1620–1684) proved that π (the circumference of a circle divided by its diameter) is four times the fraction on the right.

$$\cfrac{1}{1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \cfrac{81}{2 + \text{etc.}}}}}}}$$

(*a*) Rewrite the fraction, continuing it to $\dfrac{225}{2 + \text{etc.}}$.

(*b*) Stopping with $2 + \frac{81}{2}$, find the difference in value between four times the value of this fraction and the approximate value of π, 3.1416.

Note. William Brouncker, one of the brilliant mathematicians of his time, was an intimate friend of John Wallis (see "First Course in Algebra," page 302). These two scientists were among the pioneers in the study of expressions with a countless number of terms.

The complex fraction in the exercise, if continued indefinitely according to the law which its form suggests, is called an infinite continued fraction. Brouncker was the first to study the properties of such expressions.

CHAPTER IV

LINEAR EQUATIONS IN ONE UNKNOWN

35. Definitions. An **equation** is a statement of the equality between two equal numbers or number symbols.

Equations are of two kinds, — **identities** and **equations of condition.**

An arithmetical or an algebraic **identity** is an equation in which, if the indicated operations are performed, the two members become precisely alike, term for term.

A literal identity is true for *any* value (numerical or literal) of the letters in it.

An equation which is true only for certain values of a letter in it, or for certain sets of related values of two or more of its letters, is an **equation of condition,** or simply an equation.

A number or literal expression which, being substituted for the unknown letter in an equation, changes it to an identity, is said to **satisfy** the equation.

After the substitution is made it is usually necessary to simplify the result before the identity becomes apparent.

A **root** of an equation is any number or number symbol which satisfies the equation.

36. Axioms. An **axiom** is a statement the truth of which is accepted without proof.

Axiom I. *Adding the same number to each member of an equation does not destroy the equality.*

Axiom II. *Subtracting the same number from each member of an equation does not destroy the equality.*

Axiom III. *Multiplying each member of an equation by the same number does not destroy the equality.*

Axiom IV. *Dividing each member of an equation by the same number (not zero) does not destroy the equality.*

Two or more equations, even if of very different form, are **equivalent** if all are satisfied by every value of the unknown which satisfies any one.

Of these four axioms, or assumptions, we shall make constant use. If the "same number" referred to in each is expressed arithmetically, the result is always an equation *equivalent* to the *original* one. Further, if *identical expressions involving the unknown* be added to or subtracted from each member of an equation, the resulting equation is equivalent to the first. If, however, both members of an equation be multiplied by or divided by identical expressions containing the unknown, the resulting equation *may not* be equivalent to the original one. In other words, under the condition just stated, roots may be introduced or lost by the use of Axiom III or IV respectively.

EXAMPLES

1. Let $x - 2 = 7.$ (1)
Multiplying by $x - 3$, $x^2 - 5x + 6 = 7x - 21.$ (2)
From (2), $x^2 - 12x + 27 = 0.$ (3)
Whence $(x - 3)(x - 9) = 0.$ (4)
Therefore $x = 3$ or 9.

Since (1) has the root 9 only, and (3) has the two roots 3 and 9, (3) is not equivalent to (1), that is, a root was introduced by the use of the multiplication axiom.

2. Let $x^2 - 4 = x + 2.$ (1)
Dividing by $x + 2$, $x - 2 = 1.$ (2)
Solving (2), $x = 3.$ (3)
But from (1), $x^2 - x - 6 = 0.$ (4)
Whence $(x - 3)(x + 2) = 0.$ (5)
Therefore $x = 3$ or -2.

Here (5) shows that (1) has the two roots 3 and -2, and since (2) has but one root, 3, it is evident that a root was lost by the use of the division axiom.

The student should note the preceding illustration carefully, as the possibility of dividing each member of an equation by a common factor involving the unknown frequently arises. A very common type is the following:

3. Let	$x^2 - 2x = 0.$	(1)
Dividing by x,	$x - 2 = 0.$	(2)
Whence	$x = 2.$	(3)

But (1) has the root $x = 0$ also, which is lost by dividing both members of (1) by x.

If proper methods of solution are applied to an equation (or to a false statement in the form of an equation), and one or more values of the unknown which are thus obtained do not satisfy the original statement, such values are called **extraneous** (or extra) roots.

An extraneous root is a root of an equation which is not equivalent to the original statement, but which is derived from it in the process of solution.

37. Principles. The preceding discussion may be summed up thus:

PRINCIPLE I. *If identical expressions (which may or may not contain the unknown) be added to or subtracted from each member of an equation, the resulting equation is equivalent to the original one.*

PRINCIPLE II. *Extraneous roots may be introduced into a solution by multiplying both members of an equation by an integral expression containing the unknown.*

PRINCIPLE III. *If both members of an equation be divided by an integral expression containing the unknown, one or more roots will usually be lost by such a division.*

It can be seen from Example 1 that the root introduced is the value of the unknown obtained by setting the multiplier, $x - 3$, equal to zero and solving the resulting equation.

Similarly, it can be seen from Example 2 that the root lost is that value of the unknown obtained by setting the divisor, $x + 2$, equal to zero and solving the resulting equation.

Sometimes a statement in the form of an equation has no root; yet the ordinary method of solution appears to give one. For example, consider the statement $\frac{4x-1}{x-3}=\frac{x+8}{x-3}+5$.

If one multiplies each member by $x-3$ and solves as usual, he obtains $x=3$. This answer cannot be verified because $x-3$, the denominator, becomes zero for $x=3$. Here the multiplication by $x-3$ introduced the value 3 for x. Checking will always discover the falsity of such a result (see "First Course in Algebra," page 161).

For solving equations in one unknown which may or may not involve fractions we have the

RULE. *Free the equation of any parentheses it may contain except such as inclose factors of the denominators.*

Where polynomial denominators occur, factor them if possible.

Find the L.C.M. of the denominators of the fractions and multiply each fraction and each integral term of the equation by it, using cancellation wherever possible.

Transpose and solve as usual.

Reject all values for the unknown which do not satisfy the original equation.

Checking the solution of an equation is often called **testing,** or **verifying,** the result. For this we have the

RULE. *Substitute the value of the unknown obtained from the solution in place of the letter which represents the unknown in the original equation. Then simplify the result until the two members are seen to be identical.*

EXERCISES

1. Give an example of (*a*) a numerical identity; (*b*) a literal identity; (*c*) a conditional equation; (*d*) two equivalent equations; (*e*) a statement in the form of an equation, but which has no root.

2. Define and illustrate transposition. (*a*) On what axiom does transposition depend? (*b*) If one equation is obtained from another by transposition, are the two equivalent? Explain.

3. What extraneous roots, if any, are introduced if both members of the following equations are multiplied by the expression on the right?

(*a*) $x + 3 = 7$	$x + 4$
(*b*) $x + 5 = 0$	$x + 5$
(*c*) $x + c = 0$	$x - c$
(*d*) $x + a = 0$	x
(*e*) $x = 5$	4
(*f*) $x - 1 = 0$	$x^2 + 3x + 2$

4. What roots, if any, are lost by dividing both members of the following equations by the expression on the right?

(*a*) $x^2 - 4 = 0$	$x - 2$
(*b*) $x^2 - 4x + 3 = x - 3$	$x - 3$
(*c*) $x^2 + x - 12 = x + 4$	$x + 4$
(*d*) $x^2 - 2x = 4x$	x
(*e*) $x^4 - 16 = x^2 - 4$	$x^2 - 4$
(*f*) $(x - a)^3 = (x - a)^2$	$x - a$

Solve the following for the unknowns involved, considering *a*, *b*, *c*, and *d* as known numbers:

5. $\frac{7x}{3} - \frac{1}{6} = \frac{9x}{4}.$

6. $\frac{5x}{21} - \frac{a}{6} = \frac{x}{14}.$

7. $2 - \frac{1}{y} = \frac{3}{2}.$

8. $\frac{3}{2x} - \frac{3}{20} = \frac{7}{5x}.$

9. $\frac{5 + n}{n + 6} = \frac{3}{4}.$

10. $\frac{m - 2}{m - 3} = \frac{17}{18}.$

11. $\frac{2x + 5}{10x} - \frac{3(2x + 1)}{2x} = 3\frac{1}{5}.$

12. $\frac{5x - 7}{6} - \frac{5}{2}\left(\frac{4 - x}{10}\right) = \frac{15x - 22}{6}.$

13. $\frac{1 - 8x}{5} - \frac{2(1 - 6x)}{24x - 3} = \frac{1 - 24x}{15}.$

14. $\frac{3x - 9}{3x - 5} - 2 = \frac{3x - 5}{8 - 3x}.$

15. $\frac{1}{a} + \frac{1}{b} - \frac{1}{c} = \frac{1}{x}.$

16. $\frac{x}{a} + \frac{x + m}{b} = 1.$

17. $\frac{cy}{2d} - c^2 = \frac{2dy - c^3}{c}.$

18. $\frac{2x-3b}{a}-\frac{2a-3x}{b}+\frac{9b}{a}=\frac{4a}{b}-5.$

19. $\frac{3}{b(a-x)}+\frac{1}{a-x}=\frac{-3(b+3)}{2ab}.$

20. $\frac{1}{z-2}+\frac{3z}{z-2}=3.$

21. $\frac{3}{2y+1}-\frac{1+2y}{2y-1}=\frac{4y^2}{1-4y^2}.$

22. $\left(1-\frac{1}{c}\right)\div\left(1+\frac{1}{z}\right)+\frac{c+z}{z+1}=1.$

23. $\frac{3b+9x}{9a+6x}-\frac{a-2x}{2x+3a}=2.$

24. $\frac{8}{x-7}-\frac{2-6x}{x^2-6x-7}=\frac{27}{x+1}.$

25. $\frac{4}{x+4}-\frac{3}{x-5}=\frac{2(x-14)}{x^2-x-20}.$

26. $\frac{x}{2(a+b)}-\frac{5}{b-a}=\frac{bx}{b^2-a^2}.$

27. $\frac{2x}{3}-\frac{d}{5}\left(\frac{6x}{c}-10d\right)=2cd\left(\frac{2}{3}-\frac{d}{5c}\right).$

28. $82.4-13x=32x-52.6.$

29. $.5704-.20x=19.6512-.016x.$

30. $.01(2x+.205)-.0125(1.5x-.5)=.01955.$

31. $\frac{2x-5.5}{9}=x-1.25-\frac{7x^2+3.5x-84.4375}{9x-11.75}.$

32. $\frac{1}{x-.33a}+\frac{1}{\frac{2x}{3}-.22a}=\frac{5}{4a}.$

33. $\frac{4a}{3y-.12a}-\frac{3}{2}=\frac{5a}{6y-.24a}-\frac{4a}{y-.04a}.$

34. $\frac{c^2}{ax}+\frac{1}{c}=\frac{3c-3a}{x}+\frac{1}{a}+\frac{a^2}{cx}.$

PROBLEMS

1. At what time between 4 and 5 o'clock will the hands of a clock be together?

Solution: First, the minute hand moves twelve times as fast as the hour hand. Second, at 4 o'clock the hour hand is 20 minute spaces ahead of the minute hand. Now let x equal the number of minute spaces that the minute hand travels from its position at 4 o'clock until it overtakes the hour hand. Obviously the hour hand must travel $x - 20$ spaces before it is overtaken by the minute hand.

Therefore $x = (x - 20)12.$

Whence $x = 21\frac{9}{11}.$

Hence the hands are together at $21\frac{9}{11}$ minutes after 4 o'clock.

2. At what time between 7 and 8 o'clock are the hands of a clock together?

3. At what time between 2 and 3 o'clock are the hands of a clock in a straight line?

4. At what time between 6 and 7 o'clock is the minute hand (*a*) 10 minute spaces ahead of the hour hand? (*b*) 10 minute spaces behind the hour hand?

5. At what times between 5 and 6 o'clock are the hands of a clock at right angles?

6. If the earth is between a planet and the sun and in a line with them, the planet is said to be in *opposition*. The earth and Mars revolve about the sun in (approximately) 365 days and 687 days respectively. Mars was in opposition September 24, 1909. What is the approximate date of the next opposition?

Solution: For the sake of simplicity we will suppose in this and in similar problems that the planets move in the same plane and in circular paths, of which the sun is the center.

Let $x =$ the required number of days.

Now in x days the earth will make $\frac{x}{365}$ revolutions about the sun.

And in x days Mars will make $\frac{x}{687}$ revolutions about the sun.

But to be in opposition the earth must in x days go round the sun once more than Mars does.

Therefore $\frac{x}{365} = \frac{x}{687} + 1.$

Clearing, $687\,x = 365\,x + 250755.$

Whence $x = 779 +.$

Therefore the required date is November 11, 1911.

7. If a planet is between the earth and the sun and in a line with them, it is said to be in *conjunction.* Venus was in (superior) conjunction April 28, 1909. If Venus revolves about the sun once in 225 days, find the approximate date of the next conjunction.

8. Jupiter revolves about the sun once in 4332 days. On February 28, 1909, the planet was in opposition. Find the approximate date of the next opposition.

9. Saturn revolves about the sun once in 10,759 days. It was in opposition April 3, 1909. Find the approximate date of the next opposition.

10. Two men travel in the same direction around an island, one making the circuit every $2\frac{1}{2}$ hours and the other every 3 hours. If they start together, after how many hours will they be together again?

11. Three automobiles travel in the same direction around a circular road. They make the circuit in $2\frac{3}{4}$ hours, $3\frac{1}{3}$ hours, and $4\frac{2}{5}$ hours respectively. If they start at the same time, after how many hours are the three together again?

12. Is the answer to Exercise 10 an integral multiple of $2\frac{1}{2}$ and 3? Is it the least integral multiple?

13. Is the answer to Exercise 11 an integral multiple of $2\frac{3}{4}$, $3\frac{1}{3}$, and $4\frac{2}{5}$? Is it the least integral multiple?

14. Reduce $2\frac{3}{4}$, $3\frac{1}{3}$, and $4\frac{2}{5}$ to improper fractions and divide the L.C.M. of the numerators by the G.C.D. of the denominators. Compare the result with the answer to Exercise 11.

15. The method of finding the L.C.M. of two or more fractions or mixed numbers is hinted at in Exercise 14. State a rule therefor. Find by the rule the L.C.M. of $1\frac{1}{6}$, $2\frac{1}{3}$, and $3\frac{1}{9}$.

16. Find by the same rule the L.C.M. (*a*) of $\frac{a}{b}$ and $\frac{c}{d}$; (*b*) of $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$.

17. How many ounces of alloy must be added to 56 ounces of silver to make a composition 70% silver?

18. Gun metal of a certain grade is composed of 16% tin and 84% copper. How much tin must be added to 410 pounds of this gun metal to make a composition 18% tin?

HINT. Since the composition is 16% tin, then $\frac{16}{100} \cdot 410 =$ the number of pounds of tin in the first composition.

Let $x =$ the number of pounds of tin to be added.

Then $\frac{16 \cdot 410}{100} + x =$ the number of pounds of tin in the second composition, and $410 + x =$ the number of pounds of both metals in the second composition.

Therefore $$\frac{\frac{16 \cdot 410}{100} + x}{410 + x} = \frac{18}{100}, \text{ etc.}$$

19. A 30-gallon mixture of milk and water tests 16% cream. How many gallons of water has been added if the milk is known to test 20% cream?

20. How many gallons of alcohol 90% pure must be mixed with 10 gallons of alcohol 95% pure so as to make a mixture 92% pure?

21. The diameter of the earth is $3\frac{2}{3}$ times that of the moon, and the difference of the two diameters is 5760 miles. Find each diameter in miles.

22. The diameter of the sun is 3220 miles greater than 109 times the diameter of the earth, and the sum of the two diameters is 874,420 miles. Find each diameter in miles.

23. The distance of the earth from the sun is $387\frac{1}{2}$ times the earth's distance from the moon. Light traveling 186,000 miles per second would require 8 minutes $18\frac{2}{3}\frac{2}{1}$ seconds longer to go from the earth to the sun than from the earth to the moon. Find each distance in miles.

24. The mean distance between Mars and the earth when they are on opposite sides of the sun is 234,500,000 miles. When the two planets are nearest each other on the same side of the sun, the mean distance between them is 48,500,000 miles. Find the distance of each from the sun in miles.

25. The diameter of Jupiter is $10\frac{10}{11}$ times the diameter of the earth, and the sum of their diameters is 94,320 miles. Find each diameter in miles.

26. A can do a piece of work in 15 days and B in 25 days. After they have worked together 3 days, how many days will B require to finish the work?

27. A can do a piece of work in a days, B in b days, and C in $a + b$ days. How many days will it take them all working together to do the work?

28. A cistern has two pipes. By one it can be filled in $2m$ hours; by the other it can be emptied in $\frac{n+1}{3}$ hours. Assume $2m$ less than $\frac{n+1}{3}$ and find the number of hours required to fill the cistern if both pipes are opened.

29. Discuss Problem 28 thus: What is the relation between m and n if (a) the water runs out more slowly than it comes in; (b) the water runs out as fast as it comes in; (c) the water runs out faster than it comes in?

30. If both pipes in Problem 28 had been intake pipes, how many hours would have been required to fill the cistern one-xth full?

31. If the radius of a circle is increased 7 inches, the area is increased 440 square inches. Find the radius of the first circle ($\pi = \frac{22}{7}$ approximately).

Facts from Geometry. The area of a circle is the square of the radius multiplied by π ($\pi = 3.1416$ approximately). This is expressed by the formula $A = \pi R^2$.

The circumference of a circle equals the diameter times π. The usual formula is $C = 2\pi R$.

32. Imagine that a circular hoop one foot longer than the circumference of the earth is placed about the earth so that it is everywhere equidistant from the equator and lies in its plane. How far from the equator will the hoop be?

33. Compare the result of Exercise 32 with the one obtained when a similar process is carried out on a flagpole 6 inches in diameter, instead of the earth.

Repeating decimals. If the process of reducing a common fraction to a decimal does not end, the result is a **repeating** decimal. Thus $\frac{3}{11} = .272727\cdots$. Here, as often as 7 appears in the quotient there is a remainder of 3; that is, we return after each 7 to the original fraction. Then

$$\tfrac{3}{11} = .27\tfrac{3}{11}, \text{ or } \tfrac{3}{11} = .2727\tfrac{3}{11}, \text{ etc.}$$

Now in $.27\frac{3}{11}$ the remainder $\frac{3}{11}$ is really $\frac{3}{11}$ of .01 (since 7 stands in hundredths' place), or $\dfrac{3}{11}$ of $\dfrac{1}{100}$, or $\dfrac{\frac{3}{11}}{100}$. Therefore, if $x = \dfrac{3}{11}$, the identity $\dfrac{3}{11} = .27\frac{3}{11}$ may be written $x = .27 + \dfrac{x}{100}$.

The relation just explained will enable us to find the common fraction which generates any given repeating decimal.

Dots are used to abbreviate the writing of a repeating decimal. Thus $.\dot{3}\dot{5}$ means $.3535\cdots$, and $.04\dot{6}3\dot{2}$ means $.04632632\cdots$.

34. Find the common fraction which, reduced to a decimal, gives $.393939\cdots$.

Solution: Let x = the required fraction.

Then $\qquad x = .39 + \dfrac{x}{100}$.

Whence $\qquad 100\,x = 39 + x$.

Solving, $\qquad x = \frac{39}{99} = \frac{13}{33}$.

35. Find the common fraction which, expressed decimally, is:

(*a*) $.\dot{1}\dot{2}$. (*b*) $.\dot{7}$. (*c*) $.\dot{5}6\dot{7}$. (*d*) $.\dot{0}\dot{2}$.

(*e*) $3.\dot{2}\dot{5}$. Hint. Note that 3 does not repeat.

(*f*) $12.\dot{1}8\dot{9}$. (*g*) $.\dot{7}1428\dot{5}$. (*h*) $.\dot{1}4285\dot{7}$.

36. A passenger train whose rate is 42 miles per hour leaves a certain station a hours and b minutes after a freight train. The passenger train overtakes the freight in b hours and a minutes. Find the rate of the freight train in miles per hour.

37. The arms of a lever are 3 feet and 4 feet in length respectively. What weight on the shorter arm will balance 100 pounds on the longer?

38. A beam 12 feet long supported at each end carries a load of 3 tons at a point 5 feet from one end. Find the load in tons (excluding the weight of the beam itself) on each support.

39. The arms of a balanced lever are 8 feet and 12 feet respectively, the shorter arm carrying a load of 24 pounds. If the load on the longer arm be reduced 4 pounds, how many feet from the fulcrum must an 8-pound weight be placed on the longer arm to restore the balance?

40. A horizontal beam 12 feet long of uniform cross section is hinged at one end and rests on a support which is 4 feet from the other. The free end carries a load of 130 pounds. Excluding the weight of the beam itself, what is the weight in pounds on the support?

HINT. The products of the upward and downward pressures by their respective arms are equal.

41. A 14-foot horizontal beam of uniform cross section weighing 200 pounds is hinged at one end and rests on a support at the other end. (a) What is the weight in pounds on the support? (b) If the support is moved in 3 feet from the end of the beam, find the pressure in pounds on the support.

42. A 16-foot horizontal beam of uniform cross section weighs 300 pounds. It rests on two supports, one at one end and the other 4 feet from the other end. Find the weight in pounds on each support.

CHAPTER V

LINEAR SYSTEMS

38. Graphical solution of a linear system. The construction of the graph of a single linear equation in two variables or of a linear system in two variables depends on several assumptions and definitions. It is agreed:

I. To have at right angles to each other two lines: $X'OX$, called the ***x*-axis**; and $Y'OY$, called the ***y*-axis.**

II. To have a line of definite length for a unit of distance. Then the number 2 will correspond to a distance of twice the unit, the number $4\frac{1}{2}$ to a distance $4\frac{1}{2}$ times the unit, etc.

III. That the distance (measured parallel to the x-axis) from the y-axis to any point in the surface of the paper be the ***x*-distance** (or abscissa) of the point, and the distance (measured parallel to the y-axis) from the x-axis to the point be the ***y*-distance** (or ordinate) of the point.

IV. That the x-distance of a point to the *right* of the y-axis be represented by a *positive* number, and the x-distance of a point to the *left* by a *negative* number; also the y-distance of a point above the x-axis be represented by a *positive* number, and the y-distance of a point *below* the x-axis by a *negative* number. Briefly, *distances measured from the axis to the* ***right*** *or* ***upward*** *are* ***positive,*** *to the* ***left*** *or* ***downward, negative.***

V. That every point in the surface of the paper corresponds to a *pair of numbers,* one or both of which may be positive, negative, integral, or fractional.

VI. That of a given pair of numbers the first be the measure of the x-distance and the second the measure of the y-distance. Thus the point (2, 3) is the point whose x-distance is 2 and whose y-distance is 3.

VII. To call the point of intersection of the axes the **origin.**

The values of the x- and the y-distances of a point are often called the **coördinates** of the point.

The graph of a linear equation in two variables is a straight line. Therefore it is necessary in constructing the graph of such an equation to locate only two points whose coördinates satisfy the equation and then to draw through the two points a straight line. It is usually most convenient to locate the two points where the line cuts the axes. If these two points are very close together, however, the direction of the line will not be accurately determined. This error can be avoided by selecting two points at a greater distance apart.

The **graphical solution** of a linear system in two variables consists in plotting the two equations to the same scale and on the same axes, and obtaining from the graph the values of x and y at the point of intersection. Two straight lines can intersect in but one point. Hence but one pair of values of x and y satisfies a system of two independent linear equations in two variables.

Through the graphical study of equations we unite the subjects of geometry and algebra, which have hitherto seemed quite separate, and learn to interpret problems of the one in the language of the other.

(For further details see "First Course in Algebra," pages 187–200.)

EXAMPLE

Solve graphically the system $2x - y + 6 = 0$ and $x + 2y + 8 = 0$.

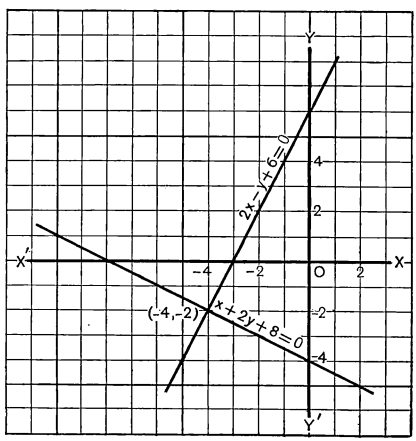
Solution: Substituting zero for x and then zero for y in each equation, we obtain for $2x - y + 6 = 0$,

$$\begin{array}{c|c|c|} x & 0 & -3 \\ \hline y & 6 & 0 \end{array},$$

and for $x + 2y + 8 = 0$,

$$\begin{array}{c|c|c|} x & 0 & -8 \\ \hline y & -4 & 0 \end{array}.$$

Then constructing the graph of each equation as indicated in the adjacent figure, we obtain for the coördinates of the point of intersection of the two lines, $x = -4$ and $y = -2$.



EXERCISES

Solve graphically:

1. $2x + y = 8$, $x + 2y = 7$.

2. $x - y = 6$, $3x + 4y = 4$.

3. $x + 2y + 11 = 0$, $y - x = 2$.

4. $x + 2y = 0$, $8y + 2x = 3$.

5. $x + 5 = -3y$, $6y + 2x - 8 = 0$.

6. $2x + 4y = 20$, $2y - 10 = -x$.

7. $x + y = 5$, $y + 2 = 0$.

8. $y + 4 = 0$, $2 - x = 0$.

In Exercise 9 graph each equation. Then add or subtract the two equations and graph the resulting equation on the same axes. Note the position of the third graph with reference to the other two. Proceed in like manner with Exercises 10 and 11.

9. $x + y = 4$, $x + 2y = 7$.

10. $x - y = 5$, $3x + 2y = 5$.

11. $3x - 4y = 12$, $4x + 3y = -6$.

12. In each of the last three exercises will the values of the x- and y-coördinates of the point of intersection of the two lines, as obtained from the graph, verify in the third equation obtained by adding the two given equations? Why?

13. Graph the equation $x - 2y = 2$. Then multiply both members by 2 or 3 and graph the resulting equation. Compare the two graphs. Then try -2 or -3 as a multiplier and graph the resulting equation. Compare the three graphs. What conclusion seems warranted?

14. What are the coördinates of the origin?

15. Is a graphical solution of a linear system in two variables ever impossible? Explain.

16. In the example on page 55 could different scales have been used on the two axes? Could the two lines have been plotted to different scales? Explain.

17. What is the form of the equation of a line parallel (*a*) to the x-axis? (*b*) to the y-axis?

18. What is the form of the equation of a line through the origin?

19. Give an example of a system in two unknowns which has (*a*) no graphical solution; (*b*) an infinite number of sets of roots.

20. The boiling point of water on a Centigrade thermometer is marked 100°, and on a Fahrenheit 212°. The freezing point on the Centigrade is zero and on the Fahrenheit is 32°. Consequently a degree on one is not equal to a degree on the other, nor does a temperature of 60° Fahrenheit mean 60° Centigrade. Show that the correct relation is expressed by the equation $C. = \frac{5}{9}(F. - 32)$, where C. represents degrees Centigrade and F. degrees Fahrenheit. Construct a graph of this equation. Can you, by means of this graph, express a Centigrade reading in degrees Fahrenheit, and vice versa?

21. By means of the graph drawn in Exercise 20 express the following Centigrade readings in Fahrenheit readings, and vice versa: (*a*) 60° C.; (*b*) 150° F.; (*c*) – 20° C.; (*d*) – 30° F.

22. What reading means the same temperature on both scales?

23. A boy starts at the southwest corner of a field and walks 20 rods, keeping twice as far from the south fence as from the west fence. He then walks east until he is three times as far from the west fence as he is from the south fence. Lastly he walks north until he is as far from one fence as he is from the other. Construct a graph of his path. Find (by measurement) the length of each portion of it and his distance from the starting point.

39. Elimination. The process of deriving from a system of n equations a system of $n-1$ equations, containing one variable less than the original system, is called **elimination.**

If one equation of a system can be obtained from one or more of the other equations of the system by the direct application of one or more of the axioms, it is called a **derived** equation; if it cannot be so obtained, it is called **independent.**

Only two methods of elimination will be considered, — that of *addition* or *subtraction*, and that of *substitution.*

40. Solution by addition or subtraction. The method of solving a system of two linear equations by addition or subtraction is illustrated in the

EXAMPLE

Solve the system $\begin{cases} 13x + 3y = 14, & (1) \\ 7x - 2y = 22. & (2) \end{cases}$

Solution: Eliminate y first as follows:

$(1) \cdot 2, \quad 26x + 6y = 28. \quad (3)$

$(2) \cdot 3, \quad 21x - 6y = 66. \quad (4)$

$(3) + (4), \quad 47x = 94. \quad (5)$

$(5) \div 47, \quad x = 2. \quad (6)$

Substituting 2 for x in (2), $14 - 2y = 22. \quad (7)$

Solving (7), $y = -4.$

Check: Substituting 2 for x and -4 for y in (1) and (2) gives

$$\begin{cases} 26 - 12 = 14, \text{ or } 14 = 14. \\ 14 + 8 = 22, \text{ or } 22 = 22. \end{cases}$$

The method of the preceding solution is stated in the

RULE. *If necessary, multiply each member of the first equation by a number and each member of the second equation by another number, such that the coefficients of the same variable in both the resulting equations will be numerically equal.*

If these coefficients have like signs, subtract one equation from the other; if they have unlike signs, add and solve the equation thus obtained.

Substitute the value just found, in the simplest of the preceding equations which contains both variables, and solve for the other variable.

CHECK. Substitute for each variable in the original equations its value as found by the rule. If the resulting equations are not obvious identities, simplify them until they become such.

Or check in the sum of the two equations unless one unknown vanishes in the addition.

41. Solution by substitution. The method of solving a system of two linear equations by substitution is illustrated in the

EXAMPLE

Solve the system $\begin{cases} 3x - 13y = 41, & (1) \\ 8x + 11y = 18. & (2) \end{cases}$

Solution: From (1), $3x = 13y + 41.$ (3)

Solving (3) for x in terms of y, $x = \dfrac{13y + 41}{3}.$ (4)

Substituting $\dfrac{13y+41}{3}$ for x in (2),

$$8 \cdot \frac{13y + 41}{3} + 11y = 18. \qquad (5)$$

(5) · 3, $8(13y + 41) + 33y = 54.$ (6)

Simplifying, $104y + 328 + 33y = 54.$ (7)

Collecting, $137y = -274.$ (8)

(8) ÷ 137, $y = -2.$ (9)

Substituting -2 for y in (4), $x = \dfrac{-26 + 41}{3} = 5.$

Check: Substituting 5 for x and -2 for y in (1) and (2) gives the identities $15 + 26 = 41$ and $40 - 22 = 18$.

The method of the preceding solution is stated in the

RULE. *Solve either equation for the value of one variable in terms of the other.*

Substitute this value for the variable in the equation from which it was not obtained and solve the resulting equation.

Substitute the definite value just found, in the simplest of the preceding equations which contains both variables, and solve, thus obtaining a definite value for the other variable.

CHECK. As in preceding example.

If either equation in a linear system contains fractions, it is usually (see exception in Exercise 22, page 61) best to clear of fractions and then to apply one of the preceding rules. Occasionally one can avoid quadratics by solving without first clearing of fractions (see Problem 7, page 77).

EXERCISES

1. What is a constant? a variable?

2. Define and give examples involving two unknowns of (*a*) a linear equation; (*b*) a system of linear equations; (*c*) a simultaneous linear system; (*d*) equivalent equations; (*e*) a determinate system; (*f*) an indeterminate equation; (*g*) an indeterminate system; (*h*) an incompatible or inconsistent system.

3. What is the graph in each case in Exercise 2?

4. What is the general form of a linear equation in two variables?

5. To what general form may any incompatible linear system in two unknowns be reduced?

6. What is the general form of a linear system in two unknowns which has an infinite number of sets of roots?

Solve by addition or subtraction:

7. $2x + 5y = 8,$
$x - 10y = 9.$

8. $5x + 38 = 12y,$
$3x + 8y = 0.$

9. $9t - 2n = 18,$
$20t = 7n + 63.$

10. $11m - 10 = -18n,$
$9m + 12n = -15.$

11. $3x - 2y = 18,\ 30 + 8y = 5x.$

Solve by substitution:

12. $3r - 8s = 13,$
$r + 6s = 0.$

13. $2(x + y) + 3y = 4,$
$5 = x + y.$

14. $16x + 7 = 15y,$
$4x + 5y = 0.$

15. $6x + \frac{14}{3} - 12y = 0,$
$7y - 3x - 4 = 0.$

16. $\frac{8m - 3n}{2} + 6n = -9,$
$4m - 1 = 3n.$

Solve by either one of the preceding methods:

17. $\frac{2x}{3} + y = -7,$
$5x - 3y = 105.$

18. $\frac{3r}{4} - \frac{7}{2} = \frac{s}{12},$
$r + 8 = -2s.$

19. $\frac{3y+1}{13} - \frac{z+20}{12} = -3,$
$\frac{z}{8} - \frac{2y}{9} = 1.$

20. $5x + .7y = -1.25,$
$.12x - .08y = 3\frac{3}{5}.$

21. $\frac{m+2n}{5} - \frac{2m-n}{10} = \frac{5}{2},$
$\frac{m+n}{4} - \frac{m-n}{7} = \frac{n}{2}.$

22. $\frac{1}{x} + \frac{1}{y} = -\frac{1}{6},$
$\frac{2}{x} - \frac{3}{y} = -\frac{4}{3}.$

23. $\frac{2r+4s}{2r-s} = \frac{38}{3},$
$\frac{5}{r} = \frac{14}{s}.$

24. $\frac{2m+3n-2}{m+n+6} = \frac{4}{3},$
$\frac{1}{m} + \frac{1}{n} = \frac{9}{m}.$

25. $\frac{3}{x-3} - \frac{4}{5y} = 0,$
$x - \frac{6y}{5} = \frac{3}{5}.$

26. $\frac{.25h + 8 + .1k}{k - 10 + h} = 2.5,$
$\frac{1}{.8h - 2.2} + \frac{20}{35 - 5.5k} = 0.$

27. $\frac{2h+5k}{\frac{2}{3}} = 39,$
$\frac{2h - \frac{3k}{4}}{k-h} = 3.$

28. $\frac{3(x+y)}{\frac{3}{5}} + \frac{x-y}{-\frac{11}{5}} = 0,$
$2x + y = 7.$

29. $\frac{\frac{x}{5} - \frac{y}{2}}{\frac{3}{4}} + \frac{\frac{5y}{2} - 7 - \frac{x}{5}}{\frac{7}{2}} = 0,$
$\frac{x}{4} - 2y = \frac{13}{2}.$

30. $\frac{3x+z}{z-x} + 2 = \frac{12}{x-z}, \quad \frac{x+7}{z} + \frac{17+x}{5-z} = 0.$

31. $\frac{m+5}{n} = \frac{m-5}{n-10}, \quad \frac{m - \frac{2}{5}(3m-2n) - \frac{1}{3}}{m-2n-1} + \frac{1}{3} = 0.$

Solve for x and y:

32. $\frac{2cx}{a} - \frac{y}{a} = 5c,$
$\frac{2x}{3} - \frac{y}{c} = a.$

33. $\frac{x}{a} - \frac{y}{c} = 6,$
$\frac{x}{2a} + \frac{y}{3c} = 13.$

34. $\frac{a}{x} + \frac{3a}{y} = 1,$
$\frac{3a}{x} + \frac{a}{y} = \frac{1}{2}.$

35. $\dfrac{x+y}{10a} = 2 - \dfrac{x-y}{4a}$,
$x = y$.

36. $\dfrac{a}{x} + \dfrac{a+b}{y} = 1$,
$\dfrac{a+b}{x} + \dfrac{b}{y} = \dfrac{4}{3}$.

37. $(c+b)y + ax = 1$,
$ay = 1 - (c+b)x$.

38. $a^4 \cdot a^{y-7} = a^{12}$,
$c^4 \cdot c^{x-3} = c^y$.

39. $ax + by = c$,
$dx + ey = f$.

42. Determinants of the second order. The arrangement of numbers $\begin{vmatrix} 4 & 2 \\ 5 & 3 \end{vmatrix}$ has been given the meaning $4 \cdot 3 - 5 \cdot 2$.

Such an arrangement is called a **determinant.**

The value of any such determinant is easily found since $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$ means $ad - bc$.

Accordingly, $\begin{vmatrix} 5 & 6 \\ 2 & 3 \end{vmatrix} = 5 \cdot 3 - 2 \cdot 6 = 15 - 12 = 3$.

Similarly, $\begin{vmatrix} 6 & -2 \\ 8 & 3 \end{vmatrix} = 6 \cdot 3 - 8 \cdot (-2) = 18 + 16 = 34$.

And $3\begin{vmatrix} 3 & 9 \\ -4 & -5 \end{vmatrix} = 3[-15 - (-36)] = 3[-15 + 36]$
$= 3 \cdot 21 = 63$.

The preceding operations can be reversed and the difference (or sum) of two products written as a determinant.

Thus $mn - rs$ can be written $\begin{vmatrix} m & s \\ r & n \end{vmatrix}$, and in a number of other ways.

Similarly, $ab - k = ab - 1 \cdot k = \begin{vmatrix} a & k \\ 1 & b \end{vmatrix}$.

EXERCISES

Find the value of the determinants:

1. $\begin{vmatrix} 4 & 1 \\ 3 & 5 \end{vmatrix}$.

2. $\begin{vmatrix} 6 & -2 \\ 8 & 1 \end{vmatrix}$.

3. $3\begin{vmatrix} 2 & 3 \\ 4 & -3 \end{vmatrix}$.

4. $\begin{vmatrix} -3 & 2 \\ 7 & 8 \end{vmatrix}$.

5. $3\begin{vmatrix} \frac{1}{2} & 6 \\ \frac{1}{3} & 8 \end{vmatrix}$.

6. $4\begin{vmatrix} -3 & 5 \\ 7 & 8 \end{vmatrix}$.

7. $\begin{vmatrix} 2a & -10b \\ 5a & 3b \end{vmatrix}$.

8. $\dfrac{5}{2a}\begin{vmatrix} a^3 & 0 \\ a^2 & 2a \end{vmatrix}$.

9. $\dfrac{7}{3}\begin{vmatrix} 3e & 0 \\ 9d & -14e^2 \end{vmatrix}$.

Write as a determinant:

10. $ax - cr.$ **12.** $\frac{8}{3} - ar.$ **14.** $ab + cd.$

11. $mz - 3d.$ **13.** $hk - c.$ **15.** $a - b.$

Find the value of the fractions:

16. $\dfrac{\begin{vmatrix} 5 & 1 \\ 3 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}}.$ **17.** $\dfrac{\begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 4 & 3 \end{vmatrix}}.$ **18.** $\dfrac{\begin{vmatrix} c & b \\ f & e \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}.$ **19.** $\dfrac{\begin{vmatrix} c & (1-c) \\ 7c & (7-12c) \end{vmatrix}}{\begin{vmatrix} c & 3 \\ 7c & 36 \end{vmatrix}}.$

Write as the quotient of two determinants:

20. $\dfrac{st - cd}{3cx - 5r}.$ **22.** $\dfrac{ax - 6}{2r - 5t}.$ **24.** $\dfrac{6x - hm}{3a - 3r}.$

21. $\dfrac{a + \dfrac{10}{c}}{a^2 - 12}.$ **23.** $\dfrac{\dfrac{m}{3} - 7}{2m - 1}.$ **25.** $\dfrac{x^2 + 0}{\dfrac{a^2}{4} + \dfrac{15}{a}}.$

43. Solution by determinants. For the system

$$\begin{cases} ax + by = c, & (1) \\ dx + ey = f, & (2) \end{cases}$$

$x = \dfrac{ce - bf}{ae - bd}$, and $y = \dfrac{af - cd}{ae - bd}$ (see Exercise 39, page 62).

Using determinants, $$x = \frac{ce - bf}{ae - bd} = \frac{\begin{vmatrix} c & b \\ f & e \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}, \qquad (3)$$

and $$y = \frac{af - cd}{ae - bd} = \frac{\begin{vmatrix} a & c \\ d & f \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}. \qquad (4)$$

The determinant expressions for x and y in (3) and (4) can be used as formulas to solve any pair of linear equations in two unknowns. This method is particularly useful in the solution of linear equations with literal coefficients. The determinant forms can be easily remembered and written down at once if we observe carefully the following points:

I. *The determinants in the denominators are identical, and each is formed by the coefficients of x and y as they stand in the original equations (1) and (2).*

II. *The determinant in the numerator of the value of x is formed from the denominator by replacing the coefficients of x,* $\begin{matrix} a \\ d \end{matrix}$, *by the constant terms* $\begin{matrix} c \\ f \end{matrix}$.

III. *The determinant in the numerator of the value of y is formed from the denominator by replacing the coefficients of y,* $\begin{matrix} b \\ e \end{matrix}$, *by the constant terms* $\begin{matrix} c \\ f \end{matrix}$.

Biographical Note. GOTTFRIED WILHELM LEIBNITZ. For the last few hundred years the study of the higher mathematics has been carried on almost entirely by professors in the universities. It is rather exceptional for a man not connected with any educational institution to achieve distinction in this field. Before this was the case, however, scholars were accustomed to devote themselves to any or all branches of learning which attracted them, and many men of wide erudition in various walks of life flourished at different times during the two or three hundred years following the fifteenth century.

But of them all, the man who perhaps most clearly deserves the title of universal genius is Leibnitz (1646–1716). He was born in Leipzig, Germany, and on account of the poor instruction in the school to which he was sent, he was obliged to learn Latin by himself, which he did at the age of eight. By the time he was twelve he read Latin with ease, and had begun Greek. Not until the age of twenty-six, when he was sent to Paris on a political errand, did he become deeply interested in mathematics. From 1676, for nearly forty years, he held the well-paid position of librarian in the ducal palace of Brunswick, serving under three princes, the last of whom became George I of England in 1714. This post afforded him time for the deep study of mathematics, philosophy, theology, law, politics, and languages, in all of which he distinguished himself. An incomplete edition of his mathematical works has been published in seven volumes.

Personally he was quick of temper, impatient of contradiction, overfond of money, and one of the few really great men who have been offensively conceited.

It is in his writings that we find the first mention of determinants. He also discovered the calculus independently of Sir Isaac Newton, and the last years of both men were embittered by a most unfortunate wrangle in which the friends of Newton accused Leibnitz of publishing as his own, results which really belonged to Newton.

EXAMPLE

Solve by determinants $\begin{cases} 2y + x = 7, \\ 5x = 2y + 11. \end{cases}$

Solution: Writing the equations in the standard form, we have

$$x + 2y = 7,$$
$$5x - 2y = 11.$$

Then
$$x = \frac{\begin{vmatrix} 7 & 2 \\ 11 & -2 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 5 & -2 \end{vmatrix}} = \frac{-14 - 22}{-2 - 10} = \frac{-36}{-12} = 3.$$

In solving for y the denominator is the same as before; hence

$$y = \frac{\begin{vmatrix} 1 & 7 \\ 5 & 11 \end{vmatrix}}{-12} = \frac{11 - 35}{-12} = \frac{-24}{-12} = 2.$$

Check: $\begin{cases} 4 + 3 = 7, \\ 15 = 4 + 11. \end{cases}$

EXERCISES

Solve by determinants and check results:

1. $2x + 3y = 7,$ $3x - 2y = 4.$

2. $4x = 3y + 8,$ $5y + 6 = 3x.$

3. $5x + 4y = 10a + 4,$ $x - 2ay = 0.$

4. $.3x + .02y = 185,$ $.5x + .04y = 335.$

5. $4x + 3y = 6,$ $\dfrac{3x}{4} + \dfrac{3y}{4} = 3.$

6. $\dfrac{x}{2} - \dfrac{y}{3} = 6,$ $\dfrac{3x}{2} + \dfrac{2y}{3} = 8.$

7. $7x + 5y = 21c,$ $\dfrac{x}{c} - \dfrac{y}{2c} = 3.$

8. $\dfrac{x}{a} + \dfrac{y}{b} = \dfrac{a + b}{ab},$ $x - y = \dfrac{a^2 - b^2}{ab}.$

44. Indeterminate equations. If numerical values are given to any two variables in the equation $m + n + p = 6$, a value for the third variable can be found, which, taken with the values assigned to the other variables, satisfies the equation.

For example, let $m = 1$ and $n = 2$. Then $m + n + p = 6$ becomes $1 + 2 + p = 6$, whence $p = 3$. Obviously $m = 1$, $n = 2$, and $p = 3$

satisfy the equation. Other values may be given to m and n (or m and p, or n and p), and the foregoing process repeated, thus obtaining set after set of roots. A few sets of roots are tabulated here.

m	1	2	0	1	6	$\frac{1}{2}$	-4	10
n	2	2	0	3	0	$\frac{5}{2}$	2	-1
p	3	2	6	2	0	3	8	-3

A

In like manner, for $2m + 3n + 4p = 16$ the following sets of roots may be obtained:

m	1	2	0	1	6	$\frac{1}{2}$	-4	10
n	2	2	0	3	0	$\frac{5}{2}$	2	-1
p	2	$1\frac{1}{2}$	4	$\frac{5}{4}$	1	$1\frac{7}{8}$	$4\frac{1}{2}$	$-\frac{1}{4}$

B

In tables *A* and *B* the values of m and n are alike, but the corresponding values of p are different in every case. Though each equation has an infinite (unlimited) number of sets of roots, no common set appears in the tables. This suggests three questions:

(1) Have the two equations a common set of roots?
(2) If so, is there more than one common set?
(3) Is the number of such sets of roots unlimited?

These questions are answered by the work which follows.

Solve the system $\begin{cases} m + n + p = 6, & (1) \\ 2m + 3n + 4p = 16. & (2) \end{cases}$

Solution: First eliminate one variable, say m, as follows:

$(1)\cdot 2,$ $\quad 2m + 2n + 2p = 12.$ (3)

$(2) - (3)$ eliminates m, giving $\quad n + 2p = 4.$ (4)

Give n or p in (4) any value, say $n = 8$, $\quad 8 + 2p = 4.$ (5)

Solving (5), $\quad p = -2.$ (6)

Substituting 8 for n and -2 for p in (1), $m + 8 - 2 = 6.$ (7)

Solving (7), $\quad m = 0.$ (8)

Therefore the set $m = 0$, $n = 8$, and $p = -2$ satisfies (1).

These values also satisfy (2),

since $\quad 0 + 24 - 8 = 16$, or $16 = 16$.

Again, let $p = 4$ in (4), $\quad n + 8 = 4.$ (9)

Solving (9), $n = -4.$ (10)
Substituting -4 for n and 4 for p in (1), $m - 4 + 4 = 6.$ (11)
Solving (11), $m = 6.$ (12)
Then $m = 6$, $n = -4$, and $p = 4$ satisfy (1).
They also satisfy (2), since $12 - 12 + 16 = 16.$

Repetition of the foregoing process gives the four following sets of roots for the system (1), (2):

m	2	4	0	$3\frac{1}{2}$							
n	4	0	8	1	2	-1	-2				
p	0	2	-2	$1\frac{1}{2}$				5	2	-1	-10

C

Since, by the method just explained, table C may be completed and extended as much farther as is desired, we conclude that the system (1), (2) has an infinite (unlimited) number of sets of roots. We must not infer, however, that every pair of linear equations in three variables has either an infinite number of sets of roots or even one set. Such an inference is easily seen to be incorrect.

For instance, let $$x + y + z = 8, \qquad (1)$$
and $$x + y + z = 24. \qquad (2)$$

An attempt to eliminate one variable by subtracting (1) from (2) causes all the variables to vanish and gives $0 = 16$, which is false. Hence the method of the last solution fails.

In general, however, a system of two independent equations of the first degree in three variables has an infinite (unlimited) number of sets of roots.

A system of three *independent equations* of the first degree in three variables, no two equations being *incompatible*, has *one* set of roots and *only* one.

Note. It is not a little remarkable that the writings of the first great algebraist, Diophantos of Alexandria (about 300 A.D.), are devoted almost entirely to the solution of indeterminate equations; that is, to finding the sets of related values which satisfy an equation in two variables, or perhaps two equations in three variables. We know practically nothing of Diophantos himself, excepting the information contained in his epitaph, which reads as follows: "Diophantos passed one sixth of his life in childhood, one twelfth in youth, one seventh more as a bachelor; five years after his marriage a son was born who died four years before his father, at half his father's age." From this statement the reader was supposed to be able to find at what age Diophantos died. As a mathematician Diophantos stood alone, without any prominent forerunner, or disciple, so far as we know. His solutions of the indeterminate equations were exceedingly skillful, but the methods which he used were so obscure that his work had comparatively little influence upon that of later times.

45. Determinate systems. The method of obtaining the set of roots of a determinate system is illustrated in the following

EXAMPLE

Solve the system
$$\begin{cases} m + 6n - 5p = 23, & (1) \\ 3m - 8n + 4p = -1, & (2) \\ 7m - 10n + 10p = 0. & (3) \end{cases}$$

Solution: Eliminate one variable, say p, between (1) and (2) thus:

(1) · 4,	$4m + 24n - 20p = 92.$	(4)
(2) · 5,	$15m - 40n + 20p = -5.$	(5)
(4) + (5),	$19m - 16n = 87.$	(6)

Now eliminate p between (1) and (3) as follows:

(1) · 2,	$2m + 12n - 10p = 46.$	(7)
(3) · 1,	$7m - 10n + 10p = 0.$	(8)
(7) + (8),	$9m + 2n = 46.$	(9)

The equations (6) and (9) contain *the same two* variables m and n.

(6) · 1,	$19m - 16n = 87.$	(10)
(9) · 8,	$72m + 16n = 368.$	(11)
(10) + (11),	$91m = 455.$	(12)
(12) ÷ 91,	$m = 5.$	(13)
Substituting 5 for m in (6),	$95 - 16n = 87.$	(14)
Solving (14),	$n = \frac{1}{2}.$	
Substituting $\frac{1}{2}$ for n and 5 for m in (1),	$5 + 3 - 5p = 23.$	(15)
Solving (15),	$p = -3.$	

Check: Substituting 5 for m, $\frac{1}{2}$ for n, and -3 for p in (1), (2), and (3),

$$5 + 3 + 15 = 23, \text{ or } 23 = 23.$$
$$15 - 4 - 12 = -1, \text{ or } -1 = -1.$$
$$35 - 5 - 30 = 0, \text{ or } 0 = 0.$$

Or we may check thus:

(1) + (2) + (3) gives $11m - 12n + 9p = 22.$

Substituting for m, n, and p, $55 - 6 - 27 = 22$, or $22 = 22$.

This last check fails if any of the variables vanish when the three equations are added.

For the solution of a simultaneous system of linear equations in three variables we have the

Rule. *Decide from an inspection of the coefficients which variable is most easily eliminated.*

Using any two equations, eliminate that variable.

With one of the equations just used, and the third equation, again eliminate the same variable.

The last two operations give two equations in the same two variables. Solve these equations by the rule, page 58.

Substitute the two values found in the simplest of the original equations and solve for the third variable.

Check. Substitute the values found in each of the original equations and simplify results.

Or check in the sum of the three original equations (unless one variable vanishes in addition).

Four or more independent equations in three variables have no common set of roots.

In general, a system of $n+1$ independent linear equations in n variables has no set of roots; a system of n independent linear equations in n variables, no two of which are incompatible, has one set of roots; and a system of $n-1$ independent linear equations in n variables, no two of which are incompatible, has an infinite number of sets of roots.

The usual proof of the preceding theorem affords a beautiful application of determinants. Though not extremely difficult, it requires greater familiarity with determinants than the student will acquire from this text.

A system of four independent equations in four variables may be solved as follows:

Use the first and second equation, then the first and third, and lastly the first and fourth, and eliminate the same variable each time. This gives a system of three equations in the same three variables, which can be solved by the rule given above.

EXERCISES

1. Find five sets of roots for $x - 2y + z = 6$.

2. Find five sets of roots for $m - n - 2p = 8$.

3. Fill out the blanks in table C, page 67.

4. Find three sets of roots for the system

$$m + n - p = 8,$$
$$3m - 2n + 4p = 6.$$

Solve the following systems:

5. $2x + 3y = -14 - 4z,$
$x - y + 3z = 0,$
$5x + z = 14 - 2y.$

6. $x + 2y + 3z = 14,$
$4x - 5y + 6z = 12,$
$x + 15y + 9z = 58.$

7. $.4r + .3s - 8t = 4,$
$.5r + t + .8s = 1.2,$
$2.6t + .3 - r = +.5s.$

8. $.25x + .05y = -1 + .10z,$
$.50x - .30y = 0,$
$.05y + .04z = 3.$

In Exercises 9, 10, and 11 consider a, b, c as known numbers:

9. $\frac{r}{a} + \frac{3s}{2a} - \frac{t}{3a} = 6,$
$7r + 4t = 6s,$
$\frac{t}{9} - \frac{s}{6} = 0.$

10. $3m + 2p = 6a - 2n,$
$m - 5n + 6p = 2a - 11b,$
$6m - 8p = 12a + 8b.$

11. $h + 2k - l = 3b + c,$
$5h - 4k - 4l = a + b - 8c,$
$\frac{h}{2} = \frac{3a + b}{6} + \frac{k}{3} - \frac{2c}{6}.$

In Exercises 12–14 solve for x, y, and z:

12. $a^x \cdot a^{2y} \cdot a^z = a^{-1},$
$c^{2x} \cdot c^{-y} \cdot c^{z+1} = c^{-19},$
$e^{-x} \cdot e^{-y} \cdot e^{-5z} = e^{13}.$

13. $b^x \cdot b^{y+2} = b^{12},$
$c^x \cdot c^{z+4} = c^{16},$
$d^y \cdot d^{z+3} = d^{17}.$

14. $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 3,$
$\frac{a}{x} - \frac{b}{y} + \frac{2c}{z} = 2,$
$\frac{5a}{x} - \frac{2c}{z} = \frac{3b}{y}.$

15. $r + s + t + u = 6,$
$2r - s + 3t - u = -16,$
$5r + 9s + 4u = 81 + 6t,$
$r + 9t - 7u = -54 - 5s.$

16. $4h - k + m = 0,$
$7k + 2m + x = 0,$
$4m + x + 8h = 0,$
$16h + 5k - x = 4.$

(In Exercise 18 solve for x only.)

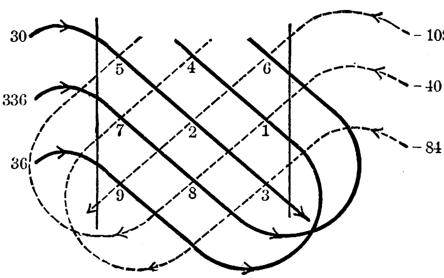
17. $4x - 3y + 2z = 20,$
$5x + 4y - 10z = 3,$
$34z - 7x - 18y = 31.$

18. $ax + by + cz = p,$
$dx + ey + fz = q,$
$gx + hy + iz = r.$

46. Determinants of the third order. The arrangement of numbers $\begin{vmatrix} 5 & 4 & 6 \\ 7 & 2 & 1 \\ 9 & 8 & 3 \end{vmatrix}$ has been given the meaning $5 \cdot 2 \cdot 3 + 7 \cdot 8 \cdot 6 + 9 \cdot 1 \cdot 4 - 6 \cdot 2 \cdot 9 - 1 \cdot 8 \cdot 5 - 3 \cdot 7 \cdot 4$, which equals $30 + 336 + 36 - 108 - 40 - 84 = 170$.

Such an arrangement is called a determinant of the **third order** because it has three **rows** (horizontal lines of numbers) and three **columns** (vertical lines of numbers). Each of the nine numbers in the determinant is called an **element.**

Every determinant of the third order is equal to a polynomial of six terms. Each of the six terms is the product of three elements so chosen that one element, and only one, is taken from each row, and one element, and only one, is taken from each column. If each element is positive, three terms of the polynomial are positive and three are negative. In connection with the preceding explanation the student should study carefully the above diagram, in which each continuous line connects three numbers whose product gives a positive term, and each dotted line connects three numbers whose product gives a negative term.



It follows, then, that the preceding determinant is equal to the polynomial

$$\begin{aligned}&5\cdot 2\cdot 3+7\cdot 8\cdot 6+9\cdot 1\cdot 4-6\cdot 2\cdot 9-1\cdot 8\cdot 5-3\cdot 7\cdot 4=\\&30 \quad +336 \quad +36 \quad -108 \quad -40 \quad -84 \quad =170.\end{aligned}$$

When finding the value of each product in a determinant, the sign of every negative element must be taken into account along with the foregoing explanations.

EXERCISES

Find the value of:

1. $\begin{vmatrix}1 & 1 & 1\\ 2 & 3 & 1\\ 2 & 4 & 1\end{vmatrix}.$

2. $\begin{vmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{vmatrix}.$

3. $\begin{vmatrix}1 & -2 & 3\\ 3 & 2 & 1\\ 1 & 1 & 1\end{vmatrix}.$

4. $\begin{vmatrix}a & b & c\\ 1 & 2 & 3\\ a & b & c\end{vmatrix}.$

5. $\begin{vmatrix}1 & 0 & 0\\ 2 & 4 & 5\\ -3 & 1 & 1\end{vmatrix}.$

6. $\begin{vmatrix}1 & 1 & 1\\ a & 1 & a\\ -a & 5 & 6\end{vmatrix}.$

7. $-\begin{vmatrix}2 & -1 & 3\\ -3 & 1 & 2\\ 4 & 5 & 1\end{vmatrix}.$

8. $\begin{vmatrix}a & 2 & 7\\ b & 3 & 8\\ c & 4 & 9\end{vmatrix}.$

9. $\begin{vmatrix}x & y & 1\\ 2 & 2 & 1\\ 3 & 4 & 1\end{vmatrix}.$

10. $\begin{vmatrix}a & b & c\\ d & e & f\\ g & h & i\end{vmatrix}.$

11. $\begin{vmatrix}x & y & 1\\ x_1 & y_1 & 1\\ x_2 & y_2 & 1\end{vmatrix}.$

12. $c\begin{vmatrix}5 & 1 & b\\ c & 0 & 0\\ 8 & 6 & a\end{vmatrix}.$

47. General linear system in three variables. For the system

$$ax+by+cz=p, \tag{1}$$
$$dx+ey+fz=q, \tag{2}$$
$$gx+hy+iz=r, \tag{3}$$
$$x=\frac{pei+qhc+rfb-cer-fhp-iqb}{aei+dhc+gfb-ceg-fha-idb}, \tag{4}$$
$$y=\frac{aqi+drc+gfp-cqg-fra-idp}{aei+dhc+gfb-ceg-fha-idb}, \tag{5}$$
$$z=\frac{aer+dhp+gqb-peg-qha-rdb}{aei+dhc+gfb-ceg-fha-idb}. \tag{6}$$

(See Exercise 18, page 71.)

In the fractions in (4), (5), and (6) observe the following points:

1. The three denominators are identical.
2. Each numerator and each denominator contains six terms, three positive and three negative.
3. Each term is the product of three letters, one of these letters, and only one, being taken from each equation.
4. Each term in the numerator differs from the term just below it in the denominator by one letter, and only one.

The preceding statements will help to make clear the reason for what now follows.

Let us write the coefficients of x, y, and z as a determinant in the order in which they occur in (1), (2), and (3), and then expand. We thus obtain

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \qquad (D)$$

But $(D) = aei + dhc + gfb - ceg - fha - idb$, which is the denominator of the fractions in (4), (5), and (6).

If in (D) we now replace the coefficients of x, $\begin{matrix}\mathbf{a}\\ \mathbf{d},\\ \mathbf{g}\end{matrix}$ by the constant terms $\begin{matrix}\mathbf{p}\\ \mathbf{q},\\ \mathbf{r}\end{matrix}$ and expand, we obtain a determinant equivalent to the *numerator* of the *fraction* (4) *whose value is* x; for

$$\begin{vmatrix} p & b & c \\ q & e & f \\ r & h & i \end{vmatrix} = pei + qhc + rfb - cer - fhp - iqb.$$

Again, if in (D) we replace the coefficients of y, $\begin{matrix}\mathbf{b}\\ \mathbf{e},\\ \mathbf{h}\end{matrix}$ by the constant terms $\begin{matrix}\mathbf{p}\\ \mathbf{q},\\ \mathbf{r}\end{matrix}$ we obtain a determinant equivalent to the *numerator* of the fraction (5) whose *value* is y.

Lastly, if in (D) we replace the coefficients of z, $\begin{matrix}\mathbf{c}\\ \mathbf{f},\\ \mathbf{i}\end{matrix}$ by the constant terms $\begin{matrix}\mathbf{p}\\ \mathbf{q},\\ \mathbf{r}\end{matrix}$ we obtain a determinant equivalent to the *numerator* of the fraction (6) whose *value* is z. (The student should perform the work outlined in the last two sentences.)

Therefore we may write the values of x, y, and z for the given system in determinant form as follows:

$$x = \frac{\begin{vmatrix} p & b & c \\ q & e & f \\ r & h & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} \quad (7) \qquad y = \frac{\begin{vmatrix} a & p & c \\ d & q & f \\ g & r & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} \quad (8) \qquad z = \frac{\begin{vmatrix} a & b & p \\ d & e & q \\ g & h & r \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} \quad (9)$$

The fractions (4), (5), and (6) are general results, and can be used as formulas to solve any three simultaneous equations in three variables, but the equivalent forms (7), (8), and (9) are far more easily remembered. These can be written down at once for *any* system of three equations in three variables, since

I. *The determinants in the denominators are identical, and each is formed by the coefficients of* x, y, *and* z, *as they stand in the original equations.*

II. *Each determinant in the numerator is formed from the denominator by putting the column of constant terms (as they stand in the original equation) in place of the column of the coefficients of the variable whose value is sought.*

The method of solution by determinants of a system of equations in three variables is illustrated in the following

EXAMPLES

1. Solve the system
$$\begin{cases} 3x + y = 14 - z, & (1) \\ x + z = 1 + 2y, & (2) \\ x + y = 15 - 2z. & (3) \end{cases}$$

Solution: Rewriting in standard form,

$$3x + y + z = 14, \quad (4)$$
$$x - 2y + z = 1, \quad (5)$$
$$x + y + 2z = 15. \quad (6)$$

From I and II preceding,

$$x = \frac{\begin{vmatrix} 14 & 1 & 1 \\ 1 & -2 & 1 \\ 15 & 1 & 2 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & 2 \end{vmatrix}} = \frac{-26}{-13} = 2, \text{ and } y = \frac{\begin{vmatrix} 3 & 14 & 1 \\ 1 & 1 & 1 \\ 1 & 15 & 2 \end{vmatrix}}{-13} = \frac{-39}{-13} = 3.$$

The value of z can now be more easily obtained by substituting the values of x and y already found in (1), (2), or (3) than by means of determinants.

Substituting in (2), $2 + z = 1 + 6$;
whence $z = 5$.

Check: (1) + (2) + (3) gives

$$5x + 2y + z = 30 + 2y - 3z.$$

Substituting, $10 + 6 + 5 = 30 + 6 - 15$,
or $21 = 21$.

2. Solve the system
$$\begin{cases} x + y = 13 + 2z, & (1) \\ x + 7 = 3y, & (2) \\ x + 4z = -14. & (3) \end{cases}$$

Solution: Rewriting in standard form and supplying zero coefficients,

$$x + y - 2z = 13, \qquad (4)$$
$$x - 3y - 0z = -7, \qquad (5)$$
$$x - 0y + 4z = -14. \qquad (6)$$

Then
$$x = \frac{\begin{vmatrix} 13 & 1 & -2 \\ -7 & -3 & 0 \\ -14 & 0 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & -2 \\ 1 & -3 & 0 \\ 1 & 0 & 4 \end{vmatrix}} = \frac{-44}{-22} = 2.$$

The value of y can now be more easily obtained by substituting 2 for x in (2) than by means of determinants.

Accordingly $2 + 7 = 3y$;
whence $y = 3$.

Similarly, by substituting 2 for x in (3), $2 + 4z = -14$;
whence $z = -4$.

Check: (1) + (2) + (3), $3x + y + 4z + 7 = -1 + 2z + 3y$.
Substituting, $6 + 3 - 16 + 7 = -1 - 8 + 9$, or $0 = 0$.

As we have seen, determinants have a very useful application in the solution of systems of linear equations in two or three variables. With some practice one can solve such equations more rapidly by determinants than by the other methods which have been given. If the student studies advanced algebra, he will learn of determinants of the fourth and higher orders, and of the usefulness of such determinants in solving linear systems in four or more variables. Moreover, he will then see that the theory of determinants is an absolute necessity for the discussion of the general theory of linear systems in n variables.

EXERCISES

Solve for x, y, and z as in the two preceding examples:

1. $x + y + z = 1,\quad x + y - z = 2,\quad x - y + z = 3.$

2. $x + 2y + z = 1,\quad 2x + y - z = 0,\quad x + 2y - z = 0.$

3. $2x + y = 5 + z,\quad x - 2z = 6,\quad 3y + 2z = x.$

4. $x + y = 1,\quad x + z = 2,\quad y + z = 3.$

5. $x + y = 3a,\quad x + z = 4a,\quad y + z = 5a.$

6. $\frac{x}{3} + \frac{y}{2} = 9,\quad \frac{x}{2} + \frac{z}{3} = 8,\quad \frac{y}{3} + \frac{z}{2} = 13.$

7. $ax + by = 0,\quad cx - bz = 2bc,\quad bx + az - cy = b^2.$

8. $hx + ky - lz = 2hk,\quad ky - hx + lz = 2kl,\quad hx - ky + lz = 2hl.$

9. $mx + mx_1 + x_2 = 0,\quad mx + x_1 + mx_2 = ma - a,\quad mx - 3mx_1 + x_2 = 4ma.$

Note. Like so many other discoveries, the determinant notation was noticed independently by two men. In a letter to a friend, written in 1693, Leibnitz outlined the method of solving equations by the means of determinants; but, so far as we know, he used the notation in his own work very little, and certainly did not publish it during his lifetime. In fact, the letter in which this reference is found, did not come to light until 1850, and the fact that Leibnitz knew anything about determinants was not generally recognized until after that time.

In 1750 Cramer, a professor in the university at Geneva, rediscovered this method of solving linear systems; and his work had the good fortune to be accepted by scholars, forming the real beginning of the development of the subject. Since that time a great many have written on the subject, and to-day determinants are used in every field of advanced mathematics.

PROBLEMS

1. A and B together can do a piece of work in $3\frac{3}{7}$ days. If they work together 2 days and A can then finish the job alone in $2\frac{1}{2}$ days more, how many days does each require alone?

2. A man and a boy can do in 18 days a piece of work which 5 men and 9 boys can do in 3 days. In how many days can one man do the work? one boy?

3. If $ax + by = 2$ is satisfied by $x = 2$ and $y = 3$, and also by $x = 6$ and $y = 5$, what values must a and b have?

4. A launch, whose rate in still water is 12 miles per hour, goes up a stream whose rate is 2 miles per hour, and returns. The entire trip requires 24 hours. Find the number of hours required for the trip upstream and the number for the return.

5. Two sums are put at interest at 5% and 6% respectively. The annual income from both together is $100. If the first sum had yielded 1% more and the second 1% less, the annual income would have been decreased $2. Find each sum.

6. A sum of $4000 is invested, a part in 5 per cent bonds at 90, and the remainder in 6 per cent bonds at 110. If the total annual income is $220, find the sum invested at each rate.

7. A train leaves M two hours late and runs from M to P at 50% more than its usual rate, arriving on time. If it had run from M to P at 25 miles per hour, it would have been 48 minutes late. Find the usual rate and the distance from M to P.

8. A train leaves M thirty minutes late. It then runs to P at a rate 20% greater than usual, and arrives 6 minutes late. Had it run 15 miles of the distance from M to P at the usual rate and the rest of the trip at the increased rate, it would have been 12 minutes late. Find the distance from M to P and the usual rate of the train.

9. The rate of a passenger train is 62 feet a second and the rate of a freight train 38 feet a second. When they run on parallel tracks in opposite directions they pass each other in 20 seconds. The length of the freight train is three times the length of the passenger train. Find the length of each.

10. The rate of a passenger train is 45 miles per hour and the rate of a freight train is 30 miles per hour. The freight train is 240 feet longer than the passenger train. When the trains run on parallel tracks in the same direction they pass each other in 1 minute and 20 seconds. Find the length of each train.

11. The length of a freight train is 1430 feet and the length of a passenger train 550 feet. When they run on parallel tracks in opposite directions they pass each other in 18 seconds, and when they run in the same direction they pass each other in 1 minute and 30 seconds. Find the rates of the trains.

12 Two contestants run over a 440-yard course. The first wins by 4 seconds when given a start of 200 feet. They finish together when the first is given a handicap of 40 yards. Find the rate of each in feet per second.

13. It is desired to have a 10-gallon mixture of 45% alcohol. Two mixtures, one of 95% alcohol and another of 15% alcohol, are to be used. How many gallons of each will be required to make the desired mixture?

14. A chemist has the same acid in two strengths. Eight liters of one mixed with 12 liters of the other gives a mixture 84% pure, and 3 liters of the first mixed with 2 liters of the second gives a mixture 86% pure. Find the per cent of purity of each acid.

15. The crown of Hiero of Syracuse, which was part gold and part silver, weighed 20 pounds, and lost $1\frac{1}{4}$ pounds when weighed in water. How much gold and how much silver did it contain if $19\frac{1}{4}$ pounds of gold and $10\frac{1}{2}$ pounds of silver each lose one pound when weighed in water?

16. One angle of a triangle is twice another, and their sum equals the third. Find the number of degrees in each angle of the triangle.

17. The sum of three numbers is 108. The sum of one third the first, one fourth the second, and one sixth the third is 25. Three times the first added to four times the second and six times the third is 504. Find the numbers.

18. The sum of three numbers is 217. The quotient of the first by the second is 5, which is also the quotient of the second by the third. Find the numbers.

19. If the tens' and units' digits of a 3-digit number be interchanged, the resulting number is 27 less than the given number. If the same interchange is made with the tens' and hundreds' digits, the resulting number is 180 less than the given number. The sum of the digits is 14. Find the number.

20. In one hour a tank which has three intake pipes is filled seven eighths full by all three together. The tank is filled in $1\frac{1}{3}$ hours if the first and second pipes are open, and in 2 hours and 40 minutes if the second and third pipes are open. Find the time in hours required by each pipe to fill the tank.

21. The sum of two adjacent sides of a quadrilateral is 140 inches. The sum of the first of these and the side opposite is 160 inches; the sum of the first side and the fourth side is 172 inches. The perimeter is 352 inches. Find each side.

22. The sum of two sides which meet at one of the vertices of a quadrilateral is 20 feet. The sum of two which meet at the next vertex is 27 feet. The sums of the two pairs of opposite sides are 23 feet and 29 feet respectively. Find each side.

23. The sums of three pairs of adjacent sides of a quadrilateral are respectively 80 feet, 108 feet, and 116 feet. The difference of the fourth pair of adjacent sides is 24 feet. Find each side.

24. Two chairs cost h dollars. The first cost m cents more than the second. Find the cost of each in cents.

25. A and B together have d dollars. A gives c dollars to B, after which B gives m dollars to A. Then A has $\frac{1}{3}$ as many dollars as B. Find the number of dollars each had at first.

26. A and B together can do a piece of work in m days. B works c times as fast as A. How many days does each require alone?

27. A man rows m miles downstream in t hours and returns in a hours. Find his rate in still water and the rate of the river.

28. A man dying leaves a widow and eleven children. The law provides that the widow shall receive one half of the estate and that the other half shall be divided equally among the children. The executor of the estate, after paying all debts, has $3400 in cash. But five of the children had borrowed from their father $400 each, for which he had accepted their notes. The executor found these notes worthless. How shall he divide the cash on hand?

29. Solve in positive integers $5x + 2y = 42$.

HINT. $x = \frac{42 - 2y}{5} = 8 + \frac{2 - 2y}{5}$.

Now if x is to be integral, $\frac{1}{5}(2 - 2y)$ must be integral or zero; that is, $2 - 2y$ must be zero or an integral multiple of 5. Hence the least value of y is 1.

The various related sets of values which satisfy this equation may be effectively represented to the eye by the graph of the equation. Then if the line whose equation is $5x + 2y = 42$ passes through any points both of whose coördinates are positive integers, each pair of these values is a set of roots. If the line does not enter the first quadrant, we can see at a glance that the equation has no set of roots which are positive integers.

30. Solve in positive integers $7x + 2y = 36$, and illustrate the result graphically.

31. In how many ways can a debt of $73 be paid with five-dollar and two-dollar bills? Illustrate the result graphically.

32. A man buys calves at $6 each and pigs at $4 each, spending $72. How many of each did he buy?

33. In how many ways can $1.75 be paid in quarters and nickels?

34. A farmer sells some calves at $6 each, pigs at $3 each, and lambs at $4 each, receiving for all $126. In how many ways could he have sold 32 animals at these prices for the same sum? Determine the various groups.

35. In how many ways can a sum of $2.40 be made up with nickels, dimes, and quarters, on the condition that the number of nickels used shall equal the number of quarters and dimes together? Determine the various groups.

CHAPTER VI

ROOTS, RADICALS, AND EXPONENTS

48. Definitions. The **square root** of any number is one of the two equal factors whose product is the number.

For a given index the **principal root** of a number is its *one real root*, if it has but one; or its *positive real* root if it has two real roots.

From the law of signs in multiplication it follows that

Every positive number or algebraic expression has two square roots which have the same absolute value but opposite signs.

It is customary to speak of the positive square root of a number as the **principal square root**; and if no sign precedes the radical, the principal root is understood. When both the positive and the negative square roots are considered, both signs must precede the radical.

For extracting the square roots of any monomial we have the

RULE. *Write the square root of the numerical coefficient preceded by the sign* $\pm$ *and followed by all the letters of the monomial, giving to each letter an exponent equal to one half its exponent in the monomial.*

A rule much like the preceding one holds for fourth root, sixth root, and other even roots.

The even roots of negative numbers are considered in the chapter on Imaginaries.

In this chapter only a single odd root of a number is considered; that is, the **principal odd root.**

The **cube root** of any number is one of the three equal factors whose product is the number.

For extracting the cube root of a monomial we have the

RULE. *Write the cube root of the numerical coefficient followed by all the letters of the monomial, giving to each letter an exponent equal to one third of its exponent in the monomial.*

A rule much like the preceding one holds for fifth root, seventh root, and other odd roots.

49. Square root of polynomials. Extracting the square root of a number is essentially an undoing of the work of multiplication. The square of any polynomial may be represented by

$$(h + t + u)^2 = h^2 + 2\,ht + t^2 + 2\,hu + 2\,tu + u^2$$
$$= h^2 + (2\,h + t)\,t + (2\,h + 2\,t + u)\,u.$$

A little study of this last form, and a comparison with the example which follows, will make clear the reason for each step of the process.

EXAMPLE

$$\begin{array}{l|l}
 & h^2 + 2\,ht + t^2 + 2\,hu + 2\,tu + u^2 \;\underline{|\, h + t + u} \\
 & h^2 \\
\text{First trial divisor, } 2\,h & + 2\,ht + t^2 \\
\text{First complete divisor, } 2\,h + t & + 2\,ht + t^2 = (2\,h + t)\,t \\
\text{Second trial divisor, } 2\,h + 2\,t & + 2\,hu + 2\,tu + u^2 \\
\text{Second complete divisor, } 2\,h + 2\,t + u & + 2\,hu + 2\,tu + u^2 = (2h + 2\,t + u)\,u
\end{array}$$

Therefore the required roots are $\pm\,(h + t + u)$.

EXERCISES

1. State the rule for the sign of (*a*) the odd root of a number; (*b*) the even root of a number.

2. State the rule for extracting the fourth root of a monomial.

3. State the rule for extracting the fifth root of a monomial.

4. How can one obtain the fourth root of a polynomial?

5. State the rule for extracting the square root of a polynomial.

6. Can one obtain the fifth root of a number (*a*) by extracting the square root of its cube root? (*b*) by extracting the cube root of its square root? Explain.

Extract the square roots of:

7. $x^4 + 4x^3 - 2x^2 - 12x + 9$.

8. $a^6 - 10a^4 - 4a^3 + 25a^2 + 20a + 4$.

9. $4a^8 + 12a^4 - 7 - 24a^{-4} + 16a^{-8}$.

10. $49c^{-6} - 28c^{-4} + 74c^{-2} - 20 + 25c^2$.

11. $9x^4 - 6x^{\frac{7}{2}} + x^3 - 66x^{\frac{5}{2}} + 22x^2 + 121x$.

12. $25x^3 - 10x^2 + 90x^{\frac{7}{4}} + x - 18x^{\frac{3}{4}} + 81x^{\frac{1}{2}}$.

13. $16m^{-7} - 8m^{-4} + 104m - 26m^4 + 169m^9 + m^{-1}$.

14. $\frac{9}{4}a^4 - 2a^3 + 7\frac{4}{9}a^2 - \frac{28}{9}a + \frac{49}{9}$.

15. $\dfrac{a^2}{9c^2} + \dfrac{16c^2}{a^2} + \dfrac{4a}{3c} + \dfrac{16c}{a} + \dfrac{20}{3}$.

16. $\dfrac{1}{4a^4} - \dfrac{3x}{a^3} + \dfrac{9x^2}{a^2} + \dfrac{a^4}{25x^2} - \dfrac{6a}{5} + \dfrac{1}{5x}$.

Find the first four terms in the square root of:

17. $1 + 2x$. 18. $\frac{25}{9} + a^3$.

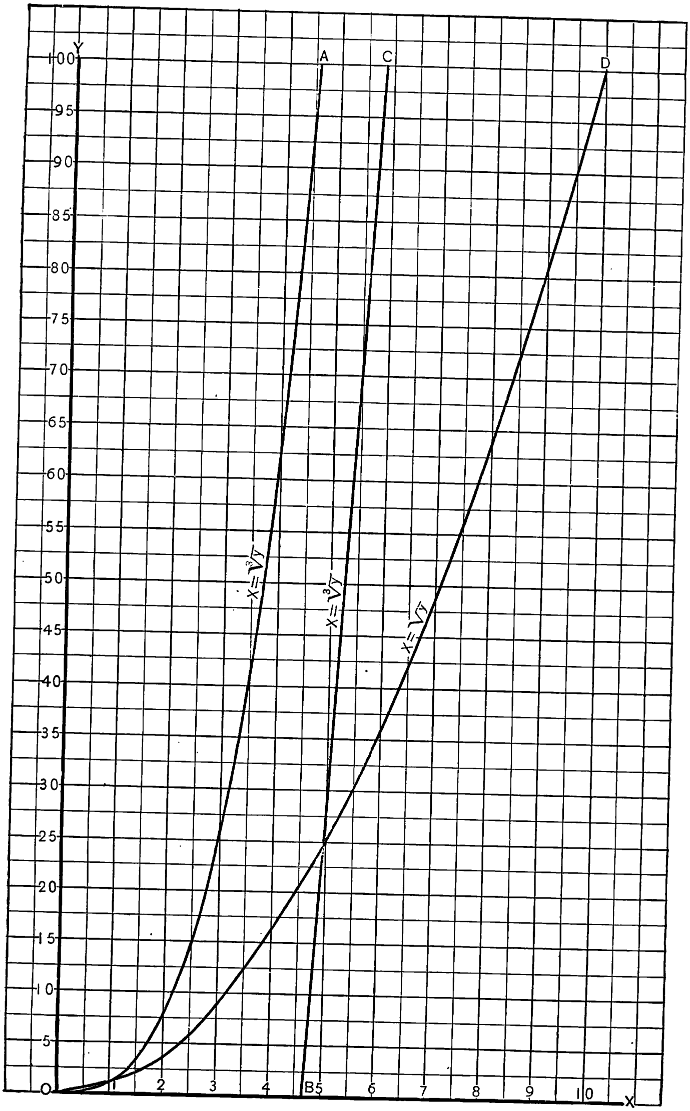
19. Find the first three terms in the fourth root of the expression in (*a*) Exercise 7; (*b*) Exercise 11.

50. Graphical method of extracting roots. When obtaining the square roots of arithmetical numbers by the graphical method we proceed as follows: Let x represent any number and y the square of that number; that is, we let $y = x^2$. Then we construct the graph of this equation, obtaining first the table:

x	$\frac{1}{2}$	1	2	3	4	5	6	7	8
y	$\frac{1}{4}$	1	4	9	16	25	36	49	64

Plotting these values, we obtain curve *OD* of page 85.

From this curve we can read off the square root of any number between 1 and 100 correct to one decimal place.


Y
100
95
90
85
80
75
70
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55
50
45
40
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25
20
15
10
5
O
1
2
3
4
B5
6
7
8
9
10
X
A
C
D
$x=\sqrt[3]{y}$
$x=\sqrt[3]{y}$
$x=\sqrt{y}$

Curve *OA* is a portion of the graph of $y = x^3$, and *BC* is a continuation of *OA*. From this curve we can obtain the cube root of any number between 1 and 200 correct to one decimal place. The cube root of numbers between 1 and 100 we obtain from curve *OA*; for numbers between 100 and 200 we obtain the cube root from *BC*.

If one desires greater precision or a larger range of numbers, or both, he can obtain them by using a large piece of cross-section paper and a different scale. Such a curve, if carefully drawn, is convenient for any computation not requiring great accuracy. The point in its favor is that one can read off the square roots or the cube roots more rapidly than he can obtain them by the methods of § 50 or § 142, or even by logarithms.

The graphical method can also be used to extract fourth and higher roots.

EXERCISES

From the graph read:

1. The square root of (*a*) 20; (*b*) 45; (*c*) 59; (*d*) 68.

2. The cube root of (*a*) 25; (*b*) 18; (*c*) 52; (*d*) 165; (*e*) 150.

3. The square of (*a*) 2.4; (*b*) 6.1; (*c*) 7.9; (*d*) 8.3.

4. The cube of (*a*) 3.2; (*b*) 3.9; (*c*) 2.8; (*d*) 5.6; (*e*) 4.8.

51. Square root of arithmetical numbers. The abbreviated process of extracting a square root of an arithmetical number is as follows:

```
        7'32'67'89 | 2706.8 +
        4
   47 | 332
      | 329
 5406 | 36789
      | 32436
54128 | 435300
      | 433024
          2276
```

Therefore the square roots of 7326789 are $\pm$ 2706.8 +.

It follows from the preceding example that the work of extracting the positive square root of a number may be a never-ending process. The number 7,326,789 has no exact square root; and no matter how far the work is carried, there is no final digit. As the work stands we know that the required root lies between 2706.8 and 2706.9. It is correct to say that 2706.8 is approximately the square root of 7,326,789, or that it is the square root correct to five figures.

The method just illustrated for extracting the positive square root of a number is the one commonly used. For it we have the

RULE. *Begin at the decimal point and point off as many periods of two digits each as possible: to the left if the number is an integer, to the right if it is a decimal; to both left and right if the number is part integral and part decimal.*

Find the greatest integer whose square is equal to or less than the left-hand period, and write this integer for the first digit of the root.

Square the first digit of the root, subtract its square from the first period, and annex the second period to the remainder.

Double the part of the root already found for a trial divisor, divide it into the remainder (omitting from the latter the right-hand digit), and write the integral part of the quotient as the next digit of the root.

Annex the root digit just found to the trial divisor to make the complete divisor, multiply the complete divisor by this root digit, subtract the result from the dividend, and annex to the remainder the next period, thus making a new dividend.

Double the part of the root already found for a new trial divisor and proceed as before until the desired number of digits of the root have been found.

After extracting the square root of a number involving decimals, point off one decimal place in the root for every decimal period in the number.

CHECK. *If the root is exact, square it. The result should be the original number. If the root is inexact, square it and add to this result the remainder. The sum should be the original number.*

EXERCISES

Find the square roots of:

1. 6889. **3.** .6724. **5.** 4.2025.

2. 56169. **4.** 1.4641. **6.** .04028049.

Extract the square roots, correct to four decimal places, of:

7. 5. **8.** .07. **9.** $\frac{13}{7}$. **10.** $\frac{237}{93}$.

11. Find the hypotenuse of the right triangle whose legs are 183 and 264 respectively.

12. A baseball diamond is a square 90 feet on each side. Find the distance from the home plate to second base, correct to .01 of a foot.

13. The hypotenuse of a right triangle is 207 feet and one leg is 83 feet. Find the other leg, correct to .01 of a foot.

14. The hypotenuse and one leg of a right triangle are respectively 292849 and 207000. Find the other leg.

15. The side of an equilateral triangle is 11 inches. Find its altitude, correct to .1 of an inch.

16. Find the side of an equilateral triangle whose altitude is 10 inches, correct to .001 of an inch.

17. Find the area of a triangle whose sides are 12, 27, and 35 inches respectively, correct to .001 of a square foot.

Fact from Geometry. If a, b, and c represent the sides of a triangle, and s equals one half of $a + b + c$, the area of the triangle equals $\sqrt{s(s-a)(s-b)(s-c)}$.

18. By the method of Exercise 17 find, correct to .01 of a square inch, the area of a triangle each side of which is 22 inches.

19. Find correct to two decimals the sum of all of the diagonal lines that can be drawn on the faces of a cube whose edge is 9 inches.

20. Find the radius of a circle whose area is 40 square feet.

21. Find the diagonal of a room whose dimensions in feet are 14, 20, and 30.

22. Find the diagonal of a cube whose edge is 1 foot.

23. A room is 24 feet by 40 feet by 14 feet. What is the length of the shortest broken line from one lower corner to the diagonally opposite upper corner, the line to be in part on the walls or the floor, but not through the air?

24. Take any two integers and form three others from them thus: find the sum of their squares, the difference of their squares, and twice their product. Is the square of one of the three resulting numbers equal to the sum of the squares of the other two? Discuss this with reference to the sides of a right triangle.

25. One leg of a right triangle is 28. Find integral values for the other two sides.

52. Fundamental laws of exponents. The laws of exponents may be stated as follows:

I. Law of Multiplication,

$$x^a \cdot x^b = x^{a+b}.$$

Law I may be stated more completely thus:

$$x^a \cdot x^b \cdot x^c \cdots = x^{a+b+c+\cdots}.$$

This follows directly from the definition of an exponent and from the Associative Law. For instance, $xx = x^2$, and $xxx = x^3$, and $xxxxx = x^5$ by definition. Hence $xx \cdot xxx = xxxxx$, or $x^2 \cdot x^3 = x^5$.

II. Law of Division,

$$x^a \div x^b = x^{a-b}.$$

This follows from Law I. For by that law $x^a = x^{a-b} \cdot x^b$. Hence, dividing both sides of this equation by x^b, we have $x^a \div x^b = x^{a-b}$.

III. Law of Involution, or raising to a power,

$$(x^a)^b = x^{ab}.$$

This follows from Law I, when instead of the distinct factors x^a, x^b, and x^c, etc., we have b factors, each equal to x^a.

Law III includes the more general forms

$$(x^a y^b)^c = x^{ac} y^{bc} \text{ and } (((x^a)^b)^c) \cdots = x^{abc\cdots}.$$

IV. Law of Evolution, or extraction of roots,

$$\sqrt[b]{x^a} = x^{\frac{a}{b}}.$$

It is assumed that these laws hold for all real values of a, b, and c, excepting under IV, where b cannot be zero.

53. Zero and negative exponents. The meaning of a zero exponent and of a negative exponent was explained on page 6.

It follows from the meaning of a negative exponent there explained that

Any **factor** *of the numerator of a fraction may be taken from the numerator and written as a* **factor** *of the denominator, and vice versa, if the sign of the exponent of the factor be changed.*

54. Meaning of a fractional exponent. The fourth law expresses the meaning of a fractional exponent. The meaning may be made clearer thus:

From Law I, $x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} = x$. Since $x^{\frac{1}{2}}$ multiplied by itself gives x, $x^{\frac{1}{2}}$ is another way of writing $\sqrt{x}$; that is, $x^{\frac{1}{2}} = \sqrt{x}$.

Similarly, $x^{\frac{1}{3}} \cdot x^{\frac{1}{3}} \cdot x^{\frac{1}{3}} = x$. Therefore $x^{\frac{1}{3}}$ is another way of writing $\sqrt[3]{x}$; that is, $x^{\frac{1}{3}} = \sqrt[3]{x}$.

In general terms $x^{\frac{1}{n}}$ is another way of writing $\sqrt[n]{x}$; that is,

$$x^{\frac{1}{n}} = \sqrt[n]{x}.$$

Further, $\quad x^{\frac{3}{2}} = x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} = (x^{\frac{1}{2}})^3 = (\sqrt{x})^3,$

and $\quad x^{\frac{3}{2}} = x^{3 \cdot \frac{1}{2}} = (x^3)^{\frac{1}{2}} = \sqrt{x^3}.$

Hence $\quad (\sqrt{x})^3 = \sqrt{x^3}.$

Similarly, $\quad x^{\frac{2}{3}} = x^{\frac{1}{3}} \cdot x^{\frac{1}{3}} = (x^{\frac{1}{3}})^2 = (\sqrt[3]{x})^2,$

and $\quad \sqrt[3]{x^2} = (x^2)^{\frac{1}{3}} = (x \cdot x)^{\frac{1}{3}} = x^{\frac{1}{3}} \cdot x^{\frac{1}{3}} = x^{\frac{2}{3}}.$

Therefore $\quad (\sqrt[3]{x})^2 = \sqrt[3]{x^2}.$

In more general terms $x^{\frac{a}{n}} = (\sqrt[n]{x})^a = \sqrt[n]{x^a}.$

EXERCISES

Write with positive exponents and then simplify results:

1. $8^{\frac{2}{3}}$.
2. $16^{\frac{3}{4}}$.
3. $32^{\frac{2}{5}}$.
4. $125^{\frac{2}{3}}$.
5. 2^{-3}.
6. $49^{-\frac{1}{2}}$.
7. $(\frac{2}{3})^{-2}$.
8. $(\frac{3}{4})^{-3}$.
9. $8^{-\frac{1}{3}}$.
10. $(-27)^{\frac{1}{3}}$.
11. $(.09)^{\frac{3}{2}}$.
12. $(32)^{.4}$.
13. $16^{-1.5}$.
14. $(.008)^{-\frac{1}{3}}$.
15. $\frac{4 \cdot 2^5 \cdot 2^{n-5}}{8 \cdot 2^{3-n}}$.
16. $(\frac{2}{3})^{-2} \cdot (\frac{3}{2})^0 \cdot 2^2$.
17. $5 \cdot 2^0 - 1^5 + (5 \cdot 2)^0 - 2(\frac{1}{8})^{\frac{1}{3}} \div (8^{-\frac{2}{3}}) + (\frac{9}{4})^{-\frac{1}{2}}$.
18. $\frac{2^{-1} - 3^{-1}}{4^{-2}}$. Hint. $\frac{2^{-1} - 3^{-1}}{4^{-2}} = \frac{\frac{1}{2} - \frac{1}{3}}{\frac{1}{4^2}} =$ etc.
19. $\frac{4^{-2} - 3^{-1}}{4^{-2} + 3^{-1}}$.
20. $\frac{3^{-1} + 4^{-1}}{3^{-3} + 4^{-3}}$.
21. $\frac{4^{-3} - 2^{-3}}{4^{-1} + 2^{-1}}$.
22. $\frac{3^e \cdot 3 - 3^{4+e}}{9 \cdot 3^{2+e}}$.
23. x^{-1}.
24. mc^{-2}.
25. $3a^{-2}x^3$.
26. $\frac{4x^{-2}}{m^{-3}}$.
27. $\frac{6x^{-2}m}{m^{-3}}$.
28. $\frac{7m^{-5}x^4}{m^{-2}x^{-3}}$.
29. $\frac{2x^{-2}a^{-5}}{a^{-6}x^5}$.
30. $\frac{6^{-1}a^{-2}c}{amc^{-3}}$.
31. $\frac{4^{-1}a^4m^{-3}}{2^{-3}a^3m^{-2}}$.
32. $\frac{2}{a^{-2} + x^{-2}}$. Hint. $\frac{2}{a^{-2} + x^{-2}} = \frac{2}{\frac{1}{a^2} + \frac{1}{x^2}} =$ etc.
33. $\frac{4}{x^{-1} - a^{-1}}$.
34. $\frac{x}{x^{-2} - a^{-2}}$.
35. $\frac{x^{-3}}{x^{-2} + a^{-2}}$.
36. $\frac{x^{-2} - a^{-2}}{x^{-1} - a^{-1}}$.
37. $\frac{a^{-1} + 2x^{-1}}{a^{-3} + 8x^{-3}}$.
38. $\frac{x^{-4} + x^{-2} + 1}{x^{-2} - x^{-1} + 1}$.

Write without a denominator:

39. $\frac{x^2}{a}$.
40. $\frac{4a^2}{cm^2}$.
41. $\frac{3m}{a^{-2}c^2}$.
42. $\frac{mc}{m + c}$.
43. $\frac{4am^{-2}}{m(a - m)^0}$.
44. $\frac{a^{-2}m^3}{m^{-2}a^3(m - a)^2}$.

Write, using positive fractional exponents instead of radical signs:

45. $\sqrt{a}$.
46. $\sqrt[3]{c}$.
47. $\sqrt[4]{m}$.
48. $\sqrt[3]{c^2}$.
49. $\sqrt[5]{c^3}$.
50. $4\sqrt[3]{a^2}$.
51. $3\sqrt[2]{c^{-4}}$.
52. $5c\sqrt[3]{c^{-2}}$.
53. $\sqrt{2 \div a}$.
54. $\sqrt{3 \div a^2}$.
55. $\sqrt[3]{4 \div a^{-2}}$.
56. $5c\sqrt[4]{a \div c^{-5}}$.

57. $\sqrt{a\sqrt{a}}$. **Solution:** $\sqrt{a\sqrt{a}} = \sqrt{a \cdot a^{\frac{1}{2}}} = \sqrt{a^{\frac{3}{2}}} = a^{\frac{3}{4}}$.

58. $\sqrt{\sqrt{m}}$.
59. $\sqrt[3]{\sqrt{c}}$.
60. $4\sqrt[2]{18\sqrt[3]{8}}$.
61. $\sqrt{c\sqrt[3]{c}}$.
62. $6\sqrt[3]{75\sqrt{9}}$.

MISCELLANEOUS EXERCISES

Perform the indicated operations:

1. $x^5 \cdot x^{-3}$.
2. $x^{\frac{2}{3}} \cdot x^{\frac{1}{2}}$.
3. $x^{\frac{1}{4}} \cdot x^{\frac{1}{3}}$.
4. $e^x \cdot e^{-x}$.
5. $e^{a-2} \cdot e^{3-2a}$.
6. $e^{a-1} \cdot e^{1-a}$.
7. $(e^2)^3$.
8. $(e^4)^3$.
9. $(e^{\frac{1}{2}})^2$.
10. $(e^{\frac{1}{3}})^4$.
11. $(e^4)^{-2}$.
12. $(e^{\frac{2}{3}})^{-3e}$.
13. $((x^4)^5)^{\frac{1}{10}}$.
14. $(m^2)^{2a}$.
15. $(x^{-1})^{-2c}$.
16. $(x^3)^{2-x}$.
17. $(x^{n-3})^{n+2}$.
18. $((x^{-3})^{4a})^{\frac{1}{2a}}$.
19. $(x^{a^2-e^2})^{\frac{1}{a-e}}$.
20. $\left[(4x^3)^0 \cdot 8 \cdot 4^{\frac{1}{2}}\right]^{\frac{1}{2}}$.
21. $x^3 \cdot x^5 \div x^6$.
22. $x^6 \cdot x^3 \div x^{18}$.
23. $x^{\frac{1}{2}} \div x^{\frac{1}{4}}$.
24. $x^{\frac{3}{4}} \div x^{\frac{2}{3}}$.
25. $e^x \div e^{-x}$.
26. $e^{a-1} \div e^2$.
27. $e^{a^2-1} \div e^{a-1}$.
28. $4^2 \div 2^2$.
29. $8^2 \div 4^3$.
30. $2^3 \cdot 4^2 \div 8^2$.
31. $4^{2n} \div 2^n$.
32. $8^{3n} \div 2^{2n}$.
33. $4^{3n} \cdot 2^{4n} \div 8^{2n}$.
34. $(x^3 - 2x^{-2})^2$.
35. $(x^{\frac{2}{3}} - 3x^{\frac{3}{2}})^2$.
36. $(a^{2n-2} - 2a^{n-1} - 8)(a^{n-1} + 4)(a^{n-1} - 2)$.
37. $(4e^x + e^{-x} - 4)^2$.
38. $(2a - 3a^{-1} - 4a^{-2})^2$.
39. $(x^2 - a^2) \div (x^{\frac{1}{2}} + a^{\frac{1}{2}})$.
40. $(x^3 - 2) \div (x - 2^{\frac{1}{3}})$.
41. $(x - y^{\frac{3}{2}}) \div (x^{\frac{1}{3}} - y^{\frac{1}{2}})$.
42. $(a^{-2} - x^{-2}) \div (x^{-\frac{1}{2}} + a^{-\frac{1}{2}})$.
43. $(x^{-3} + a^{-3}) \div (x^{-\frac{2}{3}} + a^{-\frac{2}{3}})$.
44. $(3x^{-2} - 48a^4) \div (x^{-\frac{1}{2}} + 2a)$.

45. $(x^2 - y^2) \div (x^{\frac{3}{2}} + xy^{\frac{1}{2}} + x^{\frac{1}{2}}y + y^{\frac{3}{2}}) \div (x^{\frac{1}{4}} + y^{\frac{1}{4}})$.

46. $(x + 32\,a) \div (x^{\frac{1}{5}} + 2\,a^{\frac{1}{5}}) + 2\,x^{\frac{3}{5}}a^{\frac{1}{5}} + 8\,x^{\frac{1}{5}}a^{\frac{3}{5}}$.

47. $(a^{2n+2} - a^{n+1} - 12)(4 - a^{n+1}) + 48 + 16\,a^{n+1}$.

48. $\left(\dfrac{e^{6x} - 1}{e^{3x}} - \dfrac{3\,e^{2x} - 3}{e^{x}}\right) \div (e^x - e^{-x}) + (e^x + e^{-x})^2$.

49. $(e^{5x} - 5\,e^{3x} + 10\,e^{x} - 10\,e^{-x} + 5\,e^{-3x} - e^{-5x})$
$\div (e^{2x} + e^{-2x} - 2)$.

50. Point out the error in the following:

Let x be a number such that $e^x = -1$.

Then $e^{2x} = 1$.

Hence $2\,x = 0$, or $x = 0$,

and $e^x = e^0 = 1$.

Therefore $1 = -1$.

55. Classification of numbers. All the numbers of algebra are in one or the other of two classes, **real** numbers and **imaginary** numbers.

Real numbers are of two kinds, **rational** numbers and **irrational** numbers.

A **rational** number is a positive or a negative *integer*, or a number which may be expressed as the *quotient of two such integers*.

Any real number which is not a rational number is an **irrational** number.

A **pure imaginary** number is the indicated square root of a negative number.

56. Radicals. A **radical** is an indicated root of the form $\sqrt[r]{n}$ or $c\sqrt[r]{n}$, or of the form $n^{\frac{1}{r}}$ or $cn^{\frac{1}{r}}$.

A **surd** is an irrational root of a rational number.

The **index** determines the root to be extracted and the **order** of the radical.

The **radicand** is the number, or expression, under the radical sign.

EXERCISES

Write with radical signs:

1. $x^{\frac{3}{5}}$.
2. $(ac)^{\frac{2}{3}}$.
3. $(4x)^{\frac{1}{3}}$.
4. $4x^{\frac{1}{3}}$.
5. $6cx^{\frac{3}{4}}$.
6. $a^{\frac{2}{3}}c^{\frac{3}{5}}$.
7. $3a^{\frac{1}{2}}(c^2x)^{\frac{2}{3}}$.
8. $\dfrac{4a^{\frac{2}{3}}(c-x)^{\frac{4}{3}}}{a^{\frac{1}{2}}(c-x)}$.

Find the numerical value of:

9. $4^{\frac{1}{2}}$.
10. $36^{\frac{1}{2}}$.
11. $64^{\frac{1}{3}}$.
12. $81^{\frac{1}{4}}$.
13. $4^{\frac{3}{2}}$.
14. $27^{\frac{2}{3}}$.
15. $(-8)^{\frac{2}{3}}$.
16. $(\frac{1}{9})^{\frac{1}{2}}$.
17. $(-32)^{\frac{3}{5}}$.
18. $(\frac{4}{25})^{\frac{1}{2}}\cdot(\frac{125}{8})^{\frac{1}{3}}$.
19. $(-125)^{\frac{2}{3}}\cdot(\frac{1}{25})^{\frac{3}{2}}$.
20. $(-243)^{\frac{2}{5}}\cdot(81)^{\frac{3}{4}}$.

Write with fractional exponents:

21. $\sqrt[3]{x^4}$.
22. $\sqrt{ac^3}$.
23. $3\sqrt{x^5}$.
24. $4\sqrt[3]{4x^2}$.
25. $5\sqrt[3]{8x^4}$.
26. $6\sqrt[3]{64x^5}$.
27. $5\sqrt[3]{-125c^2}$.
28. $c\sqrt{(x+a)^3}$.
29. $\dfrac{5c\sqrt{ax^3}\cdot\sqrt[3]{5x^4}}{\sqrt[3]{5x^2}\sqrt{ax}}$.
30. $\sqrt[n]{x^a}\cdot\sqrt[n]{c}$.
31. $\sqrt[n]{x^{4a}}\cdot\sqrt[n]{x^{2a}}$.

32. Give an example of (*a*) a real number; (*b*) an imaginary number; (*c*) a rational number; (*d*) an irrational number; (*e*) a radical; (*f*) a surd; (*g*) an index; (*h*) a radicand; (*i*) the principal odd root of a positive number; (*j*) the principal even root of a positive number; (*k*) the principal odd root of a negative number.

33. What is the distinction between a rational number and an irrational one?

34. Which of the numbers 8, $\frac{2}{3}$, $.\dot{3}4\dot{3}$, $\sqrt{4}$, $\sqrt{3}$, and π ($\pi = 3.14159+$) are rational? irrational?

35. Give a geometrical illustration of an irrational number by means of a right triangle.

36. Is a radical always a surd? Illustrate.

37. Is a surd always a radical? Illustrate.

38. Distinguish between a surd and a radical.

39. Which of the numbers $\sqrt{3}$, $\sqrt{4}$, $\sqrt[3]{27}$, $\sqrt{\sqrt[3]{6}}$, $\sqrt{2+\sqrt{3}}$, and $\sqrt{\pi}$ are surds? Which are radicals?

40. What is the principal root of: $\sqrt{4}$, $\sqrt[3]{8}$, and $\sqrt[3]{-8}$?

41. Name the order of: $\sqrt[2]{6}$, $a^{\frac{1}{3}}$, $\sqrt[3]{5}$, $c^{\frac{2}{3}}$, and $\sqrt[4]{m^3}$.

42. How many real numbers can be found for a designated odd root of (*a*) a positive real number? (*b*) a negative real number?

43. Change the word "odd" in (*a*) of Exercise 42 to "even," and answer.

57. Simplification of radicals. The form of a radical expression may be changed without altering its numerical value. It is often desirable to change the form of a radical so that its numerical value can be computed with the least possible labor.

The simplification of a radical is based on the general identity:

$$\sqrt[n]{a^n b} = \sqrt[n]{a^n} \cdot \sqrt[n]{b} = a\sqrt[n]{b}.$$

A radical is in its **simplest form** when the radicand

(*a*) *Is integral.*

(*b*) *Contains no rational factor raised to a power which is equal to, or greater than, the order of the radical.*

(*c*) *Is not raised to a power, unless the exponent of the power and the index of the root are prime to each other.*

For the meaning of (*a*), (*b*), and (*c*) study carefully the

EXAMPLES

Of (*a*): **1.** $\sqrt{\frac{3}{2}} = \sqrt{\frac{6}{4}} = \sqrt{\frac{1}{4} \cdot 6} = \sqrt{\frac{1}{4}}\sqrt{6} = \frac{1}{2}\sqrt{6}.$

2. $6\sqrt{\frac{1}{3}} = 6\sqrt{\frac{3}{9}} = 6\sqrt{\frac{1}{9} \cdot 3} = 6 \cdot \frac{1}{3}\sqrt{3} = 2\sqrt{3}.$

3. $\sqrt[3]{\frac{3}{16}} = \sqrt[3]{\frac{12}{64}} = \sqrt[3]{\frac{1}{64} \cdot 12} = \sqrt[3]{\frac{1}{64}}\sqrt[3]{12} = \frac{1}{4}\sqrt[3]{12}.$

4. $\sqrt{\frac{3}{5x}} = \sqrt{\frac{15x}{25x^2}} = \sqrt{\frac{1}{25x^2} \cdot 15x} = \frac{1}{5x}\sqrt{15x}.$

Of (*b*): 1. $\sqrt{4x^5} = \sqrt{4x^4 \cdot x} = \sqrt{(2x^2)^2 \cdot x} = 2x^2\sqrt{x}.$

2. $5\sqrt[3]{24x^5} = 5\sqrt[3]{8x^3 \cdot 3x^2} = 5\sqrt[3]{(2x)^3 \cdot 3x^2} = 10x\sqrt[3]{3x^2}.$

3. $\sqrt{16 - 8\sqrt{2}} = \sqrt{4(4 - 2\sqrt{2})} = 2\sqrt{4 - 2\sqrt{2}}.$

Of (*c*): 1. $\sqrt[4]{4} = \sqrt[4]{2^2} = 2^{\frac{2}{4}} = 2^{\frac{1}{2}} = \sqrt{2}.$

2. $\sqrt[6]{9} = \sqrt[6]{3^2} = 3^{\frac{2}{6}} = 3^{\frac{1}{3}} = \sqrt[3]{3}.$

3. $\sqrt[4]{a^2b^4} = a^{\frac{2}{4}}b^{\frac{4}{4}} = a^{\frac{1}{2}}b = b\sqrt{a}.$

EXERCISES

Express in simplest form:

1. $\sqrt{18}.$
2. $\sqrt[3]{16}.$
3. $2\sqrt{75}.$
4. $4\sqrt[3]{-54}.$
5. $4\sqrt{40}.$
6. $\sqrt{\sqrt{4}}.$
7. $\sqrt[3]{3\sqrt{9}}.$
8. $\sqrt{\frac{1}{3}}.$
9. $\sqrt{\frac{3}{5}}.$
10. $\sqrt{\frac{5}{8}}.$
11. $\sqrt[3]{-\frac{3}{4}}.$
12. $\sqrt[3]{\frac{1}{7}}.$
13. $6\sqrt[3]{-\frac{1}{9}}.$
14. $\sqrt{1 - (\frac{1}{3})^2}.$
15. $\sqrt{3^2 - (\frac{3}{2})^2}.$
16. $\sqrt{7^2 - (\frac{7}{2})^2}.$
17. $\sqrt{4 - 8\sqrt{3}}.$
18. $\sqrt{36 + 18\sqrt{5}}.$
19. $\sqrt[3]{54 - 9\sqrt{18}}.$
20. $\sqrt[3]{81 - 3\sqrt{243}}.$
21. $\sqrt{R^2 - 3R^2\sqrt{5}}.$
22. $\sqrt{\dfrac{R^2 - R^2\sqrt{6}}{3}}.$
23. $\sqrt{R^2 - \left(\dfrac{R}{2}\right)^2}.$
24. $\sqrt{R^2 + \left(\dfrac{R}{3}\right)^2\sqrt{3}}.$

Express entirely under the radical sign:

25. $3\sqrt{5}.$
26. $4\sqrt{3}.$
27. $2\sqrt[3]{8}.$
28. $a\sqrt{a}.$
29. $2c\sqrt[3]{c^2}.$
30. $4\sqrt[3]{\frac{1}{4}}.$
31. $x^3\sqrt{x^2}.$
32. $\dfrac{a}{3}\sqrt[3]{\dfrac{9}{a^2}}.$
33. $(2a + 1)\sqrt{\dfrac{2}{4a^2 - 1}}$
34. $\dfrac{x - 3a}{5}\sqrt[3]{\dfrac{125}{(x - 3a)^2}}.$

Express in simplest form with one radical sign:

35. $\sqrt{\sqrt{x}}.$
36. $\sqrt[3]{\sqrt[3]{x}}.$
37. $\sqrt{\sqrt[3]{x}}.$
38. $\sqrt[3]{\sqrt{x}}.$
39. $\sqrt[3]{\sqrt{8a^2x}}.$
40. $\sqrt{3\sqrt{3}}.$
41. $\sqrt{3\sqrt{3\sqrt{3}}}.$
42. $\sqrt[4]{\sqrt{8}}.$
43. $2\sqrt[3]{2\sqrt[3]{2}}.$
44. $\sqrt{\sqrt{\sqrt{x^{12}}}}.$
45. $\sqrt[a]{\sqrt[n]{x^c}}.$
46. $\sqrt[\frac{m}{n}]{x^{\frac{a}{n}}}.$

58. Addition and subtraction of radicals. Similar radicals are radicals of the same order with radicands which are identical or which can be made so by simplification.

The sum or the difference of similar radicals can be expressed as one term, while the sum or difference of dissimilar radicals can only be indicated.

EXERCISES

Simplify and collect:

1. $\sqrt{8}+\sqrt{18}$.
2. $\sqrt{50}+\sqrt{98}-\sqrt{32}$.
3. $\sqrt[3]{16}+\sqrt[3]{54}-3\sqrt[3]{2}$.
4. $\sqrt[3]{192}-4\sqrt[3]{24}+\sqrt[3]{375}$.
5. $10\sqrt{\frac{6}{5}}-\sqrt{\frac{3}{10}}+4\sqrt{\frac{15}{2}}$.
6. $3\sqrt{\frac{2}{7}}+3\sqrt{\frac{7}{2}}-2\sqrt{\frac{1}{14}}$.
7. $a\sqrt{x^3}-\sqrt{a^2x}-5\sqrt{a^2x}$.
8. $\sqrt{x^3}+\sqrt[4]{x^2}-12\sqrt[6]{x^3}$.
9. $\sqrt{\frac{3a}{x}}+\sqrt{\frac{3x}{a}}-\sqrt{\frac{ax}{3}}$.
10. $\sqrt{\frac{a}{x^3}}-\sqrt{\frac{a}{x^5}}+\sqrt{\frac{5x^3}{a}}$.
11. $\sqrt[4]{32x^5}+\sqrt[4]{1250x}-\sqrt[4]{512x}-\sqrt[4]{2x}$.
12. $\sqrt{(a+c)^3}-c\sqrt[4]{(a+c)^2}+2c\sqrt[6]{(a+c)^3}$.
13. $\sqrt[3]{(a-c)^4}+c\sqrt[6]{a^2-2ac+c^2}+(a+c)\sqrt[3]{a-c}$.
14. $\sqrt{\frac{a}{c}}-\sqrt{\frac{c}{a}}+\sqrt{\frac{a^2+c^2}{ac}+2}-\sqrt{\frac{a^2+c^2}{ac}-2}$.
15. $\sqrt[3]{24}+\sqrt[3]{(3a+9)(a+3)^2}-\sqrt[3]{81}+a\sqrt[6]{9}-4\sqrt[3]{3}$.
16. $2\sqrt{9a^3-9a^2b}-3\sqrt{9ab^2-9b^3}+\sqrt{(a^2-b^2)(a+b)}$.
17. $(a-b)\sqrt{\frac{a+b}{a-b}}+\sqrt{25a^2-25b^2}+\frac{a+b}{a-b}\sqrt{\frac{36ab^2-36b^3}{a+b}}$.

59. Multiplication of real radicals. Radicals of the same order are multiplied as follows:

Example 1. Multiply $2\sqrt{x}-3\sqrt{a}-4\sqrt{ax}$ by $2\sqrt{ax}$.

Solution:

$$\begin{array}{l} 2\sqrt{x}\quad -3\sqrt{a}\quad -4\sqrt{ax} \\ 2\sqrt{ax} \\ \hline 4x\sqrt{a}-6a\sqrt{x}-8ax \end{array}$$

Radicals of different order are multiplied as follows:

Example 2. Multiply $\sqrt{n}$ by $\sqrt[3]{x}$.

Solution:
$$\sqrt{n} = n^{\frac{1}{2}} = n^{\frac{3}{6}} = \sqrt[6]{n^3}.$$
$$\sqrt[3]{x} = x^{\frac{1}{3}} = x^{\frac{2}{6}} = \sqrt[6]{x^2}.$$

Then
$$\sqrt{n} \cdot \sqrt[3]{x} = n^{\frac{3}{6}} \cdot x^{\frac{2}{6}} = \sqrt[6]{n^3} \cdot \sqrt[6]{x^2} = \sqrt[6]{n^3x^2}.$$

The method of multiplying radicals may be stated in the

RULE. *If necessary, reduce the radicals to the same order.*

Find the products of the coefficients of the radicals for the coefficient of the radical part of the result.

Multiply together the radicands and write the product under the common radical sign.

Reduce the result to its simplest form.

The preceding rule does not hold for the multiplication of imaginary numbers; that is, for radicals of even order in which the radicands are negative. This case will be discussed in the chapter on Imaginaries.

EXERCISES

Perform the indicated multiplications and simplify the products:

1. $\sqrt{3} \cdot \sqrt{27}$.
2. $\sqrt{12} \cdot \sqrt{18}$.
3. $\sqrt[3]{4} \cdot \sqrt[3]{32}$.
4. $\sqrt[3]{20} \cdot \sqrt[3]{12}$.
5. $\sqrt{\frac{2}{3}} \cdot \sqrt{\frac{3}{4}}$.
6. $\sqrt{\frac{4}{5}} \cdot \sqrt{\frac{5}{6}} \cdot \sqrt{\frac{3}{4}}$.
7. $\sqrt{\frac{1}{2}} \cdot \sqrt{\frac{1}{3}} \cdot \sqrt{\frac{1}{5}}$.
8. $\sqrt{3} \cdot \sqrt[3]{2}$.
9. $\sqrt[4]{6} \cdot \sqrt{2}$.
10. $\sqrt[3]{12} \cdot \sqrt{\frac{1}{8}}$.
11. $\sqrt[3]{c} \cdot \sqrt{a}$.
12. $\sqrt[3]{a^2} \cdot \sqrt{a^3}$.
13. $\sqrt[4]{a^3} \cdot \sqrt{a}$.
14. $\sqrt[3]{\frac{x}{a}} \cdot \sqrt{\frac{a}{x}}$.
15. $\sqrt[4]{2x^3} \cdot \sqrt{3x}$.
16. $(\sqrt{x-3})^2$.
17. $(2\sqrt{3x-1})^2$.
18. $3\sqrt{x-3} \cdot \sqrt{4x-8}$.
19. $(\sqrt{x} - \sqrt{x-3})^2$.
20. $(\sqrt{x-3} - \sqrt{4x-7})^2$.
21. $(\sqrt{x} - \sqrt{3x})(4\sqrt{x})$.
22. $(5\sqrt{5} + 9\sqrt{3} - \sqrt{7} + 2\sqrt{105})(\sqrt{3} + \sqrt{5} - \sqrt{7})$.
23. $\left(\dfrac{6-2\sqrt{5}}{3}\right)^2 \cdot \dfrac{[(\sqrt{5}+1)(\sqrt{5}+1)]^2}{2} \cdot \dfrac{(\sqrt{2}+1)(9\sqrt{2}-9)}{16}$.

24. $(c\sqrt{x} - \sqrt{cx} + x\sqrt{c})(\sqrt{c} - \sqrt{x})$.

25. $(\sqrt{a} - \sqrt[4]{ac} + \sqrt{c})(\sqrt{a} + \sqrt[4]{ac} + \sqrt{c})$.

26. $(\sqrt{2x-1} - \sqrt{5})(2\sqrt{2x-1} + \sqrt{45})$.

Determine which of the two surds is the greater:

27. $\sqrt[3]{11}$, $\sqrt[2]{5}$.

Solution: $\sqrt[3]{11} = 11^{\frac{1}{3}} = 11^{\frac{2}{6}} = \sqrt[6]{11^2} = \sqrt[6]{121}$.

$\sqrt[2]{5} = 5^{\frac{1}{2}} = 5^{\frac{3}{6}} = \sqrt[6]{5^3} = \sqrt[6]{125}$.

Since $125 > 121$, then $\sqrt{5} > \sqrt[3]{11}$.

28. $\sqrt[3]{6}$, $\sqrt{3}$.
29. $\sqrt[3]{19}$, $\sqrt{7}$.
30. $\sqrt{3}$, $\sqrt[4]{6}$.
31. $2\sqrt{5}$, $\sqrt[3]{89}$.
32. $3\sqrt{3}$, $2\sqrt[3]{10}$.
33. $\sqrt[4]{48}$, $\sqrt[5]{64}$.

Arrange in order of magnitude:

34. $\sqrt{3}$, $\sqrt[3]{6}$, $\sqrt[9]{125}$.
35. $4\sqrt[4]{6}$, $3\sqrt[6]{25}$, $4\sqrt[9]{64}$.

Reduce to respectively equivalent surds of the same order:

36. $\sqrt{3}$, $\sqrt[8]{3x^2}$.
37. $\sqrt[3]{a+b}$, $\sqrt{a-b}$.
38. $2x\sqrt[3]{5xy}$, $5x\sqrt[4]{3xy}$.
39. $\sqrt[3]{xy}$, $\sqrt[4]{xy^2}$, $\sqrt[5]{x^2y}$.

Square:

40. $\sqrt[3]{3}$.
41. $2\sqrt[3]{4}$.
42. $\sqrt{5 - \sqrt{5}}$.
43. $4\sqrt{3 - \sqrt{5}}$.
44. $\sqrt[3]{4 - 4\sqrt{3}}$.
45. $\sqrt[4]{6 - 3\sqrt{2}}$.

Cube:

46. $\sqrt[2]{3}$.
47. $3\sqrt{5}$.
48. $\sqrt[3]{5} - \sqrt[3]{3}$.
49. $\sqrt{3 - \sqrt{5}}$.
50. $3\sqrt[3]{2} - 2\sqrt{3}$.
51. $(\sqrt[4]{1 - \sqrt{2}})^3$.

Simplify:

52. $\sqrt{R^2 - \left(\frac{R}{2}\sqrt{5} - \frac{R}{2}\right)^2}$.

53. $\left[\left(R - \frac{R}{2}\sqrt{3}\right)^2 + \left(\frac{R}{2}\right)^2\right]^{\frac{1}{2}}$.

54. $\sqrt{R^2 - \left(\frac{R\sqrt{2-\sqrt{2}}}{2}\right)^2}$.

55. $\left(\frac{R}{2}\sqrt{2+\sqrt{2}}\right)\left(\frac{R\sqrt{2-\sqrt{2}}}{2}\right)$.

56. $\sqrt{\left(e^x + \frac{2}{e^x}\right)^2 - (e^x - 2e^{-x})(e^x - 2e^{-x}) + e^{2x} + e^{-2x} - 6}$.

60. Division of radicals. Division of radicals is usually an indirect process performed by means of a rationalizing factor for the divisor.

One radical expression is a **rationalizing factor** for another if the product of the two is rational.

Thus $a\sqrt{n}$ and $\sqrt{n}$ are rationalizing factors for each other. A like relation holds between $a\sqrt[3]{n}$ and $\sqrt[3]{n^2}$.

An important pair of radicals is $\sqrt{a}+\sqrt{b}$ and $\sqrt{a}-\sqrt{b}$. Two such binomials are called **conjugate** radicals and either is the rationalizing factor for the other.

Division of one radical by another may often be performed as follows:

Example: Divide $6\sqrt{5}$ by $3\sqrt{3}$.

Solution: $6\sqrt{5} \div 3\sqrt{3} = 2\sqrt{\frac{5}{3}} = \frac{2}{3}\sqrt{15}$.

Direct division of radical expressions in which the divisor is a polynomial is very difficult. In such cases we use the

RULE. *Write the dividend over the divisor in the form of a fraction. Then multiply the numerator and denominator of the fraction by a rationalizing factor for the denominator and simplify the resulting fraction.*

This rule applies in all cases, while the rule for direct division fails when dividing a real radical by a radical of even order whose radicand is negative.

Every irrational algebraic expression containing nothing more complicated than rational numbers and radicals has a rationalizing factor. To find this factor for any given irrational expression is a problem which requires considerable algebraic training. At the present time it is wholly beyond the student to find the rationalizing factor of even so simple an expression as the denominator of the fraction $\dfrac{\sqrt{2}+\sqrt{3}}{\sqrt{2}+\sqrt[3]{2}+\sqrt[4]{2}}$. The approximate value of such a fraction can be obtained, however, by dividing the sum of the approximate values of the terms in the numerator by the sum of the approximate values of the terms in the denominator. (The table on page 262 may be used to advantage in work of this character.)

EXERCISES

Find a simple rationalizing factor for:

1. $3\sqrt{7}$.
2. $5\sqrt[3]{4}$.
3. $7\sqrt[4]{8}$.
4. $\sqrt{8}$.
5. $\sqrt[3]{16}$.
6. $\sqrt{5}-7$.
7. $\sqrt{6}-\sqrt{11}$.
8. $3\sqrt{7}-2\sqrt{13}$.
9. $\sqrt{3a}-\sqrt{2x}$.
10. $\sqrt{x-c}-\sqrt{a}$.
11. $\sqrt{2}+\sqrt{3}-\sqrt{5}$.

Perform the indicated division and simplify results:

12. $\sqrt{12}\div\sqrt{3}$.
13. $\sqrt{8}\div\sqrt{24}$.
14. $8\div 4\sqrt{3}$.
15. $8\div 2\sqrt{2}$.
16. $24\div 3\sqrt{3}$.
17. $a\div c\sqrt{x}$.
18. $(\sqrt{12}-\sqrt{18})\div 2\sqrt{3}$.
19. $(12-3\sqrt{6}-4\sqrt{24})\div 3\sqrt{2}$.
20. $\sqrt{6}\div\sqrt[3]{2}$.

HINT. $\sqrt{6}\div\sqrt[3]{2}=\frac{\sqrt{6}}{\sqrt[3]{2}}=\frac{\sqrt{6}\sqrt[3]{4}}{\sqrt[3]{2}\sqrt[3]{4}}=\frac{\sqrt{6}\sqrt[3]{4}}{2}=$ etc.

21. $\sqrt[4]{8}\div\sqrt{2}$.
22. $\sqrt{32}\div\sqrt[4]{2}$.
23. $\sqrt[3]{\frac{1}{4}}\div\sqrt{\frac{1}{2}}$.
24. $3\div(2-\sqrt{3})$.

HINT. $3\div(2-\sqrt{3})=\frac{3}{2-\sqrt{3}}=\frac{3(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})}=$ etc.

25. $4\div(\sqrt{2}-1)$.
26. $\sqrt{3}\div(\sqrt{2}+\sqrt{3})$.
27. $\sqrt{7}\div\sqrt{2-\sqrt{3}}$.
28. $\sqrt{2-\sqrt{3}}\div\sqrt{3-\sqrt{2}}$.
29. $(\sqrt{7}+\sqrt{5})\div(2\sqrt{7}-\sqrt{5})\div(19-3\sqrt{35})$.

30. Find to four decimals the numerical value of the results in Exercises (*a*) 24, (*b*) 26, and (*c*) 27.

31. In Exercise 26 divide the numerical value of the numerator by the numerical value of the denominator, each having been obtained to five decimals. Compare the quotient with the result obtained for that fraction in Exercise 30.

32. What conclusion can be drawn from Exercises 30 and 31 regarding the rationalization of the denominator of a fraction before finding its numerical value?

Change to respectively equivalent fractions having rational denominators:

33. $\dfrac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}}$. 35. $\dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}$. 37. $\dfrac{\sqrt{x-3}+\sqrt{3}}{\sqrt{x-3}-\sqrt{3}}$.

34. $\dfrac{2\sqrt{5}+3\sqrt{7}}{3\sqrt{5}-2\sqrt{7}}$. 36. $\dfrac{\sqrt{x}-2\sqrt{c}}{\sqrt{x}+\sqrt{c}}$. 38. $\dfrac{4}{\sqrt[4]{2}-\sqrt{2}}$.

Perform the indicated division:

39. $(\sqrt{10}-\sqrt{5})\div(\sqrt{10}+\sqrt{5})$. 40. $(x-\sqrt{c})\div(x-3\sqrt{c})$.

41. $(\sqrt{a+c}-\sqrt{x})\div(\sqrt{a+c}+\sqrt{x})$.

42. $(\sqrt{3}+\sqrt{2})\div(2-\sqrt{3}+\sqrt{2})$.

43. $(\sqrt{5}-\sqrt{7})\div(\sqrt{5}+\sqrt{7}-\sqrt{2})$.

44. Is there any real distinction between the direction before Exercise 33 and that before Exercise 39?

45. Does $3-\sqrt{7}$ satisfy $x^2-6x+2=0$?

46. Does $\dfrac{7-\sqrt{3}}{2}$ satisfy $2x^3-75x+161=0$?

47. Does $\frac{1}{6}(5\pm\sqrt{109})$ satisfy $3x^2-5x-7=0$?

61. Square root of surd expressions. The square of a binomial is usually a trinomial. However, the result of squaring a binomial of the form $\sqrt{a}+\sqrt{b}$ is a binomial, if a and b are rational numbers. Thus $(\sqrt{7}-\sqrt{3})^2=7-2\sqrt{21}+3=10-2\sqrt{21}$. Here in $10-2\sqrt{21}$, 10 is the sum of 7 and 3, and 21 is the product of 7 and 3. These relations, and the fact that the coefficient of the radical $\sqrt{21}$ is 2, enable us to find the square root of many expressions of the form $a\pm 2\sqrt{b}$ by writing each in the form of $x\pm 2\sqrt{xy}+y$ and then taking the square root of the trinomial square as follows:

Example: Extract the square root of $9-\sqrt{56}$.

Solution: $9-\sqrt{56}=9-2\sqrt{14}$.

We now find two numbers whose sum is 9 and whose product is 14. These are 7 and 2.

Therefore $9-2\sqrt{14}=2-2\sqrt{14}+7=(\sqrt{2}-\sqrt{7})^2$.

Hence the square roots of $9-\sqrt{56}$ are $\pm(\sqrt{2}-\sqrt{7})$.

EXERCISES

Find the positive square roots in Exercises 1–12:

1. $6 - 2\sqrt{8}$.
2. $7 + 2\sqrt{10}$.
3. $13 + \sqrt{48}$.
4. $8 - \sqrt{60}$.
5. $11 - 4\sqrt{7}$.
6. $17 + 12\sqrt{2}$.
7. $11 - 3\sqrt{8}$.
8. $65x - 20\sqrt{3x^2}$.
9. $126a - 10a\sqrt{5}$.
10. $\frac{13a}{4} - \sqrt{3a^2}$.
11. $2x + 2\sqrt{x^2 - 49}$.
12. $a + \sqrt{a^2 - 1}$.
13. $\sqrt{9 + 3\sqrt{8}} = \sqrt{?} + \sqrt{?}$.
14. $\sqrt{15 - 5\sqrt{8}} = ?$
15. $\sqrt{a + \sqrt{a^2 - 4b^2}} = ?$
16. $\sqrt{m^2 + m + 2n + 2m\sqrt{m + 2n}} = ?$

Note. In the writings of one of the later Hindu mathematicians (about 1150 A.D.) we find a method of extracting the square root of surds, which is practically the same as that given in the text. In fact, the formula for the operation is given, apart from the modern symbols, as follows: $\sqrt{a} + \sqrt{b} = \sqrt{a + b + 2\sqrt{ab}}$. The study of expressions of the type $\sqrt{\sqrt{a} \pm \sqrt{b}}$ had been carried to a most remarkable degree of accuracy by the Greek, Euclid. His researches on this subject, if original with him, place him among the keenest mathematicians of all time; but his work and all of his results are expressed in geometrical language, which is very far removed from our algebraic symbolism, and for that reason is little read now.

62. Factors involving radicals. In the chapter on Factoring it was definitely stated that (except in § 17) factors involving radicals would not then be considered. This limitation on the character of a factor is no longer necessary. Consequently many expressions which previously have been regarded as prime may now be thought of as factorable; thus

$$3x^2 - 1 = (x\sqrt{3} + 1)(x\sqrt{3} - 1) \text{ and } 4x^2 - 5 = (2x + \sqrt{5})(2x - \sqrt{5}).$$

It is not usual to allow the variable in an expression to occur under a radical sign in the factors. Hence, if x is a variable, the trinomial $x^2 + x + 1$ is not regarded as factorable into $(x + \sqrt{x} + 1)(x - \sqrt{x} + 1)$, though $(x + \sqrt{x} + 1)(x - \sqrt{x} + 1) = x^2 + x + 1$.

Therefore in this extension of our notion of a factor it must be clearly understood that the use of radicals is limited to the coefficients in the terms of the factors. Such a conception of a factor is a necessity for certain work in advanced algebra and geometry, and is very desirable in solving equations by factoring.

To restrict the use of radicals in the way just indicated is necessary for the sake of definiteness. Otherwise it would be impossible to obey a direction to factor even so simple an expression as $x^2 - y^2$; for if the variable is allowed under a radical sign in a factor, $x^2 - y^2$ has countless factors.

$$\begin{aligned}\text{Thus } x^2 - y^2 &= (x+y)(x-y)\\ &= (x+y)(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})\\ &= (x+y)(\sqrt{x}+\sqrt{y})(\sqrt[4]{x}+\sqrt[4]{y})(\sqrt[4]{x}-\sqrt[4]{y}) = \text{etc.}\end{aligned}$$

EXERCISES

Factor:

1. $x^2 - 11.$
2. $3x^2 - 16.$
3. $x^3 + 2.$
4. $x^3 - 12.$
5. $3x^3 - 27.$
6. $5x^3 + 125.$

Find the algebraic sum of:

7. $\dfrac{2\sqrt{b}}{a-b} + \dfrac{2}{\sqrt{a}+\sqrt{b}}.$
8. $\dfrac{x+c}{\sqrt{x}-\sqrt{c}} - \dfrac{x^{\frac{3}{2}}+c^{\frac{3}{2}}}{x-c}.$

Solve by factoring and check:

9. $x^2 - 5 = 0.$
10. $2x^2 - 3 = 0.$
11. $x^4 + 144 = 26x^2.$
12. $4x^4 + c = x^2 + 4cx^2.$

MISCELLANEOUS EXERCISES

Solve for n:

1. $a^3 \cdot a^2 = a^n.$
2. $2^3 \cdot 2^2 = 2^n.$
3. $2^4 \cdot 2^3 = 2^n.$
4. $2^5 \cdot 2^n = 2^{11}.$
5. $3^2 \cdot 2^4 = 9 \cdot 2^n.$
6. $4^2 \cdot 2^2 = 2^n.$
7. $4^3 \cdot 2^3 = 2^n.$
8. $4^2 \cdot 2^4 = 2^{2n}.$
9. $3^3 \cdot 9^3 = 3^n.$
10. $9^2 \cdot 3^n = 3^5.$
11. $2^2 \cdot 2^n = 32.$
12. $8^2 \cdot 4^3 = 2^n.$
13. $8^3 \cdot 4^2 = 2^n.$
14. $9^3 \cdot 27^2 = 3^n.$
15. $27^n \cdot 9^2 = 3^{10}$
16. $8^n \cdot 4^{2n} = 2^{14}.$
17. $3^6 \cdot 9^n = 81^2.$
18. $9^n \cdot 3^3 = 27^n.$
19. $2^{2n+2} \cdot 4^{n+2} = 8^{2n}.$
20. $81 \cdot 27^n = (9^n)^{\frac{n}{2}}.$
21. $(25^n)^n = \dfrac{5^{7n}}{(125)^2}.$
22. $2^{6n+3} \cdot 4^{3n+6} = (8^n)^n.$

Solve for x:

23. $x^{\frac{2}{3}} = 8.$

24. $x^{-\frac{1}{3}} = 5.$

25. $x^{-\frac{4}{3}} = 256.$

26. $x^{\frac{3}{2}} = -343.$

27. $\frac{1}{2}x^{-\frac{2}{3}} = 2.$

28. $(x^{-\frac{1}{2}})^{-4} = 49.$

29. $(ax^{\frac{1}{2}})^{-6} = 27.$

30. $\dfrac{\sqrt[3]{x^{\frac{7}{5}}}}{\sqrt[3]{x^{\frac{1}{5}}}} = \dfrac{\sqrt[5]{25}}{\sqrt[5]{16}}.$

Express in simplest form with positive exponents:

31. $\left(\sqrt[3]{-27x^9}\right)^{-2}.$

32. $\left(\sqrt[4]{16a^4x^8}\right)^{-2}.$

33. $\sqrt[2]{16^{\frac{3}{4}}x^9}.$

34. $\left(x\sqrt[3]{x^{-2}}\right)^3.$

35. $\left(x^{\frac{3}{2}}\sqrt{x^{-4}}\right)^{-\frac{2}{3}}.$

36. $\left(x^{-2}\sqrt{x^3\sqrt{4}}\right)^4.$

37. $\left(x^{-2}\sqrt{x^3\sqrt[3]{x^2}}\right)^{-\frac{1}{2}}.$

38. $\left[\left(\sqrt[3]{8x^4}\right)^{-2}\right]^{\frac{4}{3}}.$

39. $\left(\sqrt[2]{\sqrt[4]{16a^6}}\right)^4.$

40. $\left[(27x^3)^{-\frac{1}{3}}\cdot\dfrac{1}{5x^{-4}}\right]^{-3}.$

41. $\left(a^{\frac{1}{2}}x^{-\frac{1}{3}}\sqrt{ax^{-\frac{1}{4}}\sqrt[3]{x^{\frac{5}{3}}}}\right)^{\frac{1}{2}}.$

42. $\sqrt[2]{\sqrt[2]{a}}.$

43. $\sqrt[3]{\sqrt[2]{x^4}}.$

44. $\sqrt[3]{\sqrt[4]{x^5}}.$

45. $\sqrt[2]{2a^4\sqrt[5]{x^3}}.$

46. $\dfrac{\left[\sqrt{(25x^2)^2}\right]^{-\frac{1}{3}}}{\sqrt[3]{5x}}.$

Simplify:

47. $\left(\dfrac{x^2-y^2}{x-y}\right)^0$, if $x \neq y$.

48. $(a^{n-1})\cdot(a^{3-n})^3\cdot(a^{2n+5})^{-1}.$

49. $a^{\frac{n+2}{n+1}} \div a^{\frac{1}{n+1}}.$

50. $a^{\frac{n}{n-1}} \div a^{\frac{n}{n+1}}.$

51. $a^{n^2-m^2} \div a^{n-m}.$

52. $\left((a^{n+1})^{\frac{1}{n^2-1}}\right)^{n^2-n}.$

53. $[((x^{-m})^n)^{-p}] \div [((x^{-n})^m)^p].$

54. $\left(\sqrt[n]{x^{n^2}} \div \sqrt[3]{x}\right)^{\frac{3}{3n-1}}.$

55. $\dfrac{a^2\cdot 16^{\frac{1}{2}}a^{-\frac{3}{4}}}{a^{-\frac{7}{4}}+\dfrac{4}{a^{\frac{3}{4}}}}.$

56. $[(x-y)^2]^0$, (if $x \neq y$).

57. $\left(x^{\frac{1}{n^2}-1}\right)^{\frac{1}{n+1}}.$

58. $\left(\dfrac{r^{\frac{2}{3}}\cdot 2s^{\frac{4}{5}}}{2^{\frac{2}{3}}s^{-1\frac{1}{5}}}\right)^{\frac{3}{2}}.$

59. $[(2x^2)^0\cdot(2x)^{\frac{1}{2}}\cdot(2x)^{-\frac{1}{2}}]^{-\frac{1}{5}}.$

60. $[32^{-\frac{3}{5}}\cdot 9^{\frac{3}{2}}]^{-\frac{1}{3}}.$

61. $\dfrac{4^{n+1}}{2^n(4^{n-1})^n} \div \dfrac{8^{n+1}}{(4^{n+1})^{n-1}} + 3.$

62. $\left(\dfrac{8x^3}{125y^{-3}}\right)^{-\frac{1}{3}}\left(\dfrac{m^{-3}n^2}{x^3y^{-2}}\right)^{-2}\left(\dfrac{2^4x^{-6}}{x^2y^8}\right)^{\frac{1}{2}}\left(\dfrac{2y^3}{m^2}\right)^{-3}\left(\dfrac{n^6}{x^3}\right)^{\frac{1}{3}}.$

63. $[(5\,a)^{2n} \cdot (5\,a)^{p-3n}]^{\frac{2}{3}}$.

64. $(a^{\frac{2}{3}} \cdot a^{\frac{5}{11}})^{\frac{33}{5}}$.

65. $\left(r^{-\frac{1}{2}} s^{\frac{2}{3}} \sqrt{r^3 s^{-\frac{4}{3}}}\right)^{\frac{2}{5}}$.

66. $\sqrt[5]{x^{10}} + 3\,(x^{-6})^{-\frac{1}{2}}$.

67. $\dfrac{3\,a^{-b}c^{-3}}{7\,x^{-4}y^{-d}} \cdot \dfrac{21\,x^{-2}y^{5-d}}{6\,a^b c^{d-3}}$.

68. $\dfrac{2^{2n} \cdot 3^{n+1} \cdot 6^{n+\frac{1}{2}}}{(2 \cdot 3)^n \cdot 3^{n-\frac{1}{2}}} \div \dfrac{2^{\frac{3}{2}-n} \cdot 12^{2n+2}}{18^{n+2}}$.

69. $\left(\dfrac{x^{-2}y}{x^{\frac{3}{2}}y^{-\frac{5}{2}}}\right)^{\frac{2}{7}} \div \left(\dfrac{xy^{-\frac{1}{2}}}{\sqrt{x}\,y^{-1}}\right)^{-2}$.

70. $\dfrac{m^7 n}{r^{-r}s^{-1}} \cdot \dfrac{m^s r^{r+2}}{n^{-5}s^{-3}}$.

71. $\dfrac{ab^{\frac{1}{2}}c\sqrt{a}\,b\sqrt[3]{c}}{\sqrt[3]{a^2}b^{-\frac{1}{2}}a^{\frac{1}{3}}b^2c^{\frac{4}{3}}}$.

72. $\dfrac{xy^{\frac{1}{3}}z\sqrt{xy}\,\sqrt[5]{z}}{\sqrt[2]{x^3} \cdot y^{-1}zy^3\sqrt[5]{z^6}\,x^{-2}}$.

Find the square roots of:

73. $x^6 + 4\,x^3y^{\frac{2}{3}} + 9 - 4\,x^{\frac{9}{2}}y^{\frac{1}{3}} + 6\,x^3 - 12\,x^{\frac{3}{2}}y^{\frac{1}{3}}$.

74. $\dfrac{x^2}{y^2} + \dfrac{4\,y^2}{x^2} + 4 + \dfrac{3\,x^2}{y^2} + \dfrac{12\,y^2}{x^2} - 20$.

Find the indicated roots of :

75. $[(e^x - e^{-x})^2 + 4]^{\frac{1}{2}}$.

76. $[(e^x + 2^{-1}e^{-x})^2 - 2]^{\frac{1}{2}}$.

77. $(e^{2x} + e^{-2x} + 2 + 4\,e^x - 4\,e^{-x})^{\frac{1}{2}}$.

78. $(x^{-6} + 17\,x^{-2} + 16\,x^2 - 6\,x^{-4} - 24)^{\frac{1}{2}}$.

79. $\left(\dfrac{1}{x} - \dfrac{4}{\sqrt[4]{x^3}} + 6\,x^{-\frac{1}{2}} - 4\,x^{-\frac{1}{4}} + 1\right)^{\frac{1}{4}}$.

80. $\left(\dfrac{4\,a^{-8}b^8}{9} + \dfrac{9\,a^{-4}b^4}{16} + \dfrac{16}{25} + a^{-6}b^6 - \dfrac{16\,a^{-4}b^4}{15} - \dfrac{6\,a^{-2}b^2}{5}\right)^{\frac{1}{2}}$.

Simplify:

81. $\dfrac{\dfrac{x^3 \cdot a \cdot x^0 - a\,3\,x^2}{(x^3)^2}}{\dfrac{a}{x^3}}$.

82. $\dfrac{\dfrac{x^4(3\,x^2) - (x^3+5)\,4\,x^3}{(x^4)^2}}{\dfrac{x^3+20}{x^4}}$.

83. $\dfrac{\dfrac{(x^2-1)\,ax^{a-1} - x^a \cdot 2\,x}{(x^2-1)^2}}{\dfrac{x^a}{x^2-1}}$.

84. $\dfrac{\dfrac{x^2(ax^{a-1}) - (x^a+1)\,2\,x}{(x^2)^2}}{\dfrac{x^a+1}{x^2}}$.

85. $$\frac{\dfrac{x^{2a}nx^{n-1} - x^n \cdot 2\,ax^{2a-1}}{(x^{2a})^2}}{\dfrac{x^n}{x^{2a}}}.$$

86. $$\frac{\dfrac{x^{-5}(-2\,x^{-1}) - (x^2+3)(-5\,x^{-4})}{(x^{-5})^2}}{\dfrac{x^{-2}+3}{x^{-5}}}.$$

87. $$\frac{\dfrac{e^{-nx}(ne^{nx}) - (e^{nx}+1)(-ne^{-nx})}{(e^{-nx})^2}}{\dfrac{2\,e^{nx}+1}{e^{-nx}}}.$$

88. $$\frac{\dfrac{(\sqrt[3]{x}+1)\frac{1}{3}x^{-\frac{2}{3}} - \sqrt[3]{x}(\frac{1}{3}x^{-\frac{2}{3}})}{(\sqrt[3]{x}+1)^2}}{\dfrac{\sqrt[3]{x}}{\sqrt[3]{x}+1}}.$$

89. $$\frac{\dfrac{\sqrt{ax-x^2}\cdot b - bx(ax-x^2)^{-\frac{1}{2}}\cdot(a-2\,x)}{(\sqrt{ax-x^2})^2}}{bx \div \sqrt{ax-x^2}}.$$

90. $$\frac{\dfrac{(\sqrt{x^{2n}-1})\,2\,nx^{2n-1} - x^{2n}(x^{2n}-1)^{-\frac{1}{2}}(nx^{n-1})}{(\sqrt{x^{2n}-1})^2}}{x^{2n} \div \sqrt{x^{2n}-1}}.$$

91. $$\frac{\dfrac{\sqrt{x^2-10\,x}\cdot 5 - 5\,x\cdot\frac{1}{2}(x^2-10\,x)^{-\frac{1}{2}}(2\,x-10)}{(\sqrt{x^2-10\,x})^2}}{5\,x \div (x^2-10\,x)^{\frac{1}{2}}}.$$

92. Show that $\dfrac{\sqrt{m-n}}{\sqrt{m}+\sqrt{n}} = \dfrac{\sqrt{m}}{\sqrt{m-n}} - \dfrac{\sqrt{n}}{\sqrt{m-n}}.$

93. Show that $\dfrac{\sqrt[a+1]{a+2\sqrt[a+2]{x}}}{\sqrt[a]{a+1\sqrt[a+1]{x}}} = \dfrac{\sqrt[a+1]{x^2}}{\sqrt[a]{x}\cdot\sqrt[a+2]{x}}.$

PROBLEMS

(Obtain answers in simplest radical form.)

1. The side of an equilateral triangle is 12; find the altitude.

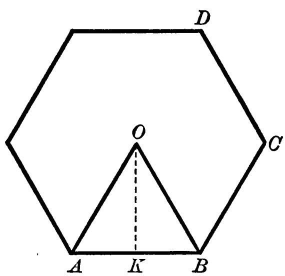
2. The side of an equilateral triangle is s; find the altitude and the area.

3. The altitude of an equilateral triangle is 20; find one side and the area.

4. Find the side of an equilateral triangle whose altitude is a.

5. Find the altitude on the shortest side of the triangle whose sides are 9, 10, and 17. Find the area of the triangle.

6. Find the altitude on the longest side of the triangle whose sides are 10, 12, and 16.



In the adjacent regular hexagon $AB = BC = CD$, etc. O is the center and OK is the apothem of the hexagon.

7. Find the apothem and the area of a regular hexagon (a) whose side is 15; (b) whose side is s.

Fact from Geometry. A regular hexagon may be divided into six equal equilateral triangles by lines from its center to the vertices.

8. Find the side and the area of a regular hexagon (a) whose apothem is 25; (b) whose altitude is h.

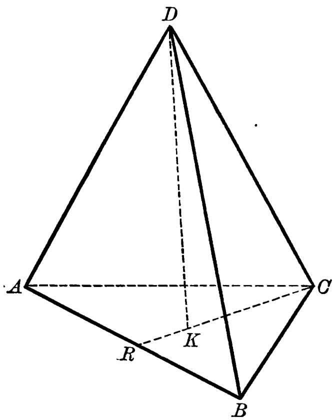
9. The base of a pyramid is a square, each side of which is 10 feet. The other four edges are each 20 feet. Find the altitude and the volume of the pyramid.

Fact from Geometry. The volume of a pyramid or cone is $\frac{a \cdot b}{3}$, where a is the altitude and b is the area of the base.

10. The side of an equilateral triangle is 18. Find the two parts into which each altitude is divided by the other altitudes.

Fact from Geometry. The altitudes of an equilateral triangle intersect at a point which divides each altitude into two parts whose ratio is 2 to 1.

The altitude of a regular tetrahedron (DK in the adjacent figure) meets the base at the point where the altitudes of the base intersect.



11. $ABCD$ is a regular tetrahedron. If each edge is 12, find CR, CK, and lastly the altitude DK.

Fact from Geometry. A regular tetrahedron is a pyramid whose four sides are equal equilateral triangles.

12. Find the altitude and volume of a regular tetrahedron whose edge is 15.

13. Show that the altitude and the volume of a regular tetrahedron whose edge is e are respectively $\frac{e}{3}\sqrt{6}$ and $\frac{e^3}{12}\sqrt{2}$.

CHAPTER VII

GRAPHICAL SOLUTION OF EQUATIONS IN ONE UNKNOWN

63. Functions. An algebraic expression involving one or more letters is a **function** of the letter or letters involved.

The letters of a function are usually referred to as **variables.**

A function is called **linear, quadratic,** or **cubic** according as its degree with respect to the variable or variables is first, second, or third respectively.

Examples of the functions just named are respectively $4x - 7$, $2x^2 - 5x + 18$, $z^3 + 8z^2 - 2z - 6$.

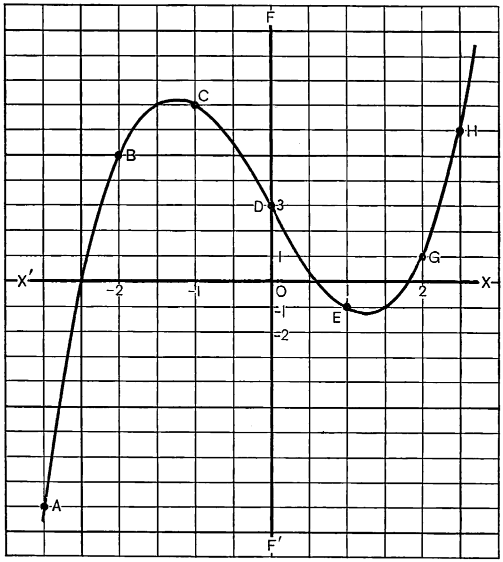
After a function of any variable, say x, has once been given, it is convenient and usual to refer to it later in the same discussion by the symbol $f(x)$, which is read *the function of* x, or more briefly f *of* x.

64. Graph of a function. A graph always shows a relation between (at least) two variables. The graph of a function of one variable is a curve showing the value of the function for any real value of the variable. This means that one axis must be the x-axis and the other the function axis, or F-axis. The method of constructing the graph of a function of x is the same for a linear, a quadratic (see "First Course in Algebra," pages 259–266), and a cubic function in one variable.

65. Graph of a cubic function. To graph the function $x^3 - 5x + 3$, first prepare a table of values as follows:

When $x =$	-4	-3	-2	-1	0	1	2	$2\frac{1}{2}$	3
$f(x)$, $x^3 - 5x + 3 =$	-41	-9	5	7	3	-1	1	$6\frac{1}{8}$	15

Plotting the points corresponding to the numbers in the table (except the first and last), we obtain A $(-3, -9)$, B, C, D, E, G, and H, in the order named. The curve crosses the x-axis three times:



once between 1 and 2; again between 0 and 1; and a third time between -2 and -3. At the points of crossing $f(x)$ is zero. Therefore the values of x at these points are the roots of $x^3 - 5x + 3 = 0$. These are approximately 1.8, .6, and -2.5.

EXERCISES

(Exercises 12–16 refer to the preceding graph.)

1. Construct the graph of $f(x) = 3x - 9$.

2. Does the x-coördinate of the point where the line crosses the x-axis satisfy the equation $3x - 9 = 0$? Why?

3. What is the graph of any linear function of x?

4. Construct the graph of $f(x) = 2x^2 - x - 6$.

5. Do the x-coördinates of the points where the curve crosses the x-axis satisfy the equation $2x^2 - x - 6 = 0$? Why?

6. What kind of a line do you expect the graph of any quadratic function in one variable to be?

7. State a rule for the graphical solution of a linear or a quadratic equation in one variable.

8. What is the effect on the graph of a quadratic function of x, if a positive number is added to the constant term?

9. What change occurs in the roots of a quadratic equation in x, if a positive number is added to its constant term?

10. When does the graphical solution of a quadratic equation give but one real root?

11. When does the graphical solution of a quadratic equation fail to give the roots of the equation?

12. If the function $x^3 - 5x + 3$ be set equal to 4, can the roots of the equation thus formed be read from the graph? If so, read them.

13. Set $x^3 - 5x + 3$ equal to -1.3 (approximately) and read the roots of the resulting equation from the graph. Explain.

14. Set the function $x^3 - 5x + 3$ equal to -4 and read the roots of the resulting equation from the graph. Explain.

15. Set $x^3 - 5x + 3$ equal to 8 and read the roots of the resulting equation from the graph. Explain.

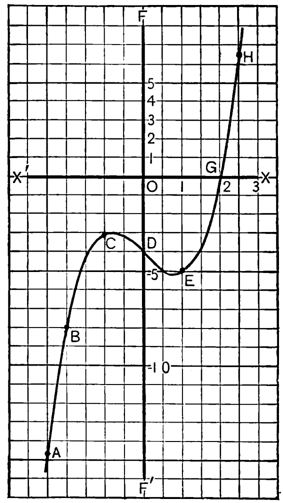
16. Set $f(x)$ equal to 9 and read the roots of the resulting equation from the graph. Explain.

17. (*a*) Is a rational function always integral? (*b*) Is an integral function always rational? (*c*) Write an example of each.

66. Imaginary roots. To make clearer the point in Exercises 14–16 preceding, we shall graph the function $x^3 - 2x - 4$.

When $x =$	-3	$-2\frac{1}{2}$	-2	-1	0	1	2	$2\frac{1}{2}$	3
$f(x)$, $x^3 - 2x - 4 =$	-25	$-14\frac{5}{8}$	-8	-3	-4	-5	0	$6\frac{5}{8}$	17

The point A corresponds to $-2\frac{1}{2}$, $-14\frac{5}{8}$. The points corresponding to the next six pairs of numbers given are B, C, D, E, G, and H in the adjacent figure. The curve through these points crosses the x-axis but *once*. This shows that the equation has but one *real* root, and that the value of this root is 2. Since the number of roots of a rational integral equation is the same as the number which indicates its degree, we conclude that the other two roots are imaginary.



Note. It required the genius of Sir Isaac Newton first to observe from the graph of a function that two of its roots become imaginary simultaneously. He also saw that an equation with two of its roots equal to each other is, in a certain sense, the limiting case between equations in which the corresponding roots appear as two real and distinct roots, and those in which they appear as imaginary roots.

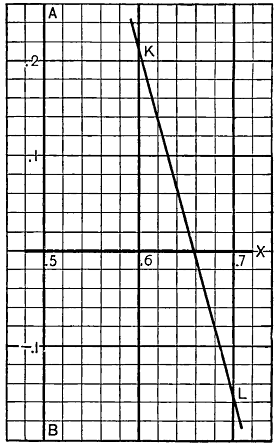
67. Graphical solution of an equation in one unknown. If the student has grasped the meaning of the preceding graphical work, he will see the correctness of the following rule for solving graphically *any* equation in one unknown.

RULE. *An equation in one unknown whose second member is zero is solved for* **real roots** *by graphing the function in the first member and then obtaining the value of x for the points where the curve crosses the x-axis.*

68. More accurate graphical solutions. By drawing the entire graph to a larger scale the student can obtain more accurately the values of the roots. If still more exact results are desired, he may proceed somewhat as follows:

The graph on page 111 shows that there is a root greater than .6 and less than .7. If we now construct on a large scale that portion of the curve between D and E (page 111) which is just above and just below the x-axis, we shall get a more precise value for the root. Substituting .6 and .7 in $x^3 - 5x + 3$, we obtain the following table:

When $x =$	.6	.7
$f(x)$, $x^3 - 5x + 3 =$	.216	$-.157$



Between $x = .6$ and $x = .7$ the function changes from + to −. Hence we are certain that the graph crosses the x-axis between these points. We now choose a much larger scale than the one used on page 111. This is indicated by the numbers on the x-axis. The scale is too large to show the y-axis in the figure, so the scale for y is indicated on the line AB. The point K corresponds to .6, .216, and the point L to .7, −.157. Since K and L are comparatively close together, the portion of the graph between them is nearly a straight line. Drawing the straight line KL, it is seen to cross the x-axis between .64 and .66, or about .658. By an algebraic method of solution it can be shown that the root, correct to three decimals, is .656. Here the graphical method gives the result to within $\frac{3}{10}$ of one per cent of the true value.

EXERCISES

Solve graphically:

(Obtain roots in Exercises 2 and 6 correct to two decimals.)

1. $x^2 + 14 = 8x$.
2. $x^3 - 3x + 4 = 0$.
3. $x^3 + x = 4$.
4. $x^3 - 2x^2 - 5x + 6 = 0$.
5. $x^3 - 8x = 0$.
6. $x^3 - 4x + 2 = 0$.
7. $x^4 - 10x^2 + 16 = 0$.
8. $x^4 - 4x^3 + 12 = 0$.

By reference to the curve obtained in Exercise 6 solve:

9. (a) $x^3 - 4x = -5$. (b) $x^3 - 10 = 4x$. (c) $x^3 - 4x - 2 = 0$.

69. Critical values of the variable. There are many practical problems involving two variables in which it is necessary to determine that value of one variable for which the other has the greatest (or least) possible value. A great number of these problems can be solved by means of a graph. The method of solution can be made clear by reference to the graph of § 65. There $f(x) = x^3 - 5x + 3$. Suppose we wish to know the value of x which gives $x^3 - 5x + 3$ the greatest possible value. Near C occurs a high point, — a turning point of the curve, — and there $x = -\frac{5}{4}$ approximately, and $f(x) = 7.3$. This value of x gives to $f(x)$ a greater value than does any other value of x between -2.5 and $+2.6$. It is true that on the portion GH above H greater values of $f(x)$ than 7.3 occur. But in practical problems similar to those of the next list it will be found that some condition of the problem will rule out of consideration any value of x which does not correspond to the turning point of the curve such as that which occurs near C or near E.

PROBLEMS

1. A manufacturer has in stock a quantity of strawboard 8 inches by 15 inches, out of which he desires to make open-top boxes by cutting equal squares out of each corner and folding up so as to make sides and ends. What must the side of the square be so as to make a box of the greatest possible volume?

HINT. Let x equal the side of the square. Then the dimensions of the box in inches are $15 - 2x$, $8 - 2x$, and x. Hence the volume $V = 4x^3 - 46x^2 + 120x$ in cubic inches. Construct the graph of $V = 4x^3 - 46x^2 + 120x$ (or that of $V/4 = x^3 - 11\frac{1}{2}x^2 + 30x$, which deals with smaller numbers); then an inspection of the turning points will give the required value of x.

2. Referring to the graph of Exercise 1: (*a*) What value has the function $4x^3 - 46x^2 + 120x$ when $x = 1\frac{2}{3}$? (*b*) What other value of x gives the function the same value? (*c*) What values of x give the function greater values than this? (*d*) What condition of the problem rules out these values as sides of the square?

Biographical Note. The notion of a function is one of the most fundamental ideas in modern mathematics. Only the simplest examples are given in this book, but many others involving expressions of the utmost complexity have been studied by mathematicians for many years. An important reason for the study of functions is found in the fact that all kinds of facts and principles which we meet in the study of nature can be expressed symbolically by means of functions, and the discovery of the properties of such functions helps us to understand the meaning of the facts. A complete understanding of the laws of falling bodies, light, electricity, or sound could never be reached without the study of the mathematical functions which these phenomena suggest.

One of the foremost living scholars who has discovered many properties of the most complicated functions is Professor Felix Klein of Göttingen, Germany. Since the time of Gauss, who was also a professor at Göttingen, the university there has been one of the leading institutions of the world in the study of mathematics. It is interesting to know that Klein's great achievements in advanced mathematics have not caused him to forget the difficulties which surround the beginner in the first years of his study, but that he has had wide influence in improving mathematical instruction in the schools not only of Germany but of other countries as well.

3. A piece of tin is 8 inches by 12 inches. From each corner a square whose side is x inches is cut out. The sides are then turned up and an open box is formed, which has the greatest possible volume. Find graphically this value of x.

4. What value of x gives $x^2 - 4x + 6$ the least possible value?

5. An open metal tank having a volume of 4 cubic yards has vertical sides and a square base. Determine the side of the base and the altitude of the tank if the inside surface is the least possible.

HINT. Let x equal the side of the base in yards and d the altitude in yards. Then the volume of the tank, 4 cubic yards, equals dx^2, and the surface equals $x^2 + 4dx$ in square yards. From these two statements we obtain surface $S = x^2 + \frac{16}{x}$. Plot the function $x^2 + \frac{16}{x}$ and the required value of x will be apparent.

6. An open metal tank having a volume of 4 cubic yards is in the form of a cylinder with a circular base. Determine the radius of the base and the altitude so that the inside surface will be the least possible.

7. The perimeter of a rectangle is 20 rods. Find the length and the width if the area is the greatest possible.

8. A boatman 6 miles from the nearest point of the beach (which is straight) wishes to reach in the shortest possible time a place 8 miles from that point along the shore. He can row 4 miles per hour and jog-trot 6 miles per hour. Determine where he must land.

HINT. Draw a right triangle ABC, AC being the shore line, B the boat, and A the point on shore nearest B. Let K on AC be the point at which he lands, and let KA in miles be x. Then $BK = \sqrt{x^2+36}$ and $CK = 8 - x$. In hours the time required to go from B to K is $\frac{\sqrt{x^2+36}}{4}$, and that from K to C is $\frac{8-x}{6}\cdot$ Therefore the total time equals $\frac{8-x}{6} + \frac{\sqrt{x^2+36}}{4}\cdot$ Plot this function and the required value of x will be apparent.

CHAPTER VIII

QUADRATIC EQUATIONS

70. Solution by completing the square. Any quadratic equation in one unknown of the general type $ax^2 + bx + c = 0$ can be solved as follows:

Example: Solve $3x^2 - 7x - 20 = 0$. (1)

Solution: Transposing, $3x^2 - 7x = 20$. (2)

Dividing (2) by the coefficient of x^2, $x^2 - \frac{7x}{3} = \frac{20}{3}$. (3)

Adding $(-\frac{7}{6})^2$ to each member of (3),

$$x^2 - \tfrac{7}{3}x + (-\tfrac{7}{6})^2 = \tfrac{20}{3} + \tfrac{49}{36} = \tfrac{289}{36}. \quad (4)$$

Then $(x - \frac{7}{6})^2 = (\frac{17}{6})^2$. (5)

Extracting the square root of each member of (5),

$$x - \tfrac{7}{6} = \pm \tfrac{17}{6}.$$

Whence $x = \frac{7}{6} \pm \frac{17}{6} = 4$ or $-\frac{5}{3}$.

Check: Substituting 4 for x in (1),

$$3 \cdot 4^2 - 7 \cdot 4 - 20 = 0.$$

$$48 - 28 - 20 = 0, \text{ or } 0 = 0.$$

Substituting $-\frac{5}{3}$ for x in (1),

$$3(-\tfrac{5}{3})^2 - 7(-\tfrac{5}{3}) - 20 = 0.$$

$$\tfrac{25}{3} + \tfrac{35}{3} - 20 = 0.$$

$$\tfrac{60}{3} - 20 = 0, \text{ or } 0 = 0.$$

A method of solving a quadratic equation of the general type in x by completing the square is stated in the

RULE. *Transpose so that the terms containing **x** are in the first member and those which do not contain **x** are in the second.*

Divide both members of the equation by the coefficient of x^2 (unless the coefficient of x^2 is $+1$).

*Then add to both members the square of one half the coefficient of **x** (in the equation just obtained), thus making the first member a perfect trinomial square.*

Rewrite the equation, expressing the first member as the square of a binomial and the second member in its simplest form.

Extract the square root of both members of the equation and write the sign ± before the square root of the second member, thus obtaining two linear equations.

Solve for **x** *the equation in which the second member is taken with the sign +, and then solve the equation in which the second member is taken with the sign −. The two results are the roots of the quadratic.*

CHECK. Substitute each result separately in place of x in the *original* equation. If the resulting equations are not obvious identities, simplify until each becomes one.

Quadratic equations often arise in which the first power of the unknown is missing. They are of the type $ax^2 = c$. Here $x = \pm\sqrt{\frac{c}{a}}$. It is evident that the solution of such equations does not require the completion of the square. The student has solved many equations of this type in Chapter VI.

EXERCISES

Solve by completing the square and check.

(Find the values of the unknown in Exercises 10–12 correct to four decimals.)

1. $x^2 - 4x - 32 = 0.$
2. $2x^2 + 5x + 3 = 0.$
3. $3s^2 + 8s + 4 = 0.$
4. $3 + 5x^2 = 8x.$
5. $6t^2 = t + 2.$
6. $\dfrac{x^2 + 3}{4} = \dfrac{5x^2 - 24}{7}.$
7. $12 + 7x - 10x^2 = 0.$
8. $39y - 14y^2 - 10 = 0.$
9. $\dfrac{3x^2}{5} - 20 = x^2 - 30.$
10. $x^2 - 5x + 2 = 0.$
11. $\dfrac{3n^2}{4} - 2n + \dfrac{1}{2} = \dfrac{9}{4}.$
12. $7x^2 - 12x - 3 = 0.$
13. $\dfrac{3x^2}{2} - 3x\sqrt{2} = 9.$
14. $(2x - 5)^2 - (x - 6)^2 = 80$

15. $\frac{1}{x}+\frac{1}{x+2}=\frac{3}{4}.$

16. $\frac{1}{w+7}-\frac{1}{w-3}=\frac{2}{5}.$

17. $\frac{x+2}{2x-1}+\frac{3}{x-5}=2\frac{5}{26}.$

18. $\frac{m}{m+1}-\frac{2m}{m+2}+\frac{9}{2}=0.$

19. $\frac{8-3x}{x}-\frac{2x}{2-x}=\frac{x+5}{3-x}.$

20. $\frac{10s^2-1-s^3}{6s-s^2-9}=\frac{s^2-9}{s+3}.$

After a student has mastered the solution of quadratics by factoring and completing the square, he should learn the formula method (§ 71) and should use thereafter the one of the three methods which is best adapted to the problem in hand.

21. $\frac{4}{x-4}-\frac{x+3}{x^2-5x+4}=2.$

22. $\frac{2(x^3+8)}{x+2}-\frac{x^3-1}{x-1}=\frac{19}{4}.$

23. $\frac{r+2\sqrt{5}}{\sqrt{15}}-\frac{2}{r}=\frac{1}{\sqrt{5}}.$

24. $x^3+7x^{\frac{3}{2}}-8=0.$

The equation $x^3+7x^{\frac{3}{2}}-8=0$ is not a quadratic equation, but it is of the general type $ax^{2n}+bx^n+c=0$. Here x occurs in but two terms and its exponent in one term is twice the exponent in the other. All equations of this form can be solved by completing the square.

Solution: $x^3+7x^{\frac{3}{2}}+\frac{49}{4}=8+\frac{49}{4}=\frac{81}{4}.$

$$x^{\frac{3}{2}}+\frac{7}{2}=\pm\frac{9}{2}.$$

$$x^{\frac{3}{2}}=1 \text{ or } -8.$$

Whence $x=1$ or $4.$

Check: Substituting 1 for x in $x^3+7x^{\frac{3}{2}}-8=0,$

$$1+7-8=0, \text{ or } 0=0.$$

Substituting 4 for x, $64+56-8=0.$

But $112\neq 0.$

Hence the equation has only one root, 1.

25. $x^3-10x^{\frac{3}{2}}-11=0.$

26. $x^4-26x^2+25=0.$

27. $x^6-7x^3-8=0.$

28. $x+x^{\frac{1}{2}}-6=0.$

29. $4x^6-7x^3=15.$

30. $4x^2+\frac{4}{x^2}-\frac{97}{9}=0.$

31. $2x-3x^{\frac{1}{2}}=2.$

32. $3x^4-11x^2+6=0.$

33. $9x^4-22x^2+8=0.$

34. $3x^{\frac{3}{2}} + 5x^{\frac{3}{4}} + 2 = 0.$

35. $2x^{\frac{2}{3}} - 9\sqrt[3]{x} + 4 = 0.$

36. $3x - 11x^{\frac{1}{2}} - 20 = 0.$

37. $6x^4 - 13x^2 + 6 = 0.$

38. $x^{-2} + 16x^{-1} - 17 = 0.$

39. $y^{-4} - 10y^{-2} + 9 = 0.$

40. $x^{-1} - 13x^{-\frac{1}{2}} = -36.$

41. $x^{2m} + 4 - 5x^m = 0.$

42. $(x^2 - 2x)^2 - 7(x^2 - 2x) = -12.$

Solution: Let $x^2 - 2x = y.$
Substituting y for $x^2 - 2x$, we obtain
$$y^2 - 7y = -12.$$
Solving, $y = 3$ or $4.$
Then $x^2 - 2x = 3.$
Whence $x = 3$ or $-2.$
Also $x^2 - 2x = 4.$
Whence $x = 1 \pm \sqrt{5}.$

In Exercises 43–48 do not expand or transpose and square. Solve as in Exercise 42.

43. $3(x^2 + 3x)^2 - 7(x^2 + 3x) - 20 = 0.$

44. $\left(x - \frac{1}{x}\right)^2 + 4\left(x - \frac{1}{x}\right) = 8\frac{1}{4}.$

45. $(4y + 5) + 2(4y + 5)^{\frac{1}{2}} = 15.$

46. $x^2 + 5x + 3\sqrt{x^2 + 5x} - 54 = 0.$

47. $x^2 - 2x - 5\sqrt{x^2 - 2x - 4} + 2 = 0.$

48. $2y(2y + 1) + 3\sqrt{8y^2 + 12y + 5} = 25 - 4y.$

49. $2x^2 + ax - 6a^2 = 0.$

50. $9x^2 - 2cx = 7c^2.$

51. $2x^3 - 17bx^2 + 8b^2x = 0.$

52. $7a^2x^2 - 4ax - 11 = 0.$

53. $ax^2 + bx + c = 0.$

54. $12kx - 4k^2 - 5x^2 = 0.$

55. $\frac{2m}{3} + \frac{8m^2}{9x} - x = 0.$

56. $c^2(x - d)^2 - d^2(c - x)^2 = 0.$

57. $mx^2 - x(m^2 + 1) = -m.$

58. $(x - r)^2 + (S - x)^2 = r^2 + S^2.$

59. $\frac{m}{m - x} - \frac{m - x}{m} = 2.$

60. $\frac{x - c}{c} - \frac{c}{x - c} = \frac{3}{2}.$

61. $\frac{x^2 - 3\,mx}{m - n} + 2\,m = \frac{nx}{n - m}.$ **62.** $\frac{x^2 + 2\,ax}{a - b} - \frac{x}{2\,b} = \frac{a}{b}.$

63. $\frac{2\,s + x}{x + s} - \frac{5\,s - x}{x - s} - \frac{s\,(x + s)}{s^2 - x^2} = 0.$

64. $2\,x^2 + 5\,x = cx^2 + 3\,cx + 3.$

HINT. $(2 - c)\,x^2 + (5 - 3\,c)\,x = 3.$

$$x^2 + \frac{5 - 3\,c}{2 - c}x = \frac{3}{2 - c}.$$

$$x^2 + \frac{5 - 3\,c}{2 - c}x + \left(\frac{5 - 3\,c}{4 - 2\,c}\right)^2 = \frac{3}{2 - c} + \frac{25 - 30\,c + 9\,c^2}{16 - 16\,c + 4\,c^2}.$$

$$\left(x + \frac{5 - 3\,c}{4 - 2\,c}\right)^2 = \frac{49 - 42\,c + 9\,c^2}{16 - 16\,c + 4\,c^2} = \left(\frac{7 - 3\,c}{4 - 2\,c}\right)^2.$$

$$x + \frac{5 - 3\,c}{4 - 2\,c} = \pm\frac{7 - 3\,c}{4 - 2\,c}, \text{ etc.}$$

65. $x^2 + 2\,s + s^2 = 2\,sx + 2\,x.$

66. $x^2 - 2\,x + 1 = ax - ax^2.$

67. $x^2 + 2\,x + 1 = hx^2 + hx.$

68. $cx^2 + 3\,x = 2\,x^2 + 2\,cx - 2.$

69. $\frac{a}{x + a} + \frac{b}{x + b} - \frac{2\,c}{x + c} = 0.$

70. $\frac{a + x}{b + x} + \frac{b + x}{a + x} = \frac{5}{2}.$

71. $\frac{ax + b}{bx + a} = \frac{mx - n}{nx - m}.$

72. $\frac{ax^2 + bx + c}{bx^2 - mx + n} = \frac{c}{n}.$

71. Solution by formula. The standard form of the general quadratic is

$$ax^2 + bx + c = 0.$$

The student solved this equation (see Exercise 53, page 121) and found

$$x = \frac{-\,b \pm \sqrt{b^2 - 4\,ac}}{2\,a}. \qquad (F)$$

The value (F) is a general result and may be used as a formula to solve any quadratic equation. The solution of a quadratic by formula requires less labor than by any other method, except for such equations as can be solved by factoring at sight. Those who have considerable experience in algebra seldom solve a quadratic by any other method than by formula.

EXAMPLES

Solve by formula and check:

1. $3x^2 - 5x = 8.$

Solution: Writing in standard form,

$$3x^2 - 5x - 8 = 0.$$

Then 3 corresponds to a, -5 to b, and -8 to c in the general quadratic $ax^2 + bx + c = 0$. Substituting these values in (F), where

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

gives

$$x = \frac{-(-5) \pm \sqrt{25 - 4 \cdot 3(-8)}}{2 \cdot 3}$$

$$= \frac{5 \pm \sqrt{25 + 96}}{6} = \frac{5 \pm 11}{6} = \frac{8}{3} \text{ or } -1.$$

Check as usual.

2. $2k^2x^2 = kx + 1.$

Solution: Writing in standard form,

$$2k^2x^2 - kx - 1 = 0.$$

Then $a = 2k^2$, $b = -k$, and $c = -1$.
Substituting these values in the formula (F),

$$x = \frac{-(-k) \pm \sqrt{(-k)^2 - 4 \cdot 2k^2(-1)}}{2 \cdot 2k^2}$$

$$= \frac{k \pm \sqrt{k^2 + 8k^2}}{4k^2} = \frac{k \pm 3k}{4k^2} = \frac{1}{k} \text{ or } -\frac{1}{2k}.$$

Check as usual.

EXERCISES

Solve for x by formula and check:

1. $2x^2 - 7x + 3 = 0.$

2. $3x^2 - x - 2 = 0.$

3. $11hx + 20h^2 = 3x^2.$

4. $5x^2 + 2cx = 16c^2.$

5. $22x^2 = 3mx + 7m^2.$

6. $6x^2 - 7rx + 2r^2 = 0.$

7. $3x^2 - 6ax + 2a^2 = 0.$

8. $3m^2 + 4mx - 7x^2 = 0.$

9. $4ax - 10a^2x^2 + 3 = 0.$

10. $12v^4 + 7v^2x - 10x^2 = 0.$

11. $x^2 + 3x = mx + 3m.$

HINT. $x^2 + (3 - m)x - 3m = 0$. Then $a = 1$, $b = 3 - m$, and $c = 3m$. Substituting these values in (F),

$$x = \frac{-(3-m) \pm \sqrt{(3-m)^2 - 4 \cdot 1(-3m)}}{2}, \text{ etc.}$$

12. $x^2 + nx = cx + cn.$

13. $3x^2 - 6cx + 2c = x.$

14. $mx^2 + kmx = kc + cx.$

15. $a^2x^2 - 2ax = b^2x^2 - 1.$

16. $m^2x^2 + mx^2 + 2x = 4.$

17. $n^3x^2 + 2nx = 5n^2x + 10.$

18. $hkx^2 - hk = h^2x + h^2x^2.$

19. $2x^2 + 5x = hx^2 + 3hx + 3.$

20. $h^2x^2 + hx + 4x = 6 + 2hx^2.$

21. $cx^2 + cmx + 5 = cx + 5(x + m).$

22. $n^2x + 3nx + 2x = nx^2 + 2n + 3n^2.$

23. $m^2x^2 + 4mx + hmx + 3hx = 9x^2 + 12x - 4h.$

PROBLEMS

1. Separate 20 into two parts, such that the first shall be the square of the second.

2. One leg of a right triangle is 8 feet and the hypotenuse is 2 feet longer than the other leg. Find the other leg, the hypotenuse, and the area.

3. The hypotenuse of a right triangle is 18 feet longer than one leg and 16 feet longer than the other. Find the three sides.

4. The number of hours required to make a trip of 112 miles was 6 more than the rate in miles per hour. Find the rate and the time.

5. The sum of the reciprocals of two consecutive numbers is $\frac{5}{6}$. Find the numbers.

6. The altitude of a triangle is 4 feet less than the base. The area of the triangle is 48 square feet. Find the base and the altitude.

7. One leg of a right triangle is 7 feet shorter than the other and the area is 30 square feet. Find the three sides of the triangle.

8. The area of a triangle is 40 square yards and the base is 2 feet more than seven times the altitude. Find the base and the altitude.

9. The area of a trapezoid is 60 square feet. One base is 2 feet more than the altitude and the other base is twice the altitude. Find the bases and the altitude.

10. One base of a trapezoid exceeds the other by 16 feet, the altitude is 2 feet more than one third of the shorter base, and the area is $116\frac{1}{9}$ square yards. Find the bases and the altitude.

11. A requires 4 more days than B to do a piece of work. If in working together they require $8\frac{8}{9}$ days, find the number of days each requires alone.

12. One diagonal of a rhombus exceeds the other by 4 inches. Find each if the area of the rhombus is 198 square inches.

13. The radius of a circle is 21 inches. How much must it be shortened so as to decrease the area of the circle 770 square inches? (Use $\pi = \frac{22}{7}$.)

14. In selling an article at 18 dollars a merchant gained a per cent 5 greater than the number of dollars the article cost. Find the cost in dollars and the gain per cent.

15. From a cask full of wine 5 gallons are drawn off. The cask is then filled by adding water, and again 5 gallons are drawn off. If, after refilling with water, 36 per cent of the mixture is water, how many gallons does the cask contain?

16. A printed page has 15 more lines than the average number of letters per line. If the number of lines is increased by 15, the number of letters per line must be decreased by 10 in order that the amount of matter on the two pages may be the same. How many letters are there on the page?

17. A sum of \$5000 is put at interest. At the end of each year the yearly interest and \$300 are added to the investment. If at the beginning of the third year the investment amounts to \$6236, find the rate of interest that the investment bears.

18. The cost of an outing was \$36. If there had been 2 more in the party, each would have been required to pay \$3 less. Find the number in the party.

19. Two bodies, A and B, move on the sides of a right triangle. A is now 123 feet from the vertex and is moving away from it at the rate of 239 feet per second. B is 239 feet from the vertex and moves toward it at the rate of 123 feet per second. At what time (past or future) are they 850 feet apart?

20. The dimensions of a rectangular box in inches are expressed by three consecutive numbers. The surface of the box is 292 square inches. Find the dimensions.

21. A three-inch square is cut from each corner of a square piece of tin. The sides are then turned up and an open box is formed, the volume of which is 300 cubic inches. Find in inches the side of the piece of tin.

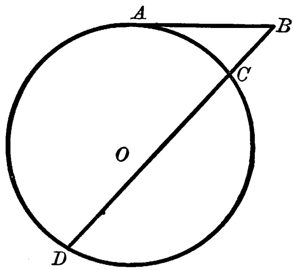
22. A piece of tin is 10 inches by 12 inches. From each corner a square is cut whose side is x inches. The sides are turned up and an open box is formed. Show that its volume is $4x^3 - 44x^2 + 120x$.

23. Now a certain value of x gives for the box in Exercise 22 the greatest possible volume. That value is one root of the equation $12x^2 - 88x + 120 = 0$. Find the value of x.

24. A rectangular box is 8 inches long. Its volume is 192 cubic inches and the area of its six faces is 208 square inches. Find the three dimensions.

25. A messenger leaves the rear of an army 28 miles long as it begins its day's march. He goes to the front and at once returns, reaching the rear as the army camps for the night. How far did he travel if the army went 28 miles during the day?

26. If AB in the adjacent figure is a tangent to the circle and BD is any secant, $\overline{AB}^2 = BC \cdot BD$.* Find BC if $AB = 12$ and $CD = 20$.



27. How high is a mountain which can just be seen from a point on the surface of the sea 80 miles distant? (Use 3960 miles for the radius of the earth.)

28. Find the distance a man can see in a straight line over a smooth lake, if his eye is 6 feet above the level of the water.

29. Two lighthouses on opposite shores of a bay are 150 and 250 feet respectively above the water. If the light from one can just be seen from the other, find the distance in miles between them.

30. A ship is 31 miles from a lighthouse which is 250 feet above the water. How high above the water is the ship's flag if it can just be seen from the lighthouse?

31. A stone dropped from a balloon which was passing over a river struck the water 12 seconds later. How high was the balloon at the time the stone was dropped?

HINT. The distance S through which a body falls from rest in t seconds is given by the equation $S = \frac{gt^2}{2}$ ($g = 32$ feet, approximately).

32. A man drops a stone over a cliff and hears it strike the ground below 13 seconds later. If sound travels 1120 feet per second, find the height of the cliff.

* If BC is small compared to DC, we may use $\overline{AB}^2 = BC \cdot CD$ as a close approximation. Thus if BC is a mountain two miles high and DC is the diameter of the earth, the equation $BC = \overline{AB}^2 \div CD$ would give the height of the mountain within one four-thousandth of the correct value.

CHAPTER IX

IRRATIONAL EQUATIONS

72. Definitions and typical solutions. An **irrational** or **radical equation** in one unknown is an equation in which the unknown letter occurs in a radicand.

Thus $3x + 2\sqrt{x} = 16$, $\sqrt{1-x} + \sqrt{x+3} = 2$, and $\sqrt[3]{x^2 - 8} = 0$ are irrational equations. Also any equation in which the unknown occurs with a fractional exponent is irrational.

The following examples illustrate the method of solution for some of the more simple irrational equations.

EXAMPLES

1. Solve $\sqrt{2x - 5} - 3 = 0$.

Solution: Transposing, $\sqrt{2x-5} = 3$.
Squaring both members, $2x - 5 = 9$.
Solving, $x = 7$.

Check: Substituting 7 for x in the original equation,

$$\sqrt{14 - 5} - 3 = 0.$$

Whence $3 - 3 = 0$.

In irrational equations it is understood that each radical expression, not preceded by the sign $\pm$, is to have *one* sign and *only one*; therefore each radical will have *one* value and *only* one. That value is the *principal root* of the radical. This fact is of importance in checking.

2. Solve $2\sqrt[3]{8x^3 + \dfrac{19x^2}{2}} - 2x - 2 = 2x - 1.$ (1)

Solution: Transposing, $2\sqrt[3]{8x^3 + \dfrac{19x^2}{2}} = 4x + 1.$ (2)

Cubing each member of (2),

$$64x^3 + 76x^2 = 64x^3 + 48x^2 + 12x + 1. \quad (3)$$

Transposing and collecting,

$$28x^2 - 12x - 1 = 0.$$

Factoring, $(2x - 1)(14x + 1) = 0.$

Therefore $x = \frac{1}{2}$ or $-\frac{1}{14}$.

Check: Substituting $\frac{1}{2}$ for x in (1),

$$2\sqrt[3]{1 + \tfrac{19}{8}} - 1 - 2 = 1 - 1.$$

$$2 \cdot \tfrac{3}{2} - 3 = 0.$$

$$3 - 3 = 0.$$

Substituting $-\frac{1}{14}$ for x in (1),

$$2\sqrt[3]{8(-\tfrac{1}{14})^3 + \tfrac{19}{2}(-\tfrac{1}{14})^2} + \tfrac{1}{7} - 2 = -\tfrac{1}{7} - 1.$$

Simplying, $2\left(+\frac{5}{14}\right) - \frac{13}{7} = -\frac{8}{7}$, or $\frac{-8}{7} = \frac{-8}{7}.$

It is easily possible to write a statement involving radical expressions which has the *form* of an equation, but is not one. Thus $\sqrt{x+1} + \sqrt{x+3} + 1 = 0$ looks like an equation, but no value of x can satisfy it. A little closer inspection shows that the statement asserts that the sum of three positive numbers is zero, a condition clearly impossible. Statements like the one given are often called "impossible equations," though, strictly speaking, they are not equations at all. In the attempt to solve an apparent equation one may resort to the usual methods of solution and obtain a result which will not satisfy the original statement. Not until one tries to verify the result is the falsity of the original statement discovered.

3. Solve $1 + \sqrt{x + 2} = \sqrt{x}.$ (1)

Solution: Transposing, $1 - \sqrt{x} = -\sqrt{x+2}.$ (2)

Squaring (2), $1 - 2\sqrt{x} + x = x + 2.$ (3)

Transposing and collecting,

$$-2\sqrt{x} = 1. \quad (4)$$

Squaring (4), $4x = 1$, or $x = \frac{1}{4}$. (5)

Check: Substituting $\frac{1}{4}$ for x in (1),

$$1 + \sqrt{\tfrac{1}{4} + 2} = +\sqrt{\tfrac{1}{4}}.$$

$1 + \frac{3}{2} = +\frac{1}{2}$, or $\frac{5}{2} = \frac{1}{2}$, which is false.

It is fairly certain that the student did not see that the statement (1) is false until the attempt was made to verify the result. It appears, then, that the method of solution may give a result which is not a root.

4. Solve $\sqrt{x-1}+\sqrt{3x+1}-2=0$. (1)

Solution: Transposing, $\sqrt{3x+1}=2-\sqrt{x-1}$. (2)

Squaring both members of (2),

$$3x+1=4-4\sqrt{x-1}+x-1. \quad (3)$$

Transposing and collecting,

$$2x-2=-4\sqrt{x-1}. \quad (4)$$

Dividing (4) by 2, $x-1=-2\sqrt{x-1}$. (5)

Squaring both members of (5),

$$x^2-2x+1=4x-4. \quad (6)$$

Transposing, $x^2-6x+5=0$. (7)

Factoring, $(x-1)(x-5)=0$.

Therefore $x=1$ or 5.

Check: Substituting 1 for x in (1),

$$\sqrt{1-1}+\sqrt{3+1}-2=0.$$
$$0+2-2=0.$$

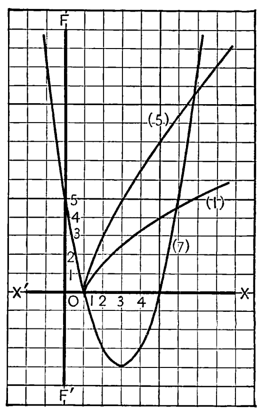
Therefore 1 is a root of (1).

Substituting 5 for x in (1),

$$\sqrt{5-1}+\sqrt{15+1}-2=0.$$
$$2+4-2=0,$$

or $4=0$; but $4\neq 0$.

Therefore 5 is not a root of (1). It was introduced by the process of squaring each member of equation (5). This process does not necessarily introduce a root. Thus 1 is a root of each of the equations (1) to (7), and while 5 is a root of (6) and (7), it is not a root of (5), as may be verified by substitution. Further, (5) was obtained by squaring (2), yet neither the root 1 nor the root 5 was introduced at that point.



Just what did happen in the course of the preceding solution is shown in the adjacent figure, where equations (1), (5), and (7) are solved graphically.

The graph shows the changes in the function due to squaring. It appears that the root 1 is common to (1), (5), and (7), while the root 5 is extraneous to (1) and (5).

As we have seen, the solution of (1) leads to the quadratic $x^2 - 6x + 5 = 0$. Since (1) and (5) have the root 1 but not 5, it is obvious that with some radical equations one may resort to squaring once without introducing an *extraneous* root.

Equation (1) is typical of many radical equations which, when solved by rationalizing, give the roots not only of the original equation, but also of such equations as may be derived from it by giving each radical therein the sign $\pm$.

It will be seen from the next example, also, that the process of rationalization does not necessarily introduce extraneous roots.

5. Solve $\sqrt{x+2} + \sqrt{3-x} = 3.$ (1)

Solution: Transposing, $\sqrt{3-x} = 3 - \sqrt{x+2}.$ (2)

Squaring (2), $3 - x = 9 - 6\sqrt{x+2} + x + 2.$ (3)

Transposing and collecting, $-2x - 8 = -6\sqrt{x+2}.$ (4)

$(4) \div -2,$ $x + 4 = 3\sqrt{x+2}.$ (5)

Squaring (5), $x^2 + 8x + 16 = 9x + 18.$ (6)

Transposing and collecting, $x^2 - x - 2 = 0.$ (7)

Factoring, $(x-2)(x+1) = 0.$

Therefore $x = 2$ or -1.

Check: Substituting 2 for x in (1),

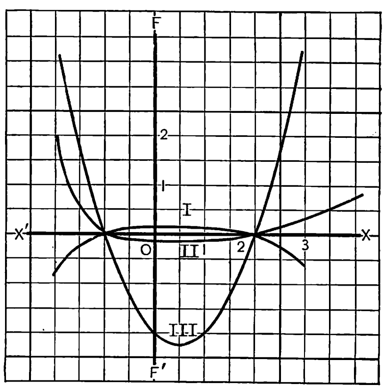
$$\sqrt{2+2} + \sqrt{3-2} = 3, \text{ or } 2 + 1 = 3.$$

Substituting -1 for x in (1),

$$\sqrt{-1+2} + \sqrt{3+1} = 3, \text{ or } 1 + 2 = 3.$$

Therefore equation (1) has two roots, 2 and -1.

The graphical solution of (1), (5), and (7) gives curves I, II, and III respectively of the adjacent figure. These graphs show the change in the function with each resort to squaring. Curves I, II, and III intersect the x-axis at the same points, showing that the roots 2 and -1 are common to (1), (5), and (7).



It should be clear from the preceding examples that we cannot determine the number of roots of a given radical equation without solving it; nor can we predict whether the given statement involving radicals is an equation. *Results obtained are roots if they satisfy the original statement, and not otherwise.*

The method of solving a radical equation may be stated in the

RULE. *Transpose the terms so that one radical expression (the least simple one) is the only term in one member of the equation.*

Next raise both members of the resulting equation to the same power as the index of this radical.

If radical expressions still remain, repeat the two preceding operations until an equation is obtained which is free from radicals. Then solve this equation.

CHECK. Substitute the values found in the *original* equation and reduce the resulting radicals to their simplest form. Whenever the radicals are rational simplify by extracting the roots indicated. Never simplify by raising both members of the equation to any power, for extraneous roots introduced by that process would not then be detected.

Finally, reject all extraneous roots.

EXERCISES

Solve, check results, and reject all extraneous roots:

1. $\sqrt{x+3}=8.$
2. $\sqrt{2x-6}+4=7.$
3. $3\sqrt{2x-8}-7=17.$
4. $\sqrt[3]{3x-4}=2.$
5. $(7x+15)^{\frac{1}{3}}+18=17.$
6. $2\sqrt[3]{3n-25}+3=7.$
7. $3x\sqrt{3x}=18\sqrt{27x}.$
8. $(9x)^{\frac{1}{2}}=x^{\frac{1}{2}}+4.$
9. $\sqrt{5n^2+19}+n=7.$
10. $(3x-4)^{\frac{1}{3}}+(4x+3)^{\frac{1}{3}}=0.$
11. $4\sqrt{v^2-v-4}+3=15.$
12. $\sqrt{x+4}=\sqrt[4]{x^2-5x+6}.$
13. $\sqrt[3]{x+1}=\sqrt{x+1}.$
14. $(s-2)^{\frac{1}{2}}=s^{\frac{1}{2}}+2^{\frac{1}{2}}.$
15. $3\sqrt{r+1}-2\sqrt{r+3}=\sqrt{2r+4}-\sqrt{r+3}+2\sqrt{r+1}.$

16. $\sqrt{2x-3}+3\sqrt{x-5}=\sqrt{3x-8}+2\sqrt{x-5}$.

17. $\sqrt{4x-12}+\sqrt{5x-2}+\sqrt{9x-14}=0$.

18. $\sqrt{4x-12}-\sqrt{5x-2}-\sqrt{9x-14}=0$.

19. $\sqrt{4x-12}+\sqrt{5x-2}-\sqrt{9x-14}=0$.

20. $\sqrt{4x-12}-\sqrt{5x-2}+\sqrt{9x-14}=0$.

21. $\dfrac{7\sqrt{n}+10}{\sqrt{4n}-2}=3$.

22. $\dfrac{2\sqrt{a}}{\sqrt{2x-a}}=\dfrac{\sqrt{2x+4a}}{3\sqrt{a}}$.

23. $\dfrac{n^{\frac{1}{2}}-3}{n^{\frac{1}{2}}}-\dfrac{5-n^{\frac{1}{2}}}{4}=0$.

24. $\dfrac{\sqrt{x+16}}{\sqrt{4-x}}+\dfrac{\sqrt{4-x}}{\sqrt{x+16}}=\dfrac{5}{2}$.

25. $5r-13r^{\frac{1}{2}}+6=0$.

26. $7x-3x^{\frac{1}{2}}-2x^{\frac{3}{2}}=0$.

27. $\dfrac{(r+5)^{\frac{1}{2}}}{(r+3)^{\frac{1}{2}}}-\dfrac{(r+3)^{\frac{1}{2}}}{(r+5)^{\frac{1}{2}}}=\dfrac{2}{\sqrt{3}}$.

28. $\sqrt{7+4x+3\sqrt{2x^2+5x+7}}-3=0$.

29. $\sqrt{17+2\sqrt{3+s+\sqrt{s+7}}}-5=0$.

30. $4x^2=10x+10-2\sqrt{4x^2-10x-2}$.

31. $3m^2=6\sqrt{3m^2-m-6}+m+22$.

32. $\sqrt{x+15}+\sqrt{x-24}-\sqrt{x-13}=\sqrt{x}$.

33. Solve for l and g, $t=\pi\sqrt{\dfrac{l}{g}}$.

34. Solve for t, $s=\dfrac{gt^2}{2}$.

35. If $a=\dfrac{R}{2}\sqrt{2}$ and $K=2R^2$, express K in terms of a.

36. If $K=\dfrac{3R^2}{2}\sqrt{3}$ and $a=\dfrac{R}{2}\sqrt{3}$, express K in terms of a.

37. If $K=2r^2\sqrt{2}$ and $a=\dfrac{r}{2}\sqrt{2+\sqrt{2}}$, express K in terms of a.

38. If $K=3r^2$ and $a=\dfrac{r}{2}\sqrt{2+\sqrt{3}}$, express K in terms of a.

39. The perimeter and the area of a certain square exceed the perimeter and area of a second square by 72 feet and 900 square feet respectively. Find the side of each square.

40. If a bullet is fired vertically upward, the least velocity V which it may have so that it will never return to the earth is given by the equation $V = \sqrt{2\,gR}$. ($g = 32$ feet per second, $R = 4000$ miles.) Find the velocity in miles per second to the nearest whole number.

41. The greatest distance x (in feet) that a ball can be thrown with velocity v (in feet per second) across a level field is given by one root of the equation $.976\,v^2x - gx^2 = 0$. ($g = 32$.) Under the conditions just stated a ball is thrown with a velocity of 100 feet per second. How far from the thrower does it strike the ground?

42. The greatest distance a baseball has been thrown is 426 feet $6\frac{1}{4}$ inches (Sheldon Lejeune, October 10, 1910). With what velocity did it leave the thrower's hand? (This velocity is called the initial velocity.)

43. Determine the initial velocity from the data: (*a*) A Lacrosse ball has been thrown 497 feet $7\frac{1}{2}$ inches (B. Quinn, 1902). (*b*) The record distance for the 16-pound shot is 51 feet (Ralph Rose, 1909). (*c*) The 16-pound hammer has been thrown 184 feet 4 inches (John Flanagan, 1910). (*d*) A football has been kicked a distance of 200 feet (W. P. Chadwick, 1887).

CHAPTER X

GRAPHS OF QUADRATIC EQUATIONS IN TWO VARIABLES

73. Graph of a quadratic equation in two variables. Before solving graphically a quadratic system, the method of graphing *one* quadratic equation in *two* variables must be clearly understood.

EXAMPLES

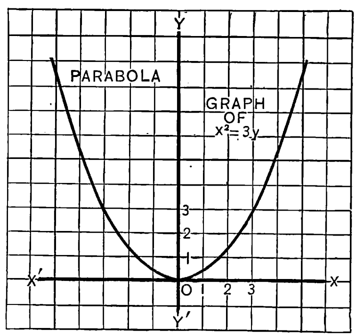
1. Construct the graph of $x^2 = 3y$.

Solution: Solving the equation for x in terms of y, $x = \pm\sqrt{3y}$.

We now assign values to y and then compute the approximate corresponding values of x. Tabulating the results gives:

$y =$	9	4	3	2	1	0	-1	Any negative value
$x =$	± 5.19	± 3.46	± 3	± 2.44	± 1.73	0	$\pm\sqrt{-3}$	Imaginary

Using an x-axis and a y-axis as in graphing linear equations, plotting the points corresponding to the real numbers in the table, and drawing the curve determined by these points, we obtain the graph of the adjacent figure. Since y is a function of x, the y-axis corresponds to the *function* axis. The curve is a **parabola.** A similar curve was always obtained in Chapter VII for the graph of a quadratic function of one variable.



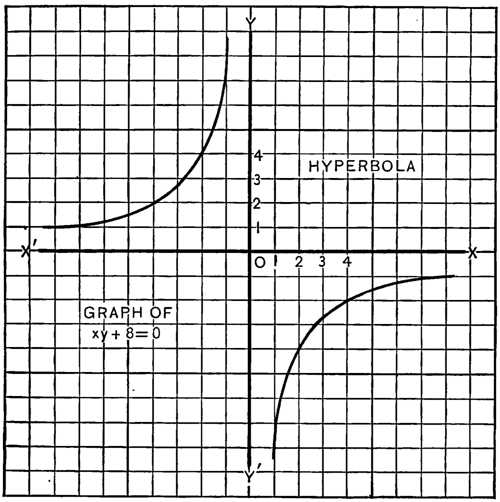
The graph of any equation of the form $y^2 = ax$ is a *parabola.*

2. Graph the equation $xy + 8 = 0$.

Solution: Solving for y in terms of x, $y = -\frac{8}{x}$.

Assigning values to x as indicated in the following table, we then compute the corresponding values of y.

$x =$	-6	-5	-4	-3	-2	-1	$-\frac{3}{4}$	$\frac{3}{4}$	1	2	3	4	5	6	8
$y =$	$\frac{4}{3}$	$\frac{8}{5}$	2	$\frac{8}{3}$	4	8	$\frac{32}{3}$	$-\frac{32}{3}$	-8	-4	$-\frac{8}{3}$	-2	$-\frac{8}{5}$	$-\frac{4}{3}$	-1



Proceeding as before with the numbers in the table, we obtain the two-branched curve of the above figure, which does not touch either axis. The curve is called an **hyperbola.**

The graph of any equation of the form $\boldsymbol{xy = K}$ is an *hyperbola.* The curve for $xy = K$ (K = any constant) is always in the same general position. That is, if K is positive, one branch of the curve lies in the first quadrant and the other branch in the third. If K is negative, one branch lies in the second quadrant and the other in the fourth.

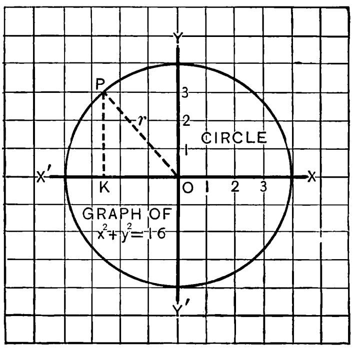
3. Graph the equation $x^2 + y^2 = 16$.

Solution: Solving for y in terms of x, $y = \pm\sqrt{16 - x^2}$.

Assigning values to x as indicated in the following table, we compute the corresponding approximate values of y.

$x =$	5	-4	-3	-2	-1	0	1	2	3	4	5
$y =$	$\pm 3\sqrt{-1}$	0	± 2.64	± 3.46	± 3.87	± 4	± 3.87	± 3.46	± 2.64	0	$\pm 3\sqrt{-1}$

For values of x numerically greater than 4, y is *imaginary*. The points corresponding to the pairs of real numbers in the table lie on the **circle** in the adjacent figure. The center of the circle is at the origin and the radius is 4.



Further, the graph of any equation of the form

$$x^2 + y^2 = r^2$$

is a *circle* whose radius is r. This can be proved from the right triangle PKO. If P represents *any* point on the circle, OK equals the x-distance of P, KP equals the y-distance, and OP equals the radius. Now $\overline{OK}^2 + \overline{KP}^2 = \overline{OP}^2$; that is, $x^2 + y^2 = r^2$. It follows, then, that the graphs of $x^2 + y^2 = 9$ and $x^2 + y^2 = 8$ are circles whose centers are at the origin and whose radii are 3 and $\sqrt{8}$ respectively. Hereafter, when it is required to graph an equation of the form $x^2 + y^2 = r^2$, the student may use compasses, and, with the origin as the center and the proper radius (the square root of the constant term), describe the circle at once.

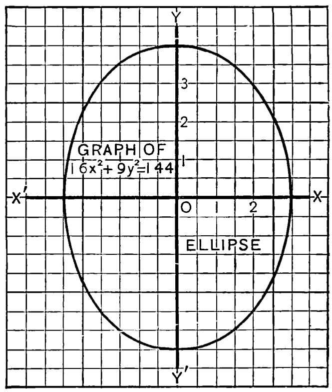
In all of the graphical work which follows it is expected that the student will save time by obtaining from the curve on page 85, or from the table on page 262, the square roots or cube roots which he may need.

4. Graph the equation $16x^2 + 9y^2 = 144$.

Solution: Solving for y in terms of x, $y = \pm \frac{4}{3}\sqrt{9 - x^2}$.

Assigning values to x as indicated in the following table, we compute the corresponding approximate values of y.

$x =$	-4	-3	-2	-1	0	$+1$	$+2$	$+3$	$+4$
$y =$	$\pm\frac{4}{3}\sqrt{-7}$	0	± 2.98	± 3.77	± 4	± 3.77	± 2.98	0	$\pm\frac{4}{3}\sqrt{-7}$



For values of x numerically greater than 3, y is *imaginary*. The points corresponding to the real numbers in the table lie on the graph of the adjacent figure. The curve is called an **ellipse.**

The graph of any equation of the form of $ax^2 + by^2 = c$, in which a and b are unequal and of the same sign as c, is an *ellipse.*

Note. These three curves, the ellipse, the hyperbola, and the parabola, were first studied by the Greeks, who proved that they are the sections which one obtains by cutting a cone by a plane. Not for hundreds of years afterwards did any one imagine that these curves actually appear in nature, for the Greeks regarded them merely as geometrical figures, and not at all as curves that have anything to do with our everyday life. One of the most important discoveries of astronomy was made by Kepler (1571–1630), who showed that the earth revolves around the sun in an ellipse, and stated the laws which govern the motion. Those comets that return to our field of vision periodically also have elliptic orbits, while those that appear once, never to be seen again, describe parabolic or hyperbolic paths.

The path of a ball thrown through the air in any direction, except vertically upward or downward, is a parabola. The approximate parabola which a projectile actually describes depends on the elevation of the gun (the angle with the horizontal), the quality of the powder, the amount of the charge, the direction of the wind, and various other conditions. This makes gunnery a complex problem.

EXERCISES

Construct the graphs of the following equations and state the name of each curve obtained:

1. $x^2 = 2y$.
2. $y^2 + 2x = 0$.
3. $x^2 + y^2 = 36$.
4. $x^2 + y^2 = 12$.
5. $x^2 - y^2 = 25$.
6. $xy = 8$.
7. $xy = -12$.
8. $9x^2 + 16y^2 = 144$.
9. $16x^2 - 9y^2 = 144$.
10. $25x^2 + 9y^2 = 225$.

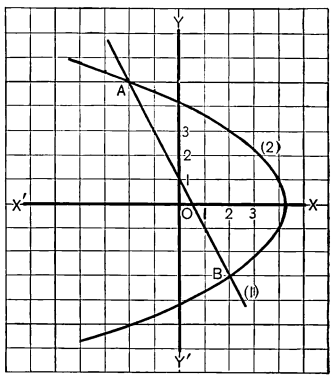
74. Graphical solution of a quadratic system in two variables. That we may solve a system of two quadratic equations by a method similar to that employed in § 38 for linear equations appears from the following

EXAMPLES

1. Solve graphically $\begin{cases} 2x + y = 1, & (1) \\ y^2 + 4x = 17. & (2) \end{cases}$

Solution: Constructing the graphs of (1) and (2), we obtain the straight line and the parabola shown in the adjacent figure. There are two sets of roots corresponding to the two points of intersection, which are:

$$A\begin{cases} x = -2, \\ y = 5, \end{cases} \quad B\begin{cases} x = 2, \\ y = -3. \end{cases}$$



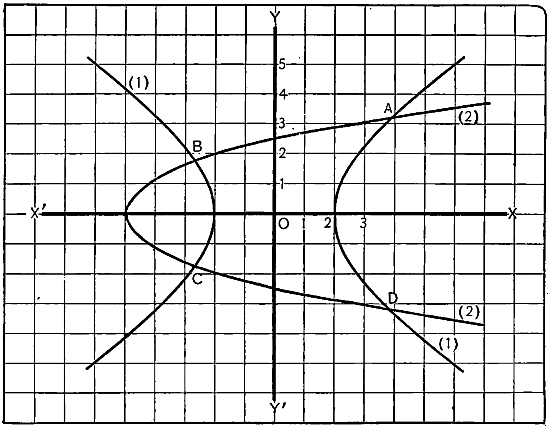
Note. If the straight line in the adjacent figure were moved to the right in such a way that it always remained parallel to its present position, the points A and B would approach each other and finally coincide. The line would then be tangent to the parabola at the point $x = 4$, $y = 1$.

Were the straight line moved still farther, it would neither touch nor intersect the parabola and there would be no graphical solution.

An illustration of these two conditions is given by the graphical solution of Exercises 8 and 9, page 142.

2. Solve graphically $\begin{cases} x^2 - y^2 = 4, & (1) \\ y^2 - x - 6 = 0. & (2) \end{cases}$

Solution: Constructing the graphs of (1) and (2), we obtain the hyperbola and the parabola of the following figure. There are four



sets of roots corresponding to the four points of intersection, which are approximately

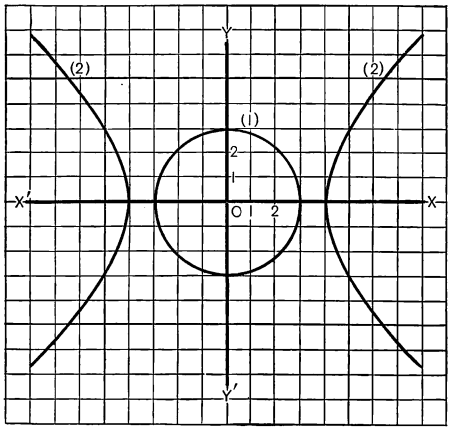
$A\begin{cases} x = 3.7, \\ y = 3.1. \end{cases}$ $B\begin{cases} x = -2.7, \\ y = 1.8. \end{cases}$ $C\begin{cases} x = -2.7, \\ y = -1.8. \end{cases}$ $D\begin{cases} x = 3.7, \\ y = -3.1. \end{cases}$

3. Solve graphically $\begin{cases} x^2 + y^2 = 9, & (1) \\ x^2 - y^2 = 16. & (2) \end{cases}$

Solution: The graphs (1) and (2) are the circle and hyperbola of the figure on page 141. These curves have no real points of intersection. There are, however, four sets of imaginary roots. Subtracting equation (2) from (1) gives $2y^2 = -7$, whence $y = \pm\sqrt{-\frac{7}{2}}$, an *imaginary* expression. Adding (1) and (2) gives $2x^2 = 25$, whence $x = \pm\frac{5}{2}\sqrt{2}$, a real expression. Using the double sign before each radical gives the four sets of imaginary roots:

$$\begin{cases} x = \frac{5}{2}\sqrt{2}, & +\frac{5}{2}\sqrt{2}, & -\frac{5}{2}\sqrt{2}, & -\frac{5}{2}\sqrt{2}. \\ y = \sqrt{-\frac{7}{2}}, & -\sqrt{-\frac{7}{2}}, & -\sqrt{-\frac{7}{2}}, & +\sqrt{-\frac{7}{2}}. \end{cases}$$

It can be shown that these sets of imaginary roots correspond to the intersections of what may be termed the imaginary branches of the curves. These branches may be represented as lines not in the same plane as the real branches but in a plane passing through the x-axis perpendicular to the plane determined by the x-axis and the y-axis.



Though the subject is not difficult, even a simple presentation of this method of constructing imaginary graphs is wholly beyond the scope of this book. The essential point to be grasped now is that real roots correspond to real intersections, and imaginary roots correspond to no intersections of real graphs.

Note. In equation (1), page 140, a greater number in place of 9 would give a larger circle than the one in the figure, and it would be easy to find a number to replace 9 such that the resulting circle would just touch the hyperbola. Were a still greater number used, the circle obtained would intersect the other curve. These varying conditions would result, respectively, in (a) no set of real roots, (b) two sets of real roots, (c) four sets of real roots.

Examples 1, 2, and 3 partially illustrate the truth of the following statement:

If in a system of two equations in two variables one equation is of the mth degree and one of the nth, there are *usually* ***mn*** sets of roots (real or imaginary) *and never more than* ***mn*** *such sets.*

EXERCISES

If possible, solve graphically each of the following systems:

1. $x^2 = 4y,$
$x + 3y = 5.$

2. $x^2 + y^2 = 25,$
$x - 2y = 10.$

3. $x^2 + y^2 = 16,$
$x^2 + y^2 = 9.$

4. $x^2 + y^2 = 4,$
$x + y = 8.$

5. $x^2 + y^2 = 25,$
$x^2 - y^2 = 16.$

6. $x^2 + y^2 = 16,$
$x^2 - y^2 = 25.$

7. $y^2 = 4x,$
$x^2 + 9y^2 = 9.$

8. $y^2 + 4x = 17,$
$2x + y = 9.$

9. $y^2 + 4x = 17,$
$2x + y = 12.$

10. $x^2 + y^2 = 16,$
$x^2 + y^2 - 2x = 8.$

11. $y - x\sqrt{8} = 0,$
$y^2 = x^3 - 9x.$

CHAPTER XI

SYSTEMS SOLVABLE BY QUADRATICS

75. Introduction. The general equation of the second degree in two variables is $ax^2 + by^2 + cxy + dx + ey + f = 0$. To solve a pair of such equations requires the solution of an equation of the fourth degree. Even the solution of $x^2 + y = 5$ and $y^2 + x = 3$ requires the solution of a biquadratic equation. In fact, only a limited number of systems of the second degree in two variables are solvable by quadratics. The student should note that he can solve graphically for real roots any system of quadratic equations, provided the terms have numerical coefficients. The algebraic solution of such systems will be possible for him only after further study of algebra.

76. Linear and quadratic. Every system of equations in two variables in which one equation is **linear** and the other **quadratic** *can be solved by the method of substitution.*

EXAMPLE

Solve the system $\begin{cases} x^2 + y^2 = 5, & (1) \\ x - y = 1. & (2) \end{cases}$

Solution: Solving (2) for x in terms of y, $\quad x = 1 + y.$ (3)
Substituting $1 + y$ for x in (1), $(1 + y)^2 + y^2 = 5.$ (4)
From (4), $\quad y^2 + y - 2 = 0.$ (5)
Solving (5), $\quad y = 1$ or $-2.$
Substituting 1 for y in (3), $\quad x = 1 + 1 = 2.$
Substituting -2 for y in (3), $\quad x = 1 - 2 = -1.$
The two sets of roots are $x = 2$, $y = 1$ and $x = -1$, $y = -2$.

Check: Substituting 2 for x and 1 for y in $\begin{cases} (1), & 4 + 1 = 5, \\ (2), & 2 - 1 = 1. \end{cases}$

Substituting -1 for x and -2 for y in $\begin{cases} (1), & 1 + 4 = 5, \\ (2), & -1 + 2 = 1. \end{cases}$

EXERCISES

Solve the following systems, pair results, and check each set of roots:

1. $x + y = 6,$
$x^2 + y^2 = 20.$

2. $4m + n = 28,$
$2m^2 + 3mn = 98.$

3. $m^2 + 2n^2 = 44,$
$m - 2n\sqrt{5} = 0.$

4. $4s + t = 6,$
$st = -10.$

5. $xy + 36 = 0,$
$4x - y = 30.$

6. $x\sqrt{3} + 5y = -72,$
$xy = -15\sqrt{3}.$

7. $3R_1 + 4R_2 = 5,$
$2R_1R_2 - 6R_1 = -3.$

8. $2xy + y^2 - 20 = 0,$
$xy + 40 = 0.$

9. $h^2 + k^2 + 2k = 40,$
$h + k + 2 = 0.$

10. $m^2 + 3mn + n^2 = 88,$
$2m = n.$

11. $x^2 + y^2 + 4x + 6y = 40,$
$x - 10 = y.$

12. $y + x\sqrt{15} = 0,$
$y^2 + x^3 = 16x.$

If the equations of a system are not one *linear* and the other *quadratic*, an attempt to solve it by *substitution* usually gives an equation of the third or fourth degree at least. In most cases such an equation could not be solved by factoring, and at the present time its solution by any other method is beyond the student. The various devices explained in the following pages are for the purpose of avoiding the necessity of solving an equation of a higher degree than the second.

77. Homogeneous equations. An equation is **homogeneous** if, on being written so that one member is zero, the terms in the other member are of the same degree with respect to the variables.

Thus $x^2 + y^2 = xy$ and $x^2 - 3xy + y^2 = 0$ are homogeneous equations of the second degree; $2x^3 + y^3 = x^2y - 3xy^2$ is a homogeneous equation of the third degree.

78. Both equations quadratic. If the system is of either type described in the following examples, it can be solved by quadratics.

The first example illustrates the type when one equation, but not necessarily both of them, is homogeneous.

EXAMPLES

1. Solve the system $\begin{cases} 3x^2 + 4y^2 = 8xy, & (1) \\ y^2 + x^2 - 5x = 3. & (2) \end{cases}$

Solution: First we solve the homogeneous equation (1) for x in terms of y.

Transposing in (1), $3x^2 - 8xy + 4y^2 = 0.$ (3)

Solving (3) by formula, $x = \dfrac{8y \pm \sqrt{64y^2 - 48y^2}}{6}.$ (4)

Whence $x = 2y$ or $\dfrac{2y}{3}.$ (5)

Substituting $2y$ for x in (2),

$$y^2 + 4y^2 - 10y = 3. \quad (6)$$

Solving (6), $y = 1 \pm \frac{2}{5}\sqrt{10}.$ (7)

By (5), $x = 2y$; then from (7), $x = 2 \pm \frac{4}{5}\sqrt{10}.$ (8)

Substituting $\dfrac{2y}{3}$ for x in (2),

$$y^2 + \frac{4y^2}{9} - \frac{10y}{3} = 3. \quad (9)$$

Solving (9), $y = 3$ or $-\frac{9}{13}.$ (10)

By (5), $x = \dfrac{2y}{3}$; then from (10), $x = 2$ or $-\frac{6}{13}.$ (11)

Pairing results,

$$\left.\begin{matrix} x = 2 \\ y = 3 \end{matrix}\right\} A, \quad \left.\begin{matrix} x = -\frac{6}{13} \\ y = -\frac{9}{13} \end{matrix}\right\} B, \quad \left.\begin{matrix} x = 2 + \frac{4}{5}\sqrt{10} \\ y = 1 + \frac{2}{5}\sqrt{10} \end{matrix}\right\} C, \quad \left.\begin{matrix} x = 2 - \frac{4}{5}\sqrt{10} \\ y = 1 - \frac{2}{5}\sqrt{10} \end{matrix}\right\} D.$$

Check:

$$A \begin{cases} 3(2)^2 + 4(3)^2 = 8 \cdot 2 \cdot 3, \text{ or } 48 = 48. \\ 3^2 + 2^2 - 5 \cdot 2 = 3, \text{ or } 3 = 3. \end{cases}$$

$$B \begin{cases} 3(-\frac{6}{13})^2 + 4(-\frac{9}{13})^2 = 8 \cdot (-\frac{6}{13})(-\frac{9}{13}), \text{ or } \frac{432}{169} = \frac{432}{169}. \\ (-\frac{9}{13})^2 + (-\frac{6}{13})^2 - 5 \cdot (-\frac{6}{13}) = 3, \text{ or } 3 = 3. \end{cases}$$

$$\begin{matrix} C \\ D \end{matrix} \begin{cases} 3(2 \pm \frac{4}{5}\sqrt{10})^2 + 4(1 \pm \frac{2}{5}\sqrt{10})^2 = 8(2 \pm \frac{4}{5}\sqrt{10})(1 \pm \frac{2}{5}\sqrt{10}). \\ (1 \pm \frac{2}{5}\sqrt{10})^2 + (2 \pm \frac{4}{5}\sqrt{10})^2 - 5(2 \pm \frac{4}{5}\sqrt{10}) = 3. \end{cases}$$

Taking both values in C and D with the sign $+$ or both with the sign $-$,

$$\begin{cases} 12 \pm \frac{48}{5}\sqrt{10} + \frac{96}{5} + 4 \pm \frac{16}{5}\sqrt{10} + \frac{32}{5} = 16 \pm \frac{64}{5}\sqrt{10} + \frac{128}{5}. \\ 1 \pm \frac{4}{5}\sqrt{10} + \frac{8}{5} + 4 \pm \frac{16}{5}\sqrt{10} + \frac{32}{5} - 10 \mp 4\sqrt{10} = 3. \end{cases}$$

If each equation of a system in two variables is quadratic and both are homogeneous with the exception of a constant term (not zero), the system is solved much like the preceding one.

2. Solve $\begin{cases} xy + 3y^2 = 6, & (1) \\ x^2 + y^2 = 10. & (2) \end{cases}$

HINT. First we combine the two equations to obtain a homogeneous equation in which the constant term is zero.

$(1)\cdot 5,\qquad 5xy + 15y^2 = 30. \qquad (3)$

$(2)\cdot 3,\qquad 3x^2 + 3y^2 = 30. \qquad (4)$

$(3)-(4),\qquad -3x^2 + 5xy + 12y^2 = 0. \qquad (5)$

Solving (5) for x in terms of y, $x = 3y$ or $-\frac{4y}{3}$. (6)

We can now substitute from (6) in (2) and proceed precisely as in the last example. The student should complete the work and obtain

$$\begin{array}{llll} x = 3, & -3, & +\frac{4}{5}\sqrt{10}, & -\frac{4}{5}\sqrt{10}. \\ y = 1, & -1, & -\frac{3}{5}\sqrt{10}, & +\frac{3}{5}\sqrt{10}. \end{array}$$

EXERCISES

Solve, pair results, and check each set of real roots:

1. $x^2 + xy = 3,$ $y^2 - xy = 10.$

2. $x^2 + y^2 = 10,$ $3y^2 + xy = 6.$

3. $u^2 + 2uv = 0,$ $2v^2 + 3uv = -16.$

4. $s^2 - 3st = 4,$ $3t^2 + 3s^2 = 12.$

5. $x(x + 2y) = 16,$ $y(y - x) = 3.$

6. $x^2 + xy + y^2 = 4,$ $x^2 - 2xy = 12.$

7. $x^2 + xy = 2y^2,$ $2x^2 + x = 2 + y^2.$

8. $x^2 + 2xy - y^2 = 32,$ $2x^2 - 3xy + y^2 = 0.$

9. $2x^2 - xy + 2y^2 = 12,$ $2x^2 + xy + 2y^2 = 8.$

10. $x^2 - xy - 5y^2 = 15,$ $x^2 - 6y^2 = 1.$

Up to this point the systems considered have been solved by a method partially described by the word "substitution." The essential step in this method is to solve one of the original equations (or one derived from the original system) for one variable in terms of the other, and substitute the value found in the other equation (or in either of the original equations). This method is applicable more frequently than those which are given later. Consequently it is much more important for the student to master the method of substitution than it is for him to master any other method.

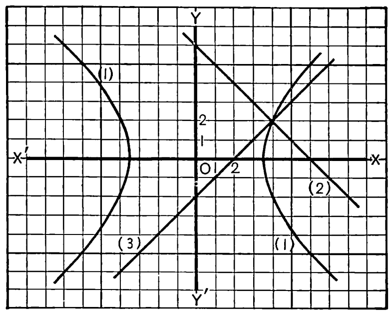
79. Equivalent systems. Equivalent systems of equations are systems which have the same set or sets of roots.

The graphs of equivalent systems have common points of intersection.

If we solve graphically the two systems

$$A\begin{cases} x^2 - y^2 = 12, & (1) \\ x + y = 6. & (2) \end{cases} \quad \text{and} \quad B\begin{cases} x - y = 2, & (3) \\ x + y = 6. & (4) \end{cases}$$

we obtain the graphs of the adjacent figure. The hyperbola (1) and the straight line (2) intersect at only one point (4, 2). The straight lines (2) and (3) intersect at this very point. Hence the systems A and B are equivalent.



80. Special devices. Systems of equations are often met which can be solved by substitution, but which are more conveniently solved as in the following illustrations. It should be observed that in every case the aim of the device is to replace the given system by an equivalent system of linear equations, or by a system in which one equation is quadratic and the other linear.

EXAMPLES

1. Solve the system $\begin{cases} x + y = 7, & (1) \\ xy = 6. & (2) \end{cases}$

Solution: Squaring (1), $x^2 + 2\,xy + y^2 = 49.$ (3)

(2) · 4, $4\,xy = 24.$ (4)

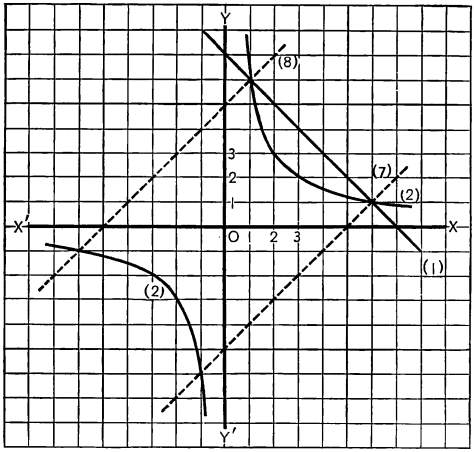
(3) − (4), $x^2 - 2\,xy + y^2 = 25.$ (5)

From (5), $x - y = \pm\, 5.$ (6)

From (6) and (1), $A\begin{cases} x + y = 7, & (1) \\ x - y = 5. & (7) \end{cases}$ $B\begin{cases} x + y = 7, & (1) \\ x - y = -5. & (8) \end{cases}$

For A, $x = 6$, $y = 1$; and for B, $x = 1$, $y = 6$.

The derived systems A and B are equivalent to the original system (1), (2). The graphs of the adjacent figure show that the straight



line (1) and the hyperbola (2) have the same points of intersection as the three straight lines (1), (7), and (8) of systems A and B.

A method similar to that of the preceding solution can be applied to the following system:

2. Solve $\begin{cases} x^2 + y^2 = 37, & (1) \\ xy = 6. & (2) \end{cases}$

Solution: (2) · 2, $2\,xy = 12.$ (3)

(1) + (3), $x^2 + 2\,xy + y^2 = 49.$ (4)

From (4), $x + y = \pm\,7.$ (5)

(1) − (3), $x^2 - 2\,xy + y^2 = 25.$ (6)

From (6), $x - y = \pm\,5.$ (7)

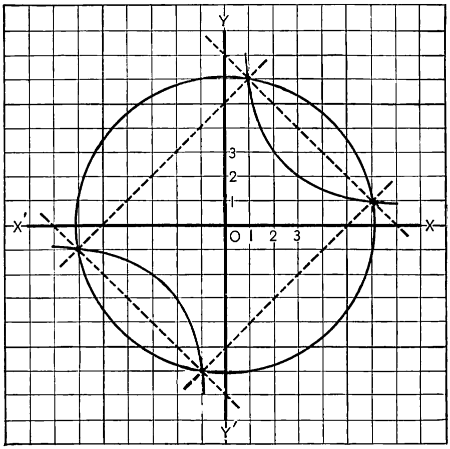
(5) and (7) combined give four systems of equations:

$A\begin{cases} x + y = 7, & (8) \\ x - y = 5. & (9) \end{cases}$ $C\begin{cases} x + y = -7, & (11) \\ x - y = 5, & (9) \end{cases}$

$B\begin{cases} x + y = 7, & (8) \\ x - y = -5. & (10) \end{cases}$ $D\begin{cases} x + y = -7, & (11) \\ x - y = -5. & (10) \end{cases}$

The solution of A, B, C, and D is left to the student.

In the figure on page 149 the graphs of (1) and (2) are the circle and the hyperbola respectively, the two curves having four points of intersection. The graphs of the four equations in the systems A, B, C, and D are the four straight lines. These four straight lines intersect in the four points in which the hyperbola and the circle



intersect. This shows that the four sets of roots belonging to the system (1), (2) are identical with the four sets belonging to the four systems A, B, C, and D; that is, the *one* system, (1) (2), is *equivalent* to the *four* systems A, B, C, and D.

EXERCISES

Solve in a manner similar to that of the two preceding examples, pair results, and check each set of real roots :

1. $x - y = 4,$
   $xy = 5.$

2. $x + 2y = 8,$
   $xy + 6 = 0.$

3. $x^2 + 4y^2 = 101,$
   $xy + 5 = 0.$

4. $6x - y = 24,$
   $36x^2 + y^2 = 288.$

5. $4x^2 - 6xy + 0y^2 = 24,$
   $xy - 20 = 0.$

6. $4x^2 + y^2 = 25,$
   $4x^2 + 4xy + y^2 = 49.$

7. $x^2 + 4y^2 = 15,$
   $x + 2y = 3\sqrt{3}.$

8. $x^2 - 2xy = 16,$
   $2y^2 - xy = -6.$

9. $\frac{1}{x^2} + \frac{1}{y^2} = 13,$
$\frac{1}{xy} = 6.$

10. $\frac{1}{x^2} - \frac{1}{xy} + \frac{1}{y^2} = 7,$
$\frac{1}{x} - \frac{1}{y} = 1.$

11. $3x - 3y = 7,$
$9x^2 + 18xy + 9y^2 = 1.$

12. $\frac{1}{x^3} + \frac{1}{y^3} = 35,$
$\frac{1}{x} + \frac{1}{y} = 5.$

13. $x^2 + 4y^2 = c,$
$4xy = d.$

81. Use of division in equations. Sometimes an equation simpler than either of those given can be derived from a system by dividing the left and right members of the first equation by the corresponding members of the second. Then the equation so obtained taken with one of the first two gives a derived system more simple than the original one but not always equivalent to it. The conditions under which the two are equivalent, however, is easily stated and explained.

THEOREM. *Let U, V, K, and R be rational integral expressions in two unknowns, x and y. Then the system*

$$UK = VR, \quad (1)$$
$$U = V \quad (2)$$

is equivalent to the two systems

$$K = R, \quad (3) \qquad U = 0, \quad (4)$$
$$U = V, \quad (2) \quad \text{and} \quad V = 0. \quad (5)$$

Proof. Substituting U for V in (1), transposing, and factoring, gives

$$U(K - R) = 0. \quad (6)$$

From (2), $$U - V = 0. \quad (7)$$

But the system (6), (7) is equivalent to the two systems (3), (2) and (4), (5). This is at once apparent since it can be seen from inspection that any set of roots which satisfies (3), (2) or (4), (5) will satisfy (6), (7); and conversely.

Now if U or V is an *arithmetical number* (not zero), the system (3), (2) *alone* is equivalent to the original one, since either (4) or (5) would not in that case involve any unknown.

Therefore in a system of the form of (1), (2) we may use division and thereby obtain one simpler equivalent system *if U or V is an arithmetical number.* In any other case we can at once write down the *two* systems which are equivalent to the original one. Either of these courses makes it easier to obtain all the sets of roots which satisfy the original system.

EXAMPLES

In Examples 1, 2, and 3 division gives in each case the *one* equivalent system on the right.

1. $x^2 - y^2 = 12,$ $\quad x + y = 6.$ $\qquad x - y = 2,$ $\quad x + y = 6.$ (See graph, p. 147.) (One set of roots.)

2. $x^2 - y^2 = 4x + 6y - 8,$ $\quad x - y = 2.$ $\qquad x + y = 2x + 3y - 4,$ $\quad x - y = 2.$ (One set of roots.)

3. $x^3 + y^3 = 28,$ $\quad x + y = 4.$ $\qquad x^2 - xy + y^2 = 7,$ $\quad x + y = 4.$ (See graph, p. 152.) (Two sets of roots.)

4. $x^3 - y^3 = 6x + 3y - 18,$ $\quad x - y = 2x + y - 6.$ Division gives the *two* systems:

$$\begin{cases} x^2 + xy + y^2 = 3, \\ x + 2y = 6, \end{cases} \text{and} \begin{cases} x - y = 0, \\ 2x + y - 6 = 0. \end{cases}$$ (Three sets of roots.)

The first system in Example 3 has *two* sets of roots, that in Example 4 has *three.* Hence the use of division without a correct use of the theorem on page 150 would frequently result in an incomplete solution. If time permits, the student should graph the equations of Example 4.

EXERCISES

Solve (using division where possible), pair results, and check each set of real roots:

1. $4x^2 - y^2 = 16,$ $\quad 2x + y = 8.$

2. $R^2h - 75 = 0,$ $\quad Rh = 15.$

3. $\dfrac{1}{x^2} - \dfrac{1}{y^2} = 15,$ $\quad \dfrac{1}{x} + \dfrac{1}{y} = 3$

4. $9h^2k - 100 = 0,$
$3k^2h + 80 = 0.$

5. $P(1+r)^2 = 224.72,$
$P + Pr = 212.$

6. $9x^2y^2 + 6 = 15xy,$
$3xy + 4 = 6.$

7. $x^4 = 9y^4 + 48,$
$x^2 = 3y^2 + 2.$

8. $\frac{gt^2}{2} = .16,\ gt = 3.2.$

9. $1 - x = y,\ 1 - x^3 = y^2.$

10. $x^2 - 2xy - 24y^2 = 32,$
$x - 6y = 2.$

11. $4x^2 + 7 = 8xy + 5y^2,$
$4x^2 = 1 + y^2.$

12. $x^3 + y^3 = 4x - 6y - 8,$
$x + y = 2x - 3y - 4.$

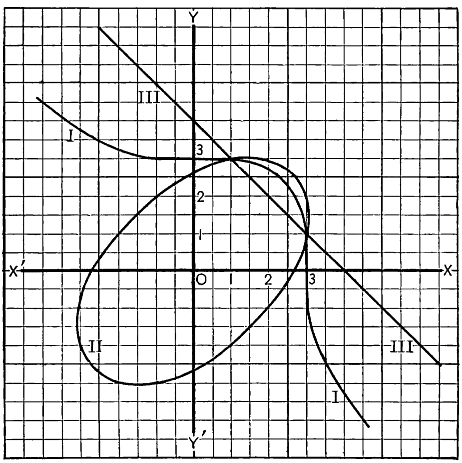
13. $x^3 - y^3 = 6x,$
$x - y = 3x.$

14. $x^3 + y^3 = 28,$
$x + y = 4.$

15. $x^2 - xy + y^2 = 7,$
$x^3 + y^3 = 28.$

16. $x + y = 4,$
$x^2 - xy + y^2 = 7.$

In the following figure I, II, and III are the graphs of $x^3 + y^3 = 28$, $x^2 - xy + y^2 = 7$, and $x + y = 4$ respectively. These equations are taken from the systems in Exercises 14, 15, and 16 which contain



but three different equations paired in three ways. Since the two sets of roots for each is the same, we know that the three systems are equivalent. The equivalence of the three systems is also shown in the preceding figure.

MISCELLANEOUS EXERCISES

Solve by any method, pair results, and check each set of real roots:

(If any system cannot be solved algebraically by the methods previously given, solve it graphically.)

1. $2x^2 + y^2 = 33,$
$x^2 + 2y^2 = 54.$

2. $3h^2 - 8k^2 = 40,$
$5h^2 + k^2 = 81.$

3. $4R_1^2 + 3 = 9R_2^2,$
$12R_1^2 + R_2^2 = \frac{31}{9}.$

4. $xy + x = 18,$
$xy + y = 20.$

5. $x^2 = y,$
$xy = 8.$

6. $x - xy = 5,$
$2y + xy = 6.$

7. $x^3 - y^3 = 19,$
$x - y = 1.$

8. $x^3 - y^3 = 19,$
$x^2 + xy + y^2 = 19.$

9. $x^2 + xy + y^2 = 19,$
$x - y = 1.$

10. Show that the systems (7), (8), and (9) are equivalent by graphing the three equations of these exercises.

11. $3s^2 - 2t^2 = 0,$
$5s^2 - 3t^2 = 1.$

12. $4n^2 + 7m^2 = 9,$
$2n^2 - \frac{9}{2} = m^2.$

13. $5W_1^2 - 6.8W_2^2 = 99.55,$
$W_1^2 - W_2^2 = 20.$

14. $xy + 2y^2 = 2,$
$3xy + 5y^2 = 2.$

15. $x^2 + 2xy + 2y^2 = 10,$
$3x^2 - xy - y^2 = 51.$

16. $y^2 + x = 7,$
$x^2 + y = 11.$

17. $x^2 + xy + y^2 = 7,$
$x^2 + y^2 = 10.$

18. $x^2 + xy + x = 0,$
$x^2 + xy + 2x = 0.$

19. $x^2 + xy + y = 0,$
$x^2 + xy + x = 0.$

20. $\frac{1}{x^2} + \frac{1}{y^2} = 13,$
$\frac{1}{x} - \frac{1}{y} = 1.$

21. $\frac{1}{x^3} - \frac{1}{y^3} = 7,$
$\frac{1}{x} - \frac{1}{y} = 1.$

22. $x^2 - 2xy + 2y^2 - y = 0,$
$2x^2 - 3xy - y^2 + 2y = 0$

23. $x^2 + z^2 = 34,$
$x^2 + y^2 = 25,$
$y^2 + z^2 = 41.$

24. $3xy = x^2y^2 - 88,$
$x - y = 6.$

25. $x^3 = y^3 + 37,$
$x^2y = xy^2 + 12.$

26. $\frac{4}{x^2} - \frac{13}{xy} + \frac{9}{y^2} = 9,$
$\frac{1}{xy} - \frac{1}{y^2} = 3.$

27. $x^2 = 8x + 6y,$
$y^2 = 6x + 8y.$

28. $x^3 - y^3 = 2x + y - 4,$
$x + 2y = 4.$

29. $xy = c,$
$x + y = a.$

30. $x^{-2} - y^{-2} = 6,$
$x^{-1} + y^{-1} = 2.$

31. $x - y = 16,$
$x^{\frac{1}{2}} - y^{\frac{1}{2}} = 2.$

32. $\frac{1}{x-2} + \frac{1}{y-2} = \frac{3}{4},$
$\frac{1}{x} - \frac{1}{y} = \frac{1}{12}.$

33. $x^2 + 6x^{-2} = 36\frac{1}{6},$
$3xy - x^2 = 36.$

34. $x - y\sqrt{x} = 24,$
$x^2 + xy^2 = 320.$

35. $\frac{x-1}{y-1} = 3,$
$\frac{y^2+y+1}{x^2-x+1} = \frac{13}{43}.$

36. $\frac{1}{x^2} + \frac{2}{xy} = 16,$
$\frac{3}{x^2} - \frac{4}{xy} + \frac{2}{y^2} = 6.$

37. $9x \div y = 18 = xy.$

38. $4x + \frac{1}{y} = 46 = \frac{26x}{5} - \frac{1}{y}.$

39. $x^2 + y^2 - (y - x) = 12,$
$x^2 - xy = 0.$

PROBLEMS

(Reject all results which do not satisfy the conditions of the problems.)

1. Find two numbers whose difference is 4 and the difference of whose squares is 88.

2. The sum of two numbers is 21 and the sum of their squares is 281. Find the numbers.

3. Find two numbers whose product is 192 and whose quotient is $\frac{3}{4}$.

4. The area of a right triangle is 150 square feet and its hypotenuse is 25 feet. Find the legs.

5. A rectangular field is 8 rods longer than it is wide and the area of the field is 8 acres. Find the length and the width.

6. The difference of the areas of two squares is 252 square feet, and the difference of their perimeters is 24 feet. Find a side of each square.

7. The area of a rectangular field is $3\frac{3}{5}$ acres and one diagonal is 60 rods. Find the perimeter of the field.

8. The perimeter of a rectangle is 112 feet and its area is 768 square feet. Find the length and the width.

9. A mean proportional between two numbers is $2\sqrt{14}$, and the sum of their squares is 113. Find the numbers.

10. The value of a certain fraction is $\frac{2}{3}$. If the fraction is squared and 44 is subtracted from both the numerator and the denominator of this result, the value of the fraction thus formed is $\frac{5}{14}$. Find the original fraction.

11. The base of a triangle is 6 inches longer than its altitude, and the area is $\frac{3}{2}$ square feet. Find the base and altitude of the triangle.

12. The volumes of two cubes differ by 1413 cubic inches and their edges differ by 3 inches. Find the edge of each.

13. The sum of the radii of two circles is 25 inches and the difference of their areas is 125π square inches. Find the radii.

14. The perimeter of a rectangle is $5C$ and its area is C^2. Find its dimensions.

15. The area of a right triangle is $8a^2 - 8b^2$ and its hypotenuse is $4\sqrt{2a^2 + 2b^2}$. Find the legs.

16. The perimeter of a right triangle is 56 feet and its area is 84 square feet. Find the legs and the hypotenuse.

17. If a 2-digit number be multiplied by the sum of its digits, the product is 324; and if three times the sum of its digits be added to the number, the result is expressed by the digits in reverse order. Find the number.

18. The yearly interest on a certain sum of money is $42. If the sum were $200 more and the interest 1% less, the annual income would be $6 more. Find the principal and the rate.

19. A wheelman leaves A and travels north. At the same time a second wheelman leaves a point 3 miles east of A and travels east. One and one-third hours after starting, the shortest distance between them is 17 miles; and $3\frac{1}{3}$ hours later the distance is 53 miles. Find the rate of each.

20. The circumference of the fore wheel of a carriage is 1 foot less and that of its rear wheel 3 feet less than the circumferences of the corresponding wheels of a farm wagon. In going 1 mile the fore wheel of the carriage makes 40 revolutions more than its rear wheel, and the fore wheel of the wagon makes 88 more than its rear wheel. Find the circumferences of the carriage wheels.

21. A starts out from P to Q at the same time B leaves Q for P. When they meet, A has gone 40 miles more than B. A then finishes the journey to Q in 2 hours and B the journey to P in 8 hours. Find the rates of A and B, and the distance from P to Q.

22. A leaves P going to Q at the same time that B leaves Q on his way to P. From the time the two meet, it requires $6\frac{2}{3}$ hours for A to reach Q, and 15 hours for B to reach P. Find the rate of each, if the distance from P to Q is 300 miles.

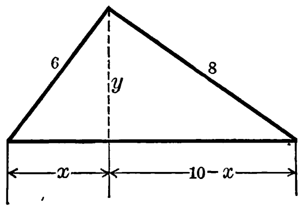
23. A man has a rectangular plot of ground whose area is 1250 square feet. Its length is twice its breadth. He wishes to divide the plot into a rectangular flower bed, surrounded by a path of uniform breadth, so that the bed and the path may have equal areas. Find the width of the path.

GEOMETRICAL PROBLEMS

1. The sides of a triangle are 6, 8, and 10. Find the altitude on the side 10.

HINT. From the adjacent figure we easily obtain the system:

$$\begin{cases} x^2 + y^2 = 36, \\ (10 - x)^2 + y^2 = 64. \end{cases}$$



2. The sides of a triangle are 8, 15, and 17. Find the altitude on the side 17 and the area of the triangle.

3. The sides of a triangle are 13, 20, and 21. Find the altitude on the side 20 and the area of the triangle.

4. The sides of a triangle are 7, 15, and 20. Find the altitude on the side 7 and the area of the triangle.

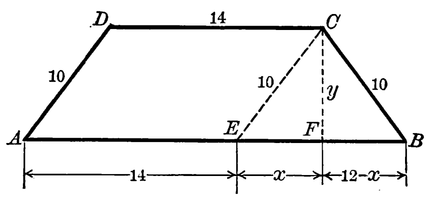
5. The sides of a triangle are 10, 17, and 21. Find the altitude on the side 10 and the area of the triangle.

6. Find correct to two decimals the altitude on the side 16 of a triangle whose sides are 12, 16, and 18 respectively.

7. The parallel sides of a trapezoid are 14 and 26 respectively, and the two nonparallel sides are 10 each. Find the altitude of the trapezoid.

HINT. Let $ABCD$ be the trapezoid. Draw CE parallel to DA and CF perpendicular to AB. Then $EC = 10$, $AE = 14$, and $EB = 26 - 14$, or 12. If we let $EF = x$, FB must equal $12 - x$; then we can obtain the system of equations:

$$\begin{cases} x^2 + y^2 = 100, \\ (12 - x)^2 + y^2 = 100. \end{cases}$$



8. The two nonparallel sides of a trapezoid are 12 and 17 respectively, and the two bases are 5 and 13 respectively. Find the altitude of the trapezoid.

9. The bases of a trapezoid are 15 and 20 respectively, and the two nonparallel sides are 29 and 30. Find the altitude of the trapezoid and the area.

10. The sides of a trapezoid are 12, 20, 17, and 45. The sides 20 and 45 are the bases. Find the altitude and the area.

11. The sides of a trapezoid are 21, 27, 40, and 30. The sides 21 and 40 are parallel. Find the altitude and the area of the trapezoid.

12. The sides of a trapezoid are 23, 85, 100, and x. The sides 23 and 100 are the bases, and each is perpendicular to the side x. Find x and the area of the trapezoid.

13. The parallel sides of a trapezoid are 42 and 250. The other sides are 123 and 325. Find the altitude and the area of the trapezoid.

14. The area of a triangle is 1 square foot. The altitude on the first side is 16 inches. The second side is 14 inches longer than the third. Find the three sides.

CHAPTER XII

PROGRESSIONS

82. Definitions. In all fields of mathematics we frequently encounter groups of three or more numbers, selected according to some law and arranged in a definite order, whose relations to each other and to other numbers we wish to study.

The individual numbers or expressions are called *terms*.

In the following examples the law of formation and the order of the terms are so obvious that the student can write down many additional terms.

EXERCISES

Write three more terms in each of the following:

1. $1, 2, 3, 4, 5, \cdots.$

2. $2, 4, 6, 8, \cdots.$

3. $9, 8, 7, 6, \cdots.$

4. $-1, -3, -5, -7, \cdots.$

5. $1^2, 3^2, 5^2, \cdots.$

6. $\sqrt[3]{1}, \sqrt[3]{2}, \sqrt[3]{3}, \cdots.$

7. $2, 4, 8, 16, \cdots.$

8. $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$

9. $1, \frac{1}{1}, \frac{1}{1\cdot 2}, \frac{1}{1\cdot 2\cdot 3}, \frac{1}{1\cdot 2\cdot 3\cdot 4}, \cdots.$

10. $1, 1+\sqrt{2}, 1+2\sqrt{2}, \cdots.$

There is an unlimited variety of such groups or successions of numbers. Only two simple types will be considered here.

83. Arithmetical progression. An **arithmetical progression** is a succession of terms in which each term after the first, minus the preceding one gives the same number.

This same number is called the **common difference** and may be any positive or negative number.

The numbers 3, 7, 11, 15, $\cdots$ form an arithmetical progression, since any term after the first, minus the preceding one gives 4. Similarly, 12, 6, 0, -6, -12, $\cdots$ is an arithmetical progression, since any term minus the preceding one gives the common difference -6. In like manner, $\frac{7}{2}$, 5, $6\frac{1}{2}$, $\cdots$ is an arithmetical progression whose common difference is $1\frac{1}{2}$.

EXERCISES

From the following select the arithmetical progressions, and in each of them find the common difference:

1. $4, \frac{20}{3}, 9\frac{1}{3}, \cdots$
2. $10, 16\frac{1}{2}, 23, \cdots.$
3. $2, 4, 8, \cdots.$
4. $9, 12\frac{1}{2}, 15, \cdots.$
5. $25, 21, 17, \cdots.$
6. $18, 8, -2, \cdots.$
7. $5a + 2, 3a + 1, a, \cdots.$
8. $3x - 5, 2x + 8, x - 7, \cdots.$
9. $\frac{9}{\sqrt{3}}, \frac{6}{\sqrt{3}}, \sqrt{3}, \cdots.$
10. $\frac{\sqrt{3}-2}{3}, \frac{2(\sqrt{3}-1)}{3}, \frac{1}{\sqrt{3}}, \cdots.$

84. The last or *n*th term of an arithmetical progression. If a denotes the first term and d the common difference, any arithmetical progression is represented by

$$a,\ a + d,\ a + 2d,\ a + 3d,\ a + 4d, \text{ etc.}$$

Here one observes that the coefficient of d in each term is one less than the number of the term. Hence the nth or general term is $a + (n - 1)d$. If l denotes the nth term, we have

$$\boldsymbol{l = a + (n - 1)d.} \qquad (A)$$

EXERCISES

1. Find the 12th term of the progression 1, 5, 9, 13, $\cdots$.
2. Find the 23d term of the progression $-18, -15, -12, \cdots$.
3. Find the 15th term of the progression 13, 7, 1, -5, $\cdots$.
4. Find the 19th term of the progression $a, 3a, 5a, \cdots$.
5. Find the 7th and 12th terms of the progression $\frac{7}{3}, 1\frac{2}{3}, 1, \cdots$.

6. Find the 5th and 20th terms of the progression $1, a+1, 1+2a, \cdots$.

7. Find the 10th term of the progression $7\sqrt{2}, 5\sqrt{2}, 3\sqrt{2}, \cdots$.

8. Find the 9th term of the progression $\frac{\sqrt{3}+5}{2}, 2, \frac{3-\sqrt{3}}{2}, \cdots$.

9. Find the $(n-1)$st term of the progression $a, a+d, a+2d, \cdots$.

10. Find the $(n-2)$d term of the progression $a, a+d, a+2d, \cdots$.

11. Find the $(n-3)$d term of the progression $\sqrt{5}-1, 2\sqrt{5}-2, 3(\sqrt{5}-1), \cdots$.

12. Find the nth term of the progression $\frac{1}{n}, \frac{n-1}{n}, 2-\frac{3}{n}, \cdots$.

13. The first and second terms of an arithmetical progression are h and k respectively. Find the third term and the nth term.

14. The first and third terms of an arithmetical progression are h and k. Find the nth term.

15. A body falls 16 feet the first second, 48 the next, 80 the next, and so on. How far does it fall during the 10th second? during the nth second?

85. Arithmetical means. The arithmetical means between two numbers are numbers which form, with the two given ones as the first and the last terms, an arithmetical progression.

The insertion of one or more arithmetical means between two given numbers is performed as in the following:

Example: Insert three arithmetical means between 5 and 69.

Solution: $l = a + (n-1)d$.
There will be five terms in all.
Therefore $69 = 5 + (5-1)d$.
Solving, $d = 16$.
The required arithmetical progression is 5, 21, 37, 53, 69.

EXERCISES

1. Insert the arithmetical mean between 3 and 15.

2. Insert the arithmetical mean between h and $4k$.

3. Insert two arithmetical means between 2 and 17.

4. Insert two arithmetical means between a and b.

5. Insert three arithmetical means between -4 and 16.

6. Insert three arithmetical means between m and n.

7. Insert six arithmetical means between 3 and 45.

8. Insert nine arithmetical means between 3 and $\frac{14}{3}$.

9. Insert four arithmetical means between $-\sqrt{2}$ and $9\sqrt{2}$.

10. Insert five arithmetical means between $7x - 3a$ and $13x + 9a$.

11. Insert six arithmetical means between $\dfrac{5}{2\sqrt{5}}$ and $\dfrac{15\sqrt{5}}{2}$.

12. Insert two arithmetical means between $\dfrac{\sqrt{2}}{\sqrt{2}-1}$ and $\sqrt{2}(1 - 2\sqrt{2})$.

13. What is the arithmetical mean between any two numbers?

14. In going a distance of 1 mile an engine increased its speed uniformly from 20 miles per hour to 30 miles per hour. What was the mean or average velocity in miles per hour during that time? How long did it require to run the mile?

15. The velocity of a falling body increases uniformly. At the beginning of the third second its velocity is 64 feet per second, and at the end of the third second it is 96 feet per second. (*a*) What is its mean or average velocity in feet per second during the third second? (*b*) How many feet does it fall during the third second?

16. The velocity of a body falling from rest is 32 feet per second at the end of the first second. What is the mean or average velocity in feet per second during the first second? How many feet does the body fall during the first second? the second second?

17. Find the mean or average length of 25 lines whose lengths in inches are the first 25 even numbers.

18. Find the mean length of 17 lines whose lengths in inches are given by the consecutive odd numbers beginning with 11.

19. With the conditions of Problem 15 determine the average velocity per second of a body which has fallen for 10 seconds.

20. A certain distance is separated into 8 intervals, the lengths of which are in arithmetical progression. If the shortest interval is 1 inch and the longest 22 inches, find the others.

86. Sum of a series. The indicated sum of several terms of an arithmetical progression is called an **arithmetical series.** The formula for the sum of n terms of an arithmetical series may be obtained as follows:

$$S = a + (a + d) + (a + 2d) + \cdots + (l - 2d) + (l - d) + l. \quad (1)$$

Reversing the order of the terms in the second member of (1),

$$S = l + (l - d) + (l - 2d) + \cdots + (a + 2d) + (a + d) + a. \quad (2)$$

Adding (1) and (2),

$$2S = (a + l) + (a + l) + (a + l) + \cdots + (a + l) + (a + l) + (a + l)$$
$$= n(a + l).$$

Therefore
$$\boldsymbol{S = \frac{n}{2}(a + l).} \quad (B)$$

Substituting for l from (A), page 160,

$$S = \frac{n}{2}(a + [a + (n - 1)d]).$$

$$\boldsymbol{S = \frac{n}{2}[2a + (n - 1)d].} \quad (C)$$

EXAMPLE

Required the sum of the integers from 11 to 99 inclusive.

Solution: $n = 89$, $a = 11$, $l = 99$.

Substituting in (B), $S = \frac{n}{2}(a + l)$ gives $S = \frac{89(11 + 99)}{2} = 4895$.

Therefore the sum of the integers from 11 to 99 is 4895.

EXERCISES

1. Find the sum of 10 terms of the series $2 + 5 + 8 + \cdots$.

2. Find the sum of 18 terms of the series $10 + 8 + 6 + \cdots$.

3. Find the sum of 10 terms of the arithmetical progression $3, 4\frac{1}{3}, \frac{17}{3}, \cdots$.

4. Find the sum of 12 terms of the arithmetical progression $18, 14\frac{1}{2}, 11, \cdots$.

5. Find the sum of the first one hundred integers.

6. Find the sum of the first one hundred even numbers.

7. Find the sum of the first one hundred odd numbers.

8. Find the sum of the even numbers between 187 and 433.

9. Find the sum of the first n odd numbers.

10. Find the sum of the first n even numbers.

11. How many of the natural numbers beginning with 1 are required to make their sum 903?

HINT. Substitute in formula (C) preceding.

12. How many terms must constitute the series $5 + 9 + 13 + \cdots$ in order that it may amount to 275?

13. Beginning with 80 in the progression 78, 80, 82, how many terms are required to give a sum of 510? Explain.

14. The second term of an arithmetical progression is -7 and the seventh term is 18. Find the eleventh term.

15. Find the sum of t terms of the arithmetical progression $\frac{1}{t}, \frac{t-1}{t}, \cdots$.

16. If $l = 29$, $a = 2$, and $d = 3$, find n and s.

17. If $a = 3$, $d = 4$, and $s = 300$, find n and l.

18. If $d = -11$, $n = 13$, and $s = 0$, find a and l.

19. The first and second terms of an arithmetical progression are h and k respectively. Find the sum of n terms of the progression.

20. If $s = -33$, $a = 5x + 2$, and $n = 11$, find l and d.

21. If $s = 40\sqrt{2}$, $a = -5\sqrt{2}$, and $d = 2\sqrt{2}$, find n and l.

22. A clock strikes the hours but not the half hours. How many times does it strike in a day?

23. A car running 30 miles an hour is started up an incline, which decreases its velocity 2 feet a second. (*a*) In how many seconds will it stop? (*b*) How far will it go up the incline?

24. A car starts down a grade and moves 4 inches the first second, 12 inches the second second, 20 inches the third second, and so on. (*a*) How fast does it move in feet per second at the end of the twenty-first second? (*b*) How far has it moved in the twenty-one seconds?

25. An elastic ball falls from a height of 20 inches. On each rebound it comes to a point $\frac{1}{2}$ inch below the height reached the time before. How often will it drop before coming to rest? Find the total distance through which it has moved.

26. The digits of a 3-digit number are in arithmetical progression. The first digit is 2 and the number is $17\frac{1}{5}$ times the sum of its digits. Find the number.

27. A clerk received \$75 a month for the first year and a yearly increase of \$50 for the next ten years. Find his salary for the eleventh year and the total amount received.

28. Fifty dollars was deposited in a bank every first of March from February 28, 1893, to March 2, 1904. If the money drew simple interest at 3%, find the amount due the depositor on March 1, 1905.

29. Assuming that a ball is not retarded by the air, determine the number of seconds it will take to reach the ground if dropped from the top of the Washington Monument, which is 555 feet high. With what velocity will it strike the ground?

30. A ball thrown vertically upward rose to a height of 256 feet. In how many seconds did it begin to fall? With what velocity was it thrown?

31. A ball thrown vertically upward returned to the ground 7 seconds later. How high did it rise? With what velocity was it thrown?

32. A pyramid of billiard balls stands on an equilateral triangle, 10 balls on a side. How many balls are there in the bottom layer? in the whole pyramid?

33. A and B start from the same place at the same time and travel in the same direction. A travels 12 miles daily. B goes 7 miles the first day, $7\frac{1}{2}$ miles the second, 8 miles the third, and so on. When are they together?

34. A leaves P and travels south 2 miles the first day, 4 the second, 6 the third, and so on. Five days later B leaves P and travels south at the uniform rate of 28 miles a day. When are they together?

Note. In the earliest mathematical work known a problem is found which involves the idea of an arithmetical progression. In the papyrus of the Egyptian priest Ahmes, who lived nearly two thousand years before Christ, we read in essence, "Divide 40 loaves among 5 persons so that the numbers of loaves that they receive form an arithmetical progression, and so that the two who receive the least bread, together have one seventh as much as the others." From that time to this, the subject has been a favorite one with mathematical writers, and has been extended so widely that it would require several volumes to record all of the discoveries regarding the various kinds of series.

87. Geometrical progression. A **geometrical progression** is a succession of terms in which each term after the first, divided by the preceding one always gives the same number.

The constant quotient is called the **ratio.**

The numbers $2, 10, 50, 250, \cdots$, form a geometrical progression, since any term after the first, divided by the preceding one gives the same number 5. Similarly, the numbers $3, -3\sqrt{2}, 6, -6\sqrt{2}, \cdots$, form a geometrical progression, since any term after the first, divided by the preceding one gives the common ratio $-\sqrt{2}$.

EXERCISES

Determine which of the following are geometrical progressions and find in each case the corresponding ratio:

1. $2, 6, 18, \cdots$.
2. $15, 5, 1, \cdots$.
3. $18, -3, \frac{1}{2}, \cdots$.
4. $2, 4, 16, \cdots$.
5. $8, -4\sqrt{2}, 4, \cdots$.
6. $\sqrt{2}, \sqrt{\frac{1}{2}}, \frac{1}{4}\sqrt{2}, \cdots$.
7. $1, 3, 9, 81, \cdots$.
8. $\sqrt{\frac{2}{3}}, \sqrt{6}, \sqrt{54}, \cdots$.
9. $\frac{1}{\sqrt{8}}, -\frac{\sqrt{2}}{2}, 2, \cdots$.
10. $7a, 35a^2, 175a^3, \cdots$.
11. $8\sqrt{5}, -2\sqrt{5}, \sqrt{5}, \cdots$.
12. $5x^2, 10x^2y, 20x^2y^2, \cdots$.
13. $3xy^{\frac{1}{2}}, 12x^{\frac{1}{2}}y, 48y^{\frac{3}{2}}, \cdots$.

14. Find the condition under which a, b, and c form a geometrical progression.

88. The *n*th term of a geometrical progression. If a denotes the first term and r the ratio, any geometrical progression is represented by $a, ar, ar^2, ar^3, \cdots$. It is evident that the exponent of r in any term is one less than the number of the term. Therefore if t_n denotes the nth or general term of any geometrical progression,

$$t_n = ar^{n-1}. \qquad (A)$$

EXERCISES

1. Find the fifth term of 4, 12, 36.

Solution: Here $a = 4$, $r = 3$, $n - 1 = 4$.

Substituting these values in the formula $t_n = ar^{n-1}$,

$$t_5 = 4 \cdot 3^4 = 324.$$

2. Find the tenth term of $3, 6, 12, \cdots$.

3. Find the eighth term of $2, 3, \frac{9}{2}, \cdots$.

4. Find the twelfth term of $5, -10, 20, \cdots$.

5. Find t_6 of the geometrical progression \$100, \$106, \$112.36, $\cdots$.

6. Find t_9 of the geometrical progression $18, -6, +2, \cdots$.

7. Find t_{10} of the geometrical progression $12a, 9a, \frac{27a}{4}, \cdots$.

8. Find t_7 of the geometrical progression $\frac{-2c}{3}, -1, \frac{-3}{2c}, \cdots$.

9. Find t_7 of the geometrical progression $4\sqrt{2}, 4, 2\sqrt{2}$.

10. Find t_6 of the geometrical progression $\frac{1}{2\sqrt{2}}, \frac{1}{6}, \frac{\sqrt{2}}{18}$.

11. Find t_8 of the geometrical progression $\frac{3}{2\sqrt{2}}, 1, \frac{2\sqrt{2}}{3}$.

12. The nth term of a geometrical progression is ar^{n-1}. What is the $(n-1)$st term? the $(n-2)$d? the $(n-3)$d? the $(n+1)$st? the $(n+2)$d?

13. The first and second terms of a geometrical progression are h and k respectively. Find the next two terms.

89. Geometrical means. Geometrical means between two numbers are numbers which form, with the two given ones as the first and the last terms, a geometrical progression.

EXERCISES

1. Insert two geometrical means between 9 and 72.

Solution: There are four terms in the geometrical progression, $a = 9$, $n = 4$, and $t_n = t_4 = 72$.

Substituting these values in $t_n = ar^{n-1}$,

$$72 = 9r^3.$$

Whence $$r = 2.$$

The required geometrical progression is 9, 18, 36, 72.

2. Insert two geometrical means between 6 and 48.

3. Insert three geometrical means between 6 and 486.

4. Insert one geometrical mean between 4 and 9.

5. Insert one geometrical mean between a^{10} and a^{20}.

6. Insert three geometrical means between -144 and -9.

7. The fifth term of a geometrical progression is 32, the ninth term is 512. Find the eleventh term.

8. The second term of a geometrical progression is $3\sqrt{2}$, the fifth term is $\frac{3}{16}$. Find the first term and the ratio.

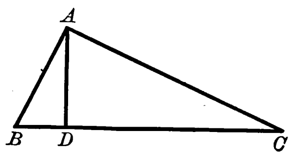
9. Show that the geometrical means between h and k are $\pm\sqrt{hk}$.

10. The first and fourth terms of a geometrical progression are h and k. Find the second and third terms.

11. Insert three geometrical means between a and c.

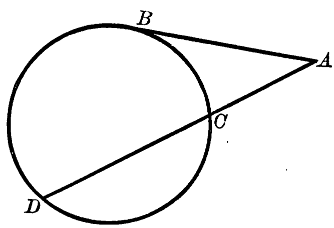
12. The sum of the first and third terms of a geometrical progression is 13 and the second term is 6. Find each term.

13. In the adjacent figure ABC is a right triangle and AD is perpendicular to the hypotenuse BC. Under these conditions the length of AD is *always* a geometric mean between the lengths of BD and DC.



(*a*) If $BD = 4$ and $DC = 9$, find AD.
(*b*) If $BC = 26$ and $AD = 12$, find BD and DC.

14. In the adjacent figure AB touches and AD cuts the circle. Under such conditions the length of AB is *always* a geometric mean between the lengths of AC and AD.



(*a*) If $AD = 16$
and $AC = 9$, find AB.
(*b*) If $DC = 24$ and $AB = 16$, find AC and AD.

90. Geometrical series. Let S_n denote the indicated sum of n terms of a geometrical progression. This indicated sum is called a **geometrical series.** Obtaining in its simplest form the expression for this sum is often called *finding the sum of the series.*

The expression for the sum is derived as follows:

$$S_n = a + ar + ar^2 + \cdots + ar^{n-3} + ar^{n-2} + ar^{n-1}. \quad (1)$$

$$(1)\cdot r, \quad rS_n = ar + ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-1} + ar^n. \quad (2)$$

The terms ar, ar^2, etc., up to ar^{n-1} in the right member of (1), occur in the right member of (2). Hence if (2) be subtracted from (1), all these terms vanish, leaving only a and ar^n.

(1) – (2), $\quad S_n - rS_n = a - ar^n.$

Whence $\quad S_n(1-r) = a - ar^n,$

and $$\mathbf{S_n = \frac{a - ar^n}{1 - r}}. \quad (B)$$

EXERCISES

1. Find the sum of the first ten terms of 5, – 10, 20, ⋯.

Solution: $\quad S_n = \dfrac{a - ar^n}{1 - r}.$

By the conditions, $a = 5$, $r = -2$, and $n = 10$.

Substituting, $\quad S_{10} = \dfrac{5 - 5(-2)^{10}}{1-(-2)} = 1705.$

2. Find the sum of 1, 5, 25, ⋯ to seven terms.

3. Find S_7 for the progression $-2, 4, -8, \cdots$.

4. Find S_8 for the progression 50, 10, 2, ⋯.

5. Find S_9 for the progression 180, – 90, 45, ⋯.

6. Find S_5 for the progression $\frac{2}{5}, 1, \frac{5}{2}, \cdots$.

7. Find S_7 for the progression $c^3, c^5, c^7, \cdots$.

8. Find S_5 for the progression $3\sqrt{2}, 6, 6\sqrt{2}, \cdots$.

9. Find S_6 for the progression $81, -27\sqrt{3}, 27, \cdots$.

10. Find S_n for the progression 3, 15, 75, ⋯.

11. Find S_{n-2} for the progression $2x, 4x^4, 8x^7, \cdots$.

12. Show that for a geometrical progression $S_n = \dfrac{a - rl}{1 - r}$.

13. What will \$100 amount to in three years, interest 4%, compounded annually? compounded semiannually?

14. A rubber ball falls from a height of 40 inches and on each rebound rises 40% of the previous height. How far does it fall on its sixth descent? Through what distance has it moved at the end of the sixth descent?

15. A vessel containing wine was emptied of one third of its contents and then filled with water. This was done six times. What portion of the original contents was then in the vessel?

16. At each stroke an air pump withdraws 40 cubic inches of the contents of a bell jar whose capacity is 400 cubic inches. After every stroke the air remaining in the jar expands and completely fills it. What portion of the original quantity of air remains in the jar at the end of the tenth stroke?

91. Infinite geometrical series. If the number of terms of a geometrical series is unlimited, it is called an infinite geometrical series.

In the progression 2, 4, 8, $\cdots$ the ratio is positive and greater than 1, and each term is greater than the term preceding it. Such a progression is said to be increasing. Obviously the sum of an unlimited number of terms of an increasing geometrical progression is unlimited. In other words, by taking enough terms the sum can be made as large as we please.

In the progression 3, $\frac{3}{2}$, $\frac{3}{4}$, $\cdots$ the ratio is positive and less than 1, and each term is less than the term preceding it. Such a progression is said to be decreasing. Though the number of terms of such a geometrical progression be unlimited, the sum is limited; that is, the sum of as many terms as we choose to take is always *less* than some definite number. The sum of the first 3 terms of the series $4+2+1+\frac{1}{2}+\frac{1}{4}+\cdots$ is 7; of 4 terms is $7\frac{1}{2}$; of 5 terms is $7\frac{3}{4}$; of 6 terms is $7\frac{7}{8}$; of 7 terms is $7\frac{15}{16}$. Here, for any number of terms, the sum is always less than 8.

The formula $$S_n=\frac{a-ar^n}{1-r} \qquad (1)$$

may be written $$S_n=\frac{a}{1-r}-\frac{ar^n}{1-r}. \qquad (2)$$

For the series $3 + \frac{3}{2} + \frac{3}{4} + \cdots$,

$$S_n = \frac{3}{1-\frac{1}{2}} - \frac{3(\frac{1}{2})^n}{1-\frac{1}{2}}. \tag{3}$$

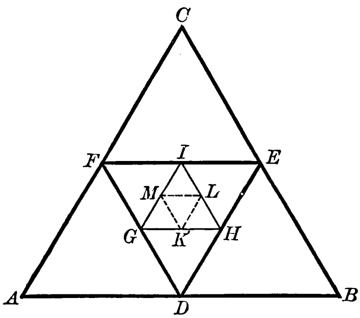
Now $(\frac{1}{2})^2 = \frac{1}{4}$, $(\frac{1}{2})^3 = \frac{1}{8}$, $(\frac{1}{2})^4 = \frac{1}{16}$, $(\frac{1}{2})^5 = \frac{1}{32}$. Consequently $(\frac{1}{2})^n$ becomes very small if n is taken very great. Therefore $3(\frac{1}{2})^n$, the numerator of the last fraction in (3), decreases and approaches zero as n increases without limit. And hence as the denominator of the fraction remains $\frac{1}{2}$ while the numerator approaches zero, the value of the fraction decreases and approaches zero as n increases. Then if S_∞ denotes S_n, where n has increased without limit, we may write

$$S_\infty \text{ approaches } \frac{3}{1-\frac{1}{2}} \text{ or } 6.$$

This means that, though n be very large, the sum of the series $3 + \frac{3}{2} + \frac{3}{4} + \cdots$ is always slightly less than 6.

The following is a geometrical illustration of the preceding series:

In the adjacent figure triangles ABC, DEF, GHI, etc., are equilateral. DEF is formed by joining the middle points of the sides of ABC, etc. Imagine this process continued until an unlimited number of triangles is so formed. Now FE is $\frac{1}{2}$ of AB, GH is $\frac{1}{2}$ of FE, ML is $\frac{1}{2}$ of GH, etc. Therefore, if $AB = 1$, $FE = \frac{1}{2}$, $GH = \frac{1}{4}$, $ML = \frac{1}{8}$, etc. Hence the perimeter of ABC is 3; of DEF, $\frac{3}{2}$; of GHI, $\frac{3}{4}$; etc. Thus the perimeters of the successive triangles form the progression $3, \frac{3}{2}, \frac{3}{4}, \cdots$, the limit of whose sum was found to be 6.



In the general case, if r is numerically less than 1, the numerical value of fraction $\frac{ar^n}{1-r}$ approaches zero as n increases without limit. Under such conditions the formula

$$S_n = \frac{a}{1-r} - \frac{ar^n}{1-r} \text{ becomes } S_\infty = \frac{a}{1-r}.$$

This means that for r numerically less than 1, S_n approaches $\frac{a}{1-r}$; but for any definite value of n it is always numerically less than this number.

Hence whenever we speak of the sum of such a series we mean the *limit* which the sum approaches as n increases indefinitely.

EXERCISES

Find the number which the sum of the first n terms of each of the following approaches as n increases without limit:

1. $3, 1, \frac{1}{3}, \cdots$.

Solution: $$S_\infty = \frac{a}{1-r}.$$

Substituting, $$S_\infty = \frac{3}{1-\frac{1}{3}} = 4\tfrac{1}{2}.$$

2. $1, \frac{1}{2}, \frac{1}{4}, \cdots$.

3. $2, -1, \frac{1}{2}, \cdots$.

4. $3, -1, \frac{1}{3}, \cdots$.

5. $2, \sqrt{2}, 1, \cdots$.

6. $5a, \frac{5a}{4}, \frac{5a}{16}, \cdots$.

7. $1, x, x^2, \cdots, (x < 1)$.

8. $3, \sqrt{3}, 1, \cdots$.

9. $1, \frac{1}{x}, \frac{1}{x^2}, \cdots, (x > 1)$.

10. $.515151\cdots$. Hint. $.515151 = \frac{51}{100} + \frac{51}{10000} + \frac{51}{1000000} + \cdots$.

11. $.666\cdots$.

12. $.272727\cdots$.

13. $.3939\cdots$.

14. $25.3636\cdots$.

15. $.72121\cdots$.

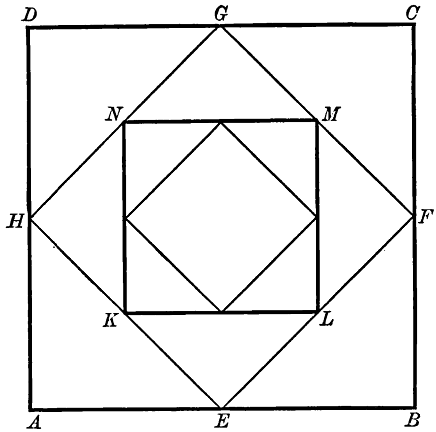
16. $.3091091\cdots$.

17. A flywheel whose perimeter is 5 feet makes 80 revolutions per second. If it makes 99% as many revolutions each second thereafter as it did the preceding second, how far will a point on its rim have moved by the time it is about to stop?

18. The area of the triangle ABC (page 172) is $\frac{9}{4}\sqrt{3}$; of triangle DEF, $\frac{9}{16}\sqrt{3}$; of triangle GHI, $\frac{9}{64}\sqrt{3}$, etc. Find the sum of the areas of all the triangles drawn as there supposed.

19. The square $EFHG$ is formed by joining the middle points of the adjacent sides of the square $ABCD$ on page 174.

If an unlimited number of squares is so formed, the perimeters of the squares will form a geometrical progression the first three terms of which may be obtained as follows: Let $AB = 2$; then $EB = BF = 1$. EF can then be found from right triangle EBF; EL is $\frac{1}{2}$ of EF, and $EK = EL$. Then KL can be found from the right triangle KEL. The perimeters of the first three squares can then be found.



(*a*) Show that the limit of the sum of the perimeters of all of the squares is

$$16 + 8\sqrt{2}.$$

(*b*) Show that the limit of the sum of the areas of all the squares is 8 square units.

20. A loan of S dollars is to be repaid in four equal annual payments of p dollars each. Find p if money is worth $r\%$.

Solution: The sum due at beginning of second year

$$= S(1 + r) - p. \quad (1)$$

The sum due at beginning of third year

$$= [S(1 + r) - p](1 + r) - p. \quad (2)$$

The sum due at beginning of fourth year

$$= \{[S(1 + r) - p](1 + r) - p\}(1 + r) - p. \quad (3)$$

The sum due at beginning of fifth year

$$= [\{[S(1 + r) - p](1 + r) - p\}(1 + r) - p](1 + r) - p. \quad (4)$$

By the conditions of the problem, $(4) = 0$, for all the debt has then been paid. Setting (4) equal to zero and simplifying,

$$S(1 + r)^4 - p(1 + r)^3 - p(1 + r)^2 - p(1 + r) - p = 0. \quad (5)$$

Solving (5) for p,

$$p = \frac{S(1 + r)^4}{(1 + r)^3 + (1 + r)^2 + (1 + r) + 1}. \quad (6)$$

But the denominator in (6) is a geometrical series whose sum by formula (*B*), page 170, is $\frac{(1 + r)^4 - 1}{r}$.

Substituting this last in (6), $p = \dfrac{Sr(1+r)^4}{(1+r)^4 - 1}$. (7)

In the general case, if we have n annual payments, the exponent 4 in (7) would be replaced by n, and then $p = \dfrac{Sr(1+r)^n}{(1+r)^n - 1}$.

21. A loan of $1000 is to be repaid in three equal annual payments, interest at 5%. Find the payment.

22. A loan of $5000 bearing interest at 6% is to be repaid in five equal annual payments. Find the payment.

Note. In the study of geometrical progressions we have seen that the sum of the infinite series $1 + x^2 + x^3 + x^4 + \cdots$ is a definite number when x has any value less than one. But it has no finite value when x is equal to or greater than one; that is, we have an expression which we cannot use arithmetically unless x has a properly chosen value. If we were studying some problem which involved such a series, it would be a matter of the most vital importance to know whether the values of x under discussion were such as to make the series meaningless.

This question of distinguishing between expressions the sum of whose terms approach a limit or converge, and those which do not, has an interesting history. Newton and his followers in the seventeenth century dealt with infinite series and always assumed that they converged, as, in fact, most of them did. But as more complicated series came into use it became more difficult to tell from inspection whether they meant anything or not for a given value of the variable.

It was not until the beginning of the nineteenth century that Gauss, Abel, and Cauchy, in Germany, Norway, and France, respectively began to study this subject effectively, and to devise far-reaching tests to determine the values of x for which certain series converge to a finite limit. It is said that on hearing a discussion by Cauchy in regard to series which do not always converge, the astronomer La Place became greatly alarmed lest he had made use of some such series in his great work on Celestial Mechanics. He hurried home and denied himself to all distractions until he had examined every series in his book. To his intense satisfaction they all converged. In fact, it has often been observed that a genius can safely take chances in the use of delicate processes, which seem very foolish and unsafe to a man of ordinary ability.

CHAPTER XIII

LIMITS AND INFINITY

92. Limits. The numerical value V of the recurring decimal $.666\cdots$ is a variable depending on the number of 6's annexed on the right. Every 6 thus repeated increases V, and the number of 6's which may be so repeated is unlimited. Still V always remains less than $\frac{2}{3}$, though constantly approaching nearer and nearer to that value. Here the fraction $\frac{2}{3}$ is called the **limit** of the variable V.

93. Definition of a limit. If a variable V takes on successively a series of values that approach nearer and nearer to a fixed number L in such a manner that the numerical value of $V - L$ becomes and remains as small as we please, then V is said to approach the *limit* L.

This may be written limit of $V = L$.

The symbol $\doteq$ gives us the equivalent notation $V \doteq L$, which is read *V approaches L as a limit.*

94. Infinity. If a variable n takes on in succession all the values $1, 2, 3, 4, \cdots$, we can conceive of no final value for n, since the system of natural numbers is unlimited. Here we may say *n increases without limit*, or n becomes **infinite.**

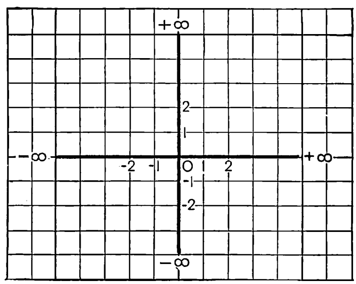
95. Definition of the term "infinite." If a variable n becomes and remains greater than any positive number k, however great, we say *n increases without limit*, or n becomes *infinite.*

The usual symbol for a variable which has become infinite is the sign ∞, read **infinity.**

Infinity is not a number in the sense in which 2, $\sqrt{6}$, and -7 are numbers. It is greater than any number. For present purposes it must be regarded as a manner of speech rather than as a number that can be added, subtracted, multiplied, or divided, as ordinary numbers are. In fact, we cannot operate with the symbol ∞ as we can with numbers.

Note. Some idea of the reason why we cannot regard ∞ as a number, and operate with it as we do with ordinary numbers, may be seen if we consider all even numbers, 2, 4, 6, 8, 10, $\cdots$. Evidently they may be continued as far as we wish, but the number of them all cannot be expressed by any integer, for it is greater than any number; that is, it is infinite. But the number of all integers, both odd and even, was also called infinite, and we symbolize both infinities by the same sign, ∞. We have, then, two infinities which are equal, or at least they are represented by the same symbol, but one contains the other. This is contrary to the axiom which we always assume for finite numbers; namely, that the whole is greater than any of its parts. Surely it is not strange that we cannot operate freely with a symbol which violates this fundamental principle.

We may have a negative infinity as well as a positive one. In order to indicate the range of values which both x and y may take in graphical work, the axes are often marked as in the adjacent figure.



A *constant* number, however large, is never spoken of as infinite.

If the variable n in $\frac{1}{n}$ takes on in succession the values 1, 2, 3, 4, $\cdots$, no final value of $\frac{1}{n}$ can be imagined. But as n increases without limit $\frac{1}{n}$ becomes very small and approaches nearer and nearer to zero without actually becoming zero.

In general, if a in the fraction $\frac{a}{n}$ is any constant not zero, and n a variable increasing without limit, $\frac{a}{n}$ approaches zero as a limit. Unfortunately, in elementary mathematics there is not in general use a symbol for a variable whose limit is zero, though such a symbol would be a great convenience.

The student will frequently meet the statement $\frac{a}{\infty}$ equals zero. This statement is, of course, meaningless until it has been defined, but it may properly be regarded as a way of saying that $\frac{a}{n}$ approaches zero as a limit when n is indefinitely increased.

96. Interpretation of $\frac{a}{0}$. Division by zero is excluded from mathematics for two reasons: (*a*) It is never necessary. (*b*) It would give rise to endless ambiguities and difficulties.

Results of the form $\frac{a}{0}$, where a is a constant not zero, frequently arise. According to the rules of computation, however, such an expression has no meaning. Though it is true that $\frac{a}{0}$ is not a definite number, results of this form may sometimes admit of interpretation.

As an illustration of this, consider the following

EXAMPLE

Solve by determinants the system $\begin{cases} x - 2y = 1, & (1) \\ \frac{1}{2}x - y = 2. & (2) \end{cases}$

Solution: $x = \dfrac{\begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & -2 \\ \frac{1}{2} & -1 \end{vmatrix}} = \dfrac{-1+4}{1-1} = \dfrac{3}{0},$

and $y = \dfrac{\begin{vmatrix} 1 & 1 \\ \frac{1}{2} & 2 \end{vmatrix}}{0} = \dfrac{2-\frac{1}{2}}{0} = \dfrac{1.5}{0}.$

The graphs of (1) and (2) are parallel lines. For such lines there is no point of intersection and consequently the system has no set of roots. Now as the results for x and y are of the form $\frac{a}{0}$, the attempt at solution by determinants fails

Therefore the interpretation of these results is that no set of roots exists for the system (1), (2).

In general, if for any system of linear equations the results obtained are of the form $\frac{a}{0}$, the system has no set of roots; that is, the system is inconsistent.

If the student meets the statement $\frac{a}{0} = \infty$, he should regard it as a loose use of the statement that $\frac{a}{n}$ becomes infinite as n approaches 0.

97. Interpretation of $\frac{0}{0}$. The fraction $\frac{x-2}{x^2-4}$ becomes $\frac{0}{0}$ when $x = 2$. For any value of x other than the critical value 2, the fraction equals a definite number. Usually we are concerned with the limit of such expressions as the variable approaches a critical value. The limit for the fraction $\frac{x-2}{x^2-4}$ is easily found. We assign to x successively the values 1.9, 1.99, 1.999, 1.9999, etc. The corresponding values of the fractions are $\frac{10}{39}$, $\frac{100}{399}$, $\frac{1000}{3999}$, $\frac{10000}{39999}$, etc. Obviously these numbers approach the limit $\frac{1}{4}$.

We may arrive at this result more easily as follows: For all values of x except 2 the terms of the fraction $\frac{x-2}{x^2-4}$ may be divided by $x - 2$, obtaining $\frac{1}{x+2}$. This result is true, however little x may differ from 2. Now if, without giving x the value 2, we make it approach 2 as a limit, $\frac{1}{x+2}$ will approach $\frac{1}{4}$ as a limit, and this is the limit of the original fraction $\frac{x-2}{x^2-4}$ as well.

By either of the preceding methods it can be shown that $\frac{x^2-9}{x-3}$, which becomes $\frac{0}{0}$ for $x = 3$, has 6 as its limit. These two fractions are simple illustrations of the important fact that the symbol $\frac{0}{0}$ is not a definite number. The truth of this can be seen more clearly from a study of the graph on the following page.

Let $$y = \frac{x^2 - 9}{x - 3}. \qquad (1)$$

Then $$y(x-3) = (x+3)(x-3). \qquad (2)$$

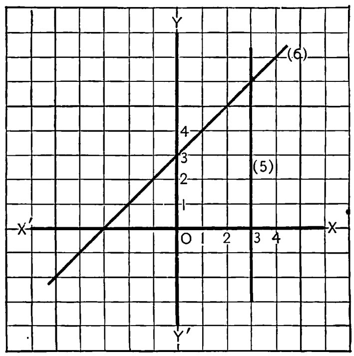
$$y(x-3) - (x+3)(x-3) = 0. \qquad (3)$$

$$(y - x - 3)(x - 3) = 0. \qquad (4)$$

Therefore $$x - 3 = 0, \qquad (5)$$

and $$y - x - 3 = 0. \qquad (6)$$

The graphs of (5) and (6) are given in the following figure. To understand what follows, it must be remembered that y and the fraction $\frac{x^2-9}{x-3}$ are *identical*, and that the graphs (5) and (6) are the complete graph of the equation (1).



For every value of x except 3 there is always one value of y, and that value is the y-distance of some point on line (6). For $x = 3$, however, the value of y is the y-distance of *any* point on line (5). Hence the fraction $\frac{x^2-9}{x-3}$ is indeterminate for $x = 3$.

It is worth noting that the limiting value of the fraction $\frac{x^2-9}{x-3}$ is seen from the graph to be 6, the y-value of the point of intersection of lines (5) and (6).

Note. The study of the limiting value of the ratio of two functions which for certain values of the variable takes on the indeterminate form 0/0 was undertaken by the Frenchman, L'Hospital, in 1696, and was carried further by John Bernouilli a few years later. A complete comprehension of the difficulties which surround this subject has been very slowly gained by mathematical writers, and even to-day it is possible to find books in which grave errors are made regarding the meaning of these expressions.

The questions involved are closely related to those regarding the nature of the infinite in mathematics. The penetration of this mystery is one of the great achievements of the latter half of the nineteenth century, and to-day well-informed mathematicians have as clear and satisfactory ideas about infinite numbers as they do about ordinary integers.

As a final illustration that $\frac{0}{0}$ may have any value, consider the following

EXAMPLE

Solve by determinants the system $\begin{cases} x - y = 1, & (1) \\ -3x + 3y = -3. & (2) \end{cases}$

Solution: $x = \dfrac{\begin{vmatrix} 1 & -1 \\ -3 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ -3 & 3 \end{vmatrix}} = \dfrac{3-3}{3-3} = \dfrac{0}{0}.$

$y = \dfrac{\begin{vmatrix} 1 & 1 \\ -3 & -3 \end{vmatrix}}{0} = \dfrac{-3+3}{0} = \dfrac{0}{0}.$

Now the graphs of equations (1) and (2) are coincident lines. Therefore any set of values of x and y which satisfies (1) will also satisfy (2), a condition indicated by the indeterminate result $\frac{0}{0}$ for the unknowns.

In general, if the solution of a system of linear equations in two or more unknowns gives $\frac{0}{0}$ as values of the unknowns, the system has an infinite number of sets of roots; that is, the system is indeterminate.

The symbol $\frac{0}{0}$ then is a symbol of indetermination.

EXERCISES

Solve by determinants and interpret results:

1. $x - y = 1,$ $2y - 2x = -2.$

2. $x - y = 0,$ $y - x = 3.$

3. $x + y + z = 0,$ $x - 2y + 3z = 1,$ $2x - y + 4z = 1.$

4. $x + y + z = 1,$ $x - y - 2z = 2,$ $0x + 0y + 0z = 0.$

5. From the results obtained in Exercise 4 what conclusion is warranted regarding the number of sets of roots belonging to a system of *two* linear equations in *three* variables?

6. $x + y + z = 2,\ 0x + 0y + 0z = 0,\ 0x + 0y + 0z = 0.$

7. What do the results obtained in Exercise 6 show in regard to the number of sets of roots belonging to *one* equation in *three* variables ?

What limit does each of the following expressions approach as n becomes ∞ ?

8. $\frac{1}{n}$. **10.** $\frac{3}{n}$. **12.** $\frac{n}{n+2}$. **14.** $\frac{4}{\frac{1}{n}}$.

9. $\frac{2}{n}$. **11.** $\frac{n}{n+1}$. **13.** $\frac{n+1}{n}$.

15. $\frac{n(n+1)}{n^2}$. **16.** $\frac{n(n+1)(n+2)}{n^3}$.

What limit does each of the following expressions approach as $n \doteq 0$?

17. $\frac{1}{n}$. **18.** $\frac{4}{\frac{1}{n}}$. **19.** $\frac{6}{\frac{1}{n}}$. **20.** $2n^2$. **21.** $\frac{n}{n^2}$.

Find the limit of :

22. $\frac{1-x}{1-x^2}$ as $x \doteq 1$. **23.** $\frac{x^2-5x+6}{x^2-4}$ as $x \doteq 2$.

24. $\frac{x-2}{x^3-8}$ as $x \doteq 2$.

CHAPTER XIV

LOGARITHMS

98. Introduction. **Logarithms** were invented to shorten the work of extended numerical computations which involve one or more operations of multiplication, division, involution, and evolution. They have decreased the labor of computing to such an extent that many calculations which would require hours without logarithms can be performed by their aid in one tenth of that time.

A logarithm is an exponent. A table of common logarithms is a table of exponents of the number 10. The greater portion of these exponents are approximate values of irrational numbers. It follows, then, that computation by means of logarithms gives only approximate results. Tables exist, however, in which each logarithm is given to twenty or more decimals; hence practically any desired degree of accuracy can be obtained by using the proper table.

It can be proved that the laws given on pages 89–90, governing the use of rational exponents, hold for irrational exponents. In the work on logarithms this fact will be assumed.

99. Graphical explanation of logarithms. The theory of computation by logarithms is simple, yet considerable time is needed to master its practical details. These details and the fact that a logarithm is an exponent will be grasped more readily if the student gets from a graph a first view of the meaning and use of logarithms. For this we shall construct the graph of the logarithmic or exponential equation,

$$N = 10^L.$$

In this equation N represents any positive number, and L, the exponent of 10, is its common logarithm.

It will be more convenient to assign values to L and compute the corresponding values of N, than to use the reverse process. Moreover, we shall restrict L to values from $+1$ to -1 inclusive, and to such fractional values that N can be obtained by the use of square root.

First, $10^1 = 10,\ 10^0 = 1,$ and $10^{-1} = .1,$

and $10^{\frac{1}{2}} = \sqrt{10} = 3.16227 +.$

Also $10^{\frac{1}{4}} = (10^{\frac{1}{2}})^{\frac{1}{2}} = \sqrt{\sqrt{10}} = \sqrt{3.16227} = 1.778 +.$

Similarly, $10^{\frac{1}{8}} = (10^{\frac{1}{4}})^{\frac{1}{2}} = \sqrt[2]{\sqrt[4]{10}} = \sqrt[2]{1.778} = 1.33 +.$

Now $10^{\frac{3}{4}} = (10^{\frac{1}{2}})(10^{\frac{1}{4}}) = (3.16227)(1.778) = 5.62 +.$

Again, $10^{\frac{3}{8}} = (10^{\frac{1}{4}})(10^{\frac{1}{8}}) = (1.778)(1.33) = 2.37 +.$

In like manner,

$$10^{\frac{5}{8}} = (10^{\frac{3}{8}})(10^{\frac{1}{4}}) = 4.21 +.$$

Lastly, $10^{\frac{7}{8}} = (10^{\frac{1}{8}})(10^{\frac{3}{4}}) = 7.49 +.$

Tabulating the values of N and L just obtained, gives

L	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$	1
N	1	1.33	1.78	2.37	3.16	4.21	5.62	7.49	10

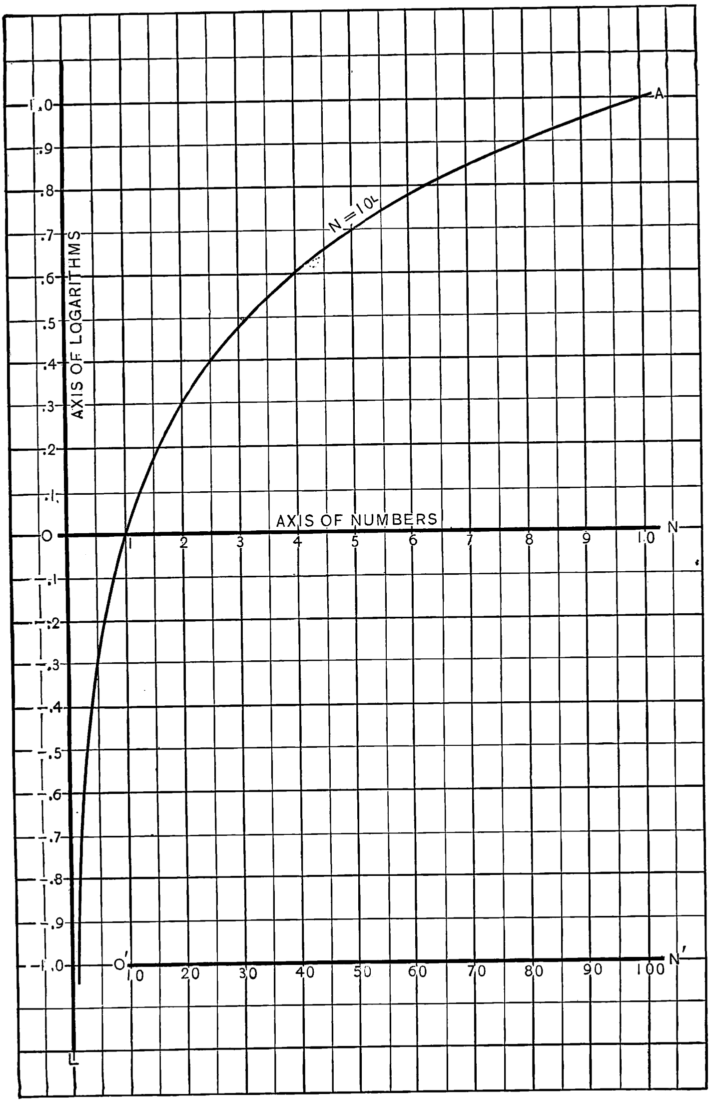
Since $$10^{-L} = \frac{1}{10^L},$$

the value of $$10^{-\frac{7}{8}} = \frac{1}{10^{\frac{7}{8}}} = \frac{10^{\frac{1}{8}}}{10^{\frac{7}{8}} \cdot 10^{\frac{1}{8}}} = \frac{10^{\frac{1}{8}}}{10} = .133.$$

Similarly, $$10^{-\frac{3}{4}} = \frac{1}{10^{\frac{3}{4}}} = \frac{10^{\frac{1}{4}}}{10} = .1778.$$

In this manner we obtain from the preceding table the following one for negative values of L between 0 and -1.

L	$-\frac{1}{8}$	$-\frac{1}{4}$	$-\frac{3}{8}$	$-\frac{1}{2}$	$-\frac{5}{8}$	$-\frac{3}{4}$	$-\frac{7}{8}$	-1
N	.749	.562	.421	.316	.237	.178	.133	.1


A
N = 10^L
AXIS OF LOGARITHMS
AXIS OF NUMBERS
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
O
-.1
-.2
-.3
-.4
-.5
-.6
-.7
-.8
-.9
-1.0
1
2
3
4
5
6
7
8
9
10
N
O'
10
20
30
40
50
60
70
80
90
100
N'
L

From the values of L and N in the foregoing tables the logarithmic (or exponential) curve on the preceding page is constructed. As only positive values of N are considered, the curve never reaches the L-axis. The approximate values of L for numbers from .1 to 10 are measured from ON to the curve.

From the curve, $\log 2 = .3$; that is, $2 = 10^{.3}$.

Then $20 = 10 \cdot 2 = 10^{1} \cdot 10^{.3} = 10^{1.3}$.

Thus the logarithm of 20 is 1 greater than the logarithm of 2. Similarly, the logarithm of 30 is 1 greater than the logarithm of 3, and so on.

Therefore, if line $O'N'$ be drawn one unit below ON, the logarithms of numbers from 10 to 100 are the values of distances to the curve from points on $O'N'$ which correspond to these numbers. This practically gives us a considerable portion of the curve beyond point A.

EXERCISES

Find from the curve the logarithm of:

1. 2.	**6.** 7.4.	**11.** .5.	**16.** 96.
2. 3.	**7.** 9.6.	**12.** .9.	**17.** 100.
3. 4.	**8.** 11.	**13.** 25.	**18.** 10.
4. 5.	**9.** 15.	**14.** 50.	**19.** 1.
5. 6.2.	**10.** .2.	**15.** 64.	**20.** 32.

Find from the curve the number whose logarithm is:

21. .3.	**25.** .95.	**29.** $-.25$.	**33.** 1.8.
22. .4.	**26.** $-.1$.	**30.** 1.3.	**34.** 2.
23. .6.	**27.** $-.4$.	**31.** 1.7.	**35.** .84.
24. .7.	**28.** $-.5$.	**32.** 1.6.	**36.** 1.2.

The preceding exercises should familiarize the student with the meaning of the curve. We shall now use it to explain logarithmic multiplication, division, raising to a power, and extracting a root. It must be remembered that the curve is on too small a scale to give very close approximations. To draw a logarithmic curve which would give results sufficiently accurate for most practical purposes would require a piece of cross-ruled paper about two feet square.

100. Logarithmic multiplication. Multiplication by means of the logarithmic curve is illustrated in the

Example: Multiply 3 by 2.

Solution: From the curve, $\log 3 = .47$; hence $3 = 10^{.47}$.
From the curve, $\log 2 = .3$; hence $2 = 10^{.3}$.
Then $3 \cdot 2 = 10^{.47} \cdot 10^{.3} = 10^{.77}$.
From the curve, $10^{.77} = 6$.

EXERCISES

Compute as in the preceding example:

1. $2 \cdot 5$.	**4.** $.8 \cdot 8$.	**7.** $.5 \cdot 80$.	**10.** $.6 \cdot 80$.
2. $3 \cdot 3$.	**5.** $30 \cdot 2$.	**8.** $22 \cdot 4.8$.	**11.** $.7 \cdot 97$.
3. $4 \cdot 4$.	**6.** $25 \cdot 4$.	**9.** $14 \cdot 6$.	**12.** $1.5 \cdot 7.2$.

101. Logarithmic division. Division by means of the logarithmic curve is illustrated in the

Example: Divide 40 by 8.

Solution: From the curve, $\log 40 = 1.6$; hence $40 = 10^{1.6}$.
From the curve, $\log 8 = .9$; hence $8 = 10^{.9}$.
Then $40 \div 8 = 10^{1.6} \div 10^{.9} = 10^{.7}$.
From the curve, $10^{.7} = 5$.

EXERCISES

Compute as in the preceding example:

1. $8 \div 2$.	**5.** $16 \div .8$.	**9.** $\frac{4 \cdot 6}{3}$.
2. $6 \div 3$.	**6.** $48 \div 6$.	
3. $40 \div 5$.	**7.** $22 \div 11$.	**10.** $\frac{80 \cdot 40}{25}$.
4. $18 \div 8$.	**8.** $56 \div 8$.	

102. Logarithmic involution. A number is raised to a power by means of the logarithmic curve, as in the

Example: Find 2^3.

Solution: From the curve, $\log 2 = .3$; hence $2 = 10^{.3}$.

Therefore $2^3 = (10^{.3})^3 = 10^{.9}$.

From the curve, $10^{.9} = 8$.

EXERCISES

Compute as in the preceding example:

1. 3^2.
2. 5^2.
3. 3^3.
4. 4^3.
5. $\dfrac{5^2 \cdot 10^3}{25^2}$.
6. $\dfrac{2^2 \cdot 3^3}{6}$.
7. $4^2 \cdot 3^3$.

103. Logarithmic evolution. Roots are extracted by means of the logarithmic curve, as in the

Example: Find $\sqrt[3]{40}$.

Solution: From the curve, $\log 40 = 1.6$; hence $40 = 10^{1.6}$.

Therefore $\sqrt[3]{40} = (40)^{\frac{1}{3}} = (10^{1.6})^{\frac{1}{3}} = 10^{.53}$.

From the curve, $10^{.53} = 3.45$, which is approximately $\sqrt[3]{40}$.

EXERCISES

Compute as in the preceding example:

1. $\sqrt{3}$.
2. $\sqrt[3]{4}$.
3. $\sqrt[4]{81}$.
4. $6^{\frac{1}{2}}$.
5. $7^{\frac{1}{3}}$.
6. $5^2 \cdot \sqrt{3}$.
7. $\dfrac{3^3 \cdot \sqrt{2}}{.7}$.

From the foregoing work the student should see that *a logarithm is an exponent,* and that by the use of logarithms multiplication is effected by addition, division by subtraction, involution by a single multiplication, and evolution by a single division. The values of N and L, which up to this time have been taken from the curve, will hereafter be obtained much more accurately from the table on pages 200–201.

104. Steps preceding computation. Before computation by means of the table can be taken up, two processes requiring considerable explanation and practice must be mastered.

I. *To find from the table the logarithm of a given number.*

II. *To find from the table the number corresponding to a given logarithm.*

105. Base. If $N = b^L$, the logarithm of N to the **base** b is L. This last is expressed by the equation $\log_b N = L$. Therefore $N = b^L$ and $\log_b N = L$ are two ways of expressing the same fact.

Consequently $2 = 10^{.301}$ and $\log_{10} 2 = .301$ are equivalent statements.

The base of the **common** or **Briggs** system of logarithms is 10. The base 10 is often omitted. Thus log 2 means $\log_{10} 2$. This system is used in numerical work to the exclusion of all others.

The base of the *natural* system of logarithms is the irrational number 2.7182 +, which is usually denoted by e.

The natural system of logarithms is used for theoretical purposes only.

EXERCISES

Write in the notation of logarithms:

1. $300 = 10^{2.47}$.
2. $65 = 10^{1.81}$.
3. $4 = 10^{.60}$.
4. $1 = 10^0$.
5. $.10 = 10^{-1}$.
6. $1730 = 10^{3.238}$.
7. $173 = 10^{2.238}$.
8. $1.73 = 10^{.238}$.
9. $.173 = 10^{-1+.238}$.
10. $.0173 = 10^{-2+.238}$.

Write as powers of 10:

11. $\log 3 = .47$.
12. $\log 20 = 1.301$.
13. $\log 4.9 = .69$.
14. $\log 490 = 2.69$.
15. $\log .0049 = -3 + .69$.
16. $\log 381 = 2.58$.

106. Characteristic and mantissa. Unless a number is an exact power of 10, its logarithm consists of an *integer* and a *decimal*.

The integral part of a logarithm is called its **characteristic.**

The decimal part of a logarithm is called its **mantissa.**

The word "mantissa" means an addition; that is, a decimal portion which is added on to the integral part, or characteristic, of the logarithm. This term was used at one time to indicate the decimal part of any number, but for over a century it has been applied almost exclusively to logarithms.

$\log 200 = 2.301$. Here 2 is the characteristic and .301 is the mantissa.

Biographical Note. JOHN NAPIER. Although many scientists have been honored with titles on account of their discoveries, very few of the titled aristocracy have become distinguished for their mathematical achievements. A notable exception to this rule is found in John Napier, Lord of Merchiston (1550–1617), who devoted most of his life to the problem of simplifying arithmetical operations.

Napier was a man of wide intellectual interests and great activity. In connection with the management of his estate he applied himself most seriously to the study of agriculture, and experimented with various kinds of fertilizer in a somewhat scientific manner, in order to find the most effective means of reclaiming soil. He spent several years in theological writing. When the danger of an invasion by Philip of Spain was imminent he invented several devices of war. Among these were powerful burning mirrors, and a sort of round musket-proof chariot, the motion of which was controlled by those within, and from which guns could be discharged through little portholes.

But by far the most serious activity of Napier's life was the effort to shorten the more tedious arithmetical processes. He invented the first approximation to a computing machine, and also devised a set of rods, often called Napier's bones, which were of assistance in multiplication. His crowning achievement, however, was the invention of logarithms, to which he devoted fully twenty years of his life.

No characteristics are given in the table on pages 200–201. The characteristic of any number is obtained from an inspection of the number itself according to rules which will now be derived.

$$10^4 = 10000; \text{ that is, the } \log 10000 = 4.$$
$$10^3 = 1000; \text{ that is, the } \log 1000 = 3.$$
$$10^2 = 100; \text{ that is, the } \log 100 = 2.$$
$$10^1 = 10; \text{ that is, the } \log 10 = 1.$$
$$10^{-1} = .1; \text{ that is, the } \log .1 = -1.$$
$$10^{-2} = .01; \text{ that is, the } \log .01 = -2.$$
$$10^{-3} = .001; \text{ that is, the } \log .001 = -3.$$

The preceding table indicates between what two integers the logarithm of a given number lies. This determines the characteristic.

Since 542 lies between 10^2 and 10^3, $\log 542 = 2$ plus a decimal.

And since .0045 lies between 10^{-3} and 10^{-2}, $\log .0045 = -3$ plus a positive decimal (or -2 plus a negative decimal).

For the determination of the characteristic of a positive number we have the rules

I. *The characteristic of a number greater than 1 is one less than the number of digits to the left of the decimal point.*

II. *The characteristic of a number less than 1 is negative and numerically one greater than the number of zeros between the decimal point and the first significant figure.*

Accordingly the characteristic of 25 is 1; of 2536 is 3; of 6 is 0; of .4 is -1; of .032 is -2; of .00036 is -4.

The table on pages 200–201 gives the mantissas of numbers from 10 to 999. Before each mantissa a decimal point is understood.

The numbers 5420, 542, 5.42, .0542, and .000542 are spoken of as composed of the same *significant* digits in the same order. They differ only in the position of the decimal point, and consequently their logarithms will differ only in their characteristics. If the base of the system is 10, however, such numbers will have the same mantissa.

The last two points are easily illustrated by any two numbers which have the same significant digits in the same order.

$$\text{Log } 5.42 = .734, \text{ or } 5.42 = 10^{.734}.$$
$$5.42 \cdot 10^2 = 542 = 10^{.734} \cdot 10^2 = 10^{2.734}.$$

Therefore $\log 542 = 2 \cdot 734$.

The property just explained does not belong to a system of logarithms in which the base is any number other than 10. It is a very convenient property, as tables of a given accuracy are far shorter when the base is 10 than they would be with any other base. For example, the table on pages 200–201 gives the mantissas of all numbers from 1 to 999. But these mantissas are just the same as the mantissas of the three-figure decimals from .001 to .999, or another set of a thousand numbers. Were the base any number other than 10, the mantissas of the numbers from 1 to 999 would be different from those of the numbers from .001 to .999. Four pages or more would then be required to print a table equivalent to the one which is here put on two.

107. Use of the tables. To obtain the logarithm of a number of three or fewer significant figures from the tables, we have the

RULE. *Determine the characteristic by inspection.*

Find in column N *the first two significant figures. In the row with these and in the column headed by the third figure of the given number find the required mantissa.*

EXERCISES

Find the logarithm of:

1. 271.	**4.** 65.	**7.** 2.7.	**10.** 6.
2. 344.	**5.** 650.	**8.** 2700.	**11.** 932.
3. 982.	**6.** 27.	**9.** 3.	**12.** .932.

Solution: The characteristic of .932 is -1 and the mantissa is .9694. Hence $\log .932 = -1 + .9694$. This is usually written in the abbreviated form $\bar{1}.9694$. The mantissa is always kept positive in order to avoid the addition and subtraction of both positive and negative decimals, which in ordinary practice contain from three to five figures. Negative characteristics, being integers, are comparatively easy to take care of. (The student should note that log .932 is really negative, being $-1 + .9694$, or $-.0306$.)

13. .643.	**15.** .00267.	**17.** .0101.
14. .0532.	**16.** .00579.	**18.** 825000.

108. Interpolation. The process of finding the logarithm of a number not found in the table, from the logarithms of two numbers which are found there, or the reverse of this process, is called **interpolation.**

If we desire the logarithm of a number not in the table, say 7635, we know that it lies between the logarithms of 7630 and 7640, which are given in the table. Since 7635 is halfway between 7630 and 7640, we assume, though it is not *strictly* true, that the required logarithm is halfway between their logarithms, 3.8825 and 3.8831. To find log .7635 we first look up log 7630 and log 7640 and then take half (or .5) their difference (this difference may be taken from the column headed D) and add it to log 7630. This gives

$$\log 7635 = 3.8825 + .5 \times .0006 = 3.8828.$$

Were we finding log 7638, we should take .8 of the difference between log 7630 and log 7640 and add it to log 7630.

The preceding solution illustrates the general

RULE. *Prefix the proper characteristic to the mantissa of the first three significant figures.*

Then multiply the difference between this mantissa and the next greater mantissa in the table (called the tabular difference, column D *of the table) by the remaining figures of the number preceded by a decimal point.*

Add the product to the logarithm of the first three figures and reject all decimals beyond the fourth place.

In this method of interpolation we have assumed that the increase in the logarithm is directly proportional to the increase in the number. As has been said, this is not strictly true, yet the results here obtained are nearly always correct to the fourth decimal place.

EXERCISES

Find the logarithm of:

1. 4625.	**6.** 72.543.	**11.** .00386.
2. 364.7.	**7.** 10.101.	**12.** .0007777.
3. 42.73.	**8.** 700.35.	**13.** 3.1416.
4. 32.75.	**9.** 505.50.	**14.** 2.71828.
5. 546.8.	**10.** 2.0075.	**15.** .023456.

109. Antilogarithms. An **antilogarithm** is the number corresponding to a given logarithm. Thus antilog 2 equals 100.

If we desire the antilogarithm of a given logarithm, say 4.7308, we proceed as follows: The mantissa .7308 is found in the *row* which has 53 in column *N*, and in the *column* which has 8 at the top. Hence the first three significant figures of the antilogarithm are 538. Since the characteristic is 4, the number must have five digits to the left of the decimal point.

Thus antilog 4.7308 = 53,800. Therefore if the mantissa of a given logarithm is found in the table, its antilogarithm is obtained by the

RULE. *Find the row and the column in which the given mantissa lies.*

In the row found, take the two figures which are in column N for the first two significant figures of the antilogarithm, and for the third figure the number at the top of the column in which the mantissa stands.

Place the decimal point as indicated by the characteristic.

EXERCISES

Find the antilogarithm of:

1. 3.9309.	**6.** 8.5740 − 10.	**10.** $\bar{4}.6345$.
2. 1.8162.	HINT. $8.5740 - 10 = \bar{2}.5740$.	**11.** 6.9232.
3. .6284.	**7.** 9.7292 − 10.	**12.** 8.2148.
4. $\bar{1}.3541$.	**8.** 4.8136 − 10.	**13.** 5.7832 − 6.
5. $\bar{2}.5740$.	**9.** 0.4533.	**14.** $\bar{5}.9996$.

If the mantissa of a given logarithm, as 2.5271, is not in the table, the antilogarithm is obtained by interpolation as follows:

The mantissa 5271 lies just between

.5263, the mantissa of 336,

and .5276, the mantissa of 337.

Therefore the antilogarithm of 1.5271 lies between 33.6 and 33.7. Since the tabular difference is 13 and the difference between .5263 and .5271 is 8, the mantissa .5271 lies $\frac{8}{13}$ of the way from .5263 to .5276. Therefore the required antiilogarithm lies $\frac{8}{13}$ of the way from 33.6 to 33.7.

Then antilog $1.5271 = 33.6 + \frac{8}{13} \times .1.$
$$33.6 + .061 = 33.66.$$

Therefore when the mantissa is not found in the table, we have the

RULE. *Write the number of three figures corresponding to the lesser of the two mantissas between which the given mantissa lies.*

Subtract the lesser mantissa from the given mantissa and divide the remainder by the tabular difference to one decimal place.

Annex this figure to the three already found and place the decimal point where indicated by the characteristic.

EXERCISES

Find the antilogarithms of:

1. 1.5723.	**5.** $\bar{1}.2586$.	**9.** 9.2654 − 10.
2. 2.3921.	**6.** 7.3472 − 10.	**10.** .7829.
3. 0.6690.	**7.** 9.8527 − 10.	**11.** 7.1050 − 10.
4. 2.5728.	**8.** 5.9616 − 8.	**12.** 6.2308 − 10.

110. Multiplication and division. Multiplication by logarithms depends on the

THEOREM. *The logarithm of the product of two numbers is the sum of the logarithms of the numbers.*

Proof. Let $\log_b N_1 = l_1,$ (1)

and $\log_b N_2 = l_2.$ (2)

From (1), $N_1 = b^{l_1}.$ (3)

From (2), $N_2 = b^{l_2}.$ (4)

(3) × (4), $N_1 N_2 = b^{l_1 + l_2}.$ (5)

Therefore $\log_b N_1 N_2 = l_1 + l_2.$

Division by logarithms depends on the

THEOREM. *The logarithm of the quotient of two numbers is the logarithm of the dividend minus the logarithm of the divisor.*

Proof. Let $\log_b N_1 = l_1$, (1)

and $\log_b N_2 = l_2$. (2)

From (1), $N_1 = b^{l_1}$. (3)

From (2), $N_2 = b^{l_2}$. (4)

(3) ÷ (4); $\dfrac{N_1}{N_2} = b^{l_1 - l_2}$.

Therefore $\log_b \dfrac{N_1}{N_2} = l_1 - l_2$.

EXERCISES

Perform the indicated operation by logarithms:

1. $18 \cdot 25$.

Solution:

$$\begin{array}{lr} & \log 18 = 1.2553 \\ & \log 25 = 1.3979 \\ \hline \text{Adding,} & \log 18 \cdot 25 = 2.6532 \\ \text{Antilog} & 2.6532 = 450. \end{array}$$

2. $37 \cdot 23$.

3. $28 \cdot 8$.

4. $9.8 \cdot 5$.

5. $42 \cdot 2.2$.

6. $386 \cdot 27$.

7. $432 \cdot 361$.

8. $589 \cdot 734$.

9. $4326 \cdot 638$.

10. $2870 \cdot 3654$.

11. $286.7 \cdot 2.341$.

12. $3.412 \cdot 2.596$.

13. $432 \cdot .574$.

Solution:

$$\begin{array}{lr} & \log 432 = 2.6355 = 2.6355 \\ & \log .574 = \bar{1}.7589 = 9.7589 - 10 \\ \hline \text{Adding,} & \log 432 \cdot .574 = 2.3944 = 12.3944 - 10 \\ \text{Antilog} & 2.3944 = 247.9. \end{array}$$

Since the *mantissa is always positive*, any number carried over from the tenth's column to the units column is positive. This occurs in the preceding solution where $.6 + .7 = 1.3$, giving $+1$ to be added to the sum of the characteristics $+2$ and -1, in the units column. Mistakes in such cases will be few if the logarithms with negative characteristics be written as in the $9 - 10$ notation on the right.

In the preceding example and in others which follow, two methods are given for writing the logarithms which have negative characteristics. This is done to illustrate those cases in which the second of the two ways is preferable. It should be understood that in practice one, but not necessarily both, of these methods is to be used.

14. 385 · .617. **17.** .0876 · .673. **20.** 675 · .0236.

15. 541 · .073. **18.** .07325 · 6.384. **21.** .437 · .0076.

16. 37.6 · .00835. **19.** .6381 · .01897. **22.** 891 ÷ .27.

Solution:

$$\begin{aligned} \log 891 &= 2.9499 \\ \log 27 &= 1.4314 \\ \hline \text{Subtracting,} \quad \log(891 \div 27) &= 1.5185 \\ \text{Antilog} \quad 1.5185 &= 33. \end{aligned}$$

23. 96 ÷ 12. **26.** 489 ÷ 27.1. **29.** 9876 ÷ 56.78.

24. 888 ÷ 37. **27.** 3460 ÷ 4.32. **30.** 6432 ÷ 7.81.

25. 976 ÷ 321. **28.** 4697 ÷ 281. **31.** 3.26 ÷ .0482.

Solution:

$$\begin{aligned} \log 3.26 &= 0.5132 = 10.5132 - 10 \\ \log .0482 &= \bar{2}.6830 = 8.6830 - 10 \\ \hline \text{Subtracting,} \quad \log(3.26 \div .0482) &= 1.8302 = 1.8302 - 0 \\ \text{Antilog} \quad 1.8302 &= 67.64. \end{aligned}$$

32. 2.35 ÷ .0673.

33. 4.86 ÷ .721.

34. .0635 ÷ .287.

35. .2674 ÷ 3.86.

36. 7635 ÷ 8692.

37. .07382 ÷ 68.72.

38. $\dfrac{256 \cdot 372}{128}$.

39 $\dfrac{347 \cdot (-625)}{346}$.

40. $\dfrac{463.2 \cdot 4.78}{-68.3}$.

41. $\dfrac{9.63 \cdot .0872}{.00635}$.

42. .078 ÷ 4.267.

111. Involution and evolution. Involution by logarithms depends on the

THEOREM. *The logarithm of the mth power of a number is m times the logarithm of the number.*

Proof. Let $\log_b N = l.$ (1)

Then $N = b^l.$ (2)

Raising both members of (2) to the mth power,

$$N^m = b^{ml}.$$

Therefore $\log_b N^m = ml.$

Evolution by means of logarithms depends on the

THEOREM. *The logarithm of the real mth root of a number is the logarithm of the number divided by m.*

Proof. Let $\log_b N = l.$ (1)

Then $N = b^l.$ (2)

Extracting the mth root of the members of (2),

$$(N)^{\frac{1}{m}} = (b^l)^{\frac{1}{m}} = b^{\frac{l}{m}}. \quad (3)$$

Therefore $$\log (N)^{\frac{1}{m}} = \frac{l}{m}. \quad (4)$$

EXERCISES

Compute, using logarithms:

1. $(2.73)^3$.

Solution: $\log 2.73 = .4362.$
Multiplying by 3, $\log (2.73)^3 = 1.3086.$
Antilog $1.3086 = 20.35.$

2. $(6.32)^4$. **3.** $(34.26)^2$. **4.** $(6.715)^3$. **5.** $(.425)^3$.

Solution: $\log .425 = \bar{1}.6284 = (9.6284 - 10).$
Multiplying by 3, $\log (.425)^3 = \bar{2}.8852 = (28.8852 - 30).$
Antilog $\bar{2}.8852 = .07676.$

Since the mantissa is always positive, we have in the preceding solution $+1$ (from $3 \cdot .6$) to unite with -3 (from $3 \cdot \bar{1}$). No confusion of positive and negative parts need arise, if the logarithms are written as indicated in the parenthesis.

6. $(.352)^4$. **7.** $(.0672)^2$. **8.** $(.003567)^5$. **9.** $\sqrt[3]{376}$.

Solution: $\log 376 = 2.5752.$
Dividing by 3, $\log \sqrt[3]{376} = .8584.$
Antilog $.8584 = 7.218 = \sqrt[3]{376}.$

10. $\sqrt[3]{583}$. **11.** $\sqrt[5]{1235}$. **12.** $\sqrt[3]{.000639}$.

Solution: $\log .000639 = \bar{4}.8055.$

If one divided $\bar{4}.8055$ as it stands by 3, he would be almost certain to confuse the negative characteristic and the positive mantissa. This and other difficulties may easily be avoided by adding to the characteristic and subtracting from the resulting logarithm any integral multiple of the index of the root which will make the characteristic positive.

Thus	$\log .000639 = 2.8055 - 6.$
Dividing by 3,	$\log \sqrt[3]{.000639} = .9351 - 2.$
Antilog	$\bar{2}.9351 = .08612 = \sqrt[3]{.000639}.$

13. $\sqrt{.0786}.$ **15.** $\sqrt[4]{.002679}.$ **17.** $(4.965)^{\frac{3}{2}}.$

14. $\sqrt[5]{.0007324}.$ **16.** $(38.4)^{\frac{2}{3}}.$ **18.** $(-6.387)^{\frac{5}{3}}.$

19. $\sqrt{\dfrac{283 \cdot 4.627}{(8.423)^3}}.$

20. $\sqrt{\dfrac{(23.56)^2 \cdot 7.384}{(4.623)^3}}.$

21. $\sqrt[11]{209}.$

22. $\sqrt{87 - \sqrt[4]{163}}.$

23. $\frac{25}{361}\sqrt[5]{\frac{127}{67}\sqrt[3]{872}}.$

Note. The following four-place table will usually give results correct to one half of one per cent. Five-place tables give the mantissa to five decimal places of the numbers from 1 to 9999, and, by interpolation, the mantissa of numbers from 1 to 99999. Five-place tables give results correct to one twentieth of one per cent, an accuracy which is sufficient for most engineering work.

Six-place tables give the mantissa to six decimals for the same range of numbers as a five-place table. The labor of using a six-place table is about fifty per cent more than that of using a five-place one. For this reason and for other reasons a six-place table is of small practical value.

Seven-place tables give the mantissas of the numbers from 1 to 99999, and by interpolation give the mantissa of numbers from 1 to 999999. Seven-place tables are seldom needed in engineering, but are of constant use in astronomy.

In place of a table of logarithms engineers often use an instrument called a "slide rule." This is really a mechanical table of logarithms arranged ingeniously for rapid practical use. Results can be obtained with such an instrument far more quickly than with an ordinary table of logarithms, and that without recording or even thinking of a single logarithm. A "slide rule" ten inches long gives results correct to three figures. In work requiring greater accuracy a larger and more elaborate instrument which gives a five-figure accuracy is used.

N	0	1	2	3	4	5	6	7	8	9	D
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	42
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	38
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	35
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	32
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	30
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	28
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	26
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	25
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	24
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	22
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	21
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	20
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	19
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	18
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	18
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	17
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	16
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	16
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	15
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	15
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	14
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	14
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	13
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	13
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	13
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	12
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	12
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	12
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	11
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	11
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	11
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	10
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	10
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	10
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	10
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	10
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	9
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	9
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	9
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	8
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	8
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	8

N	0	1	2	3	4	5	6	7	8	9	D
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	8
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	8
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	8
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	7
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	7
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	7
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	7
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	7
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	7
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	7
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	6
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	6
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	6
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	6
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	6
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	6
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	6
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	6
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	5
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	5
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	5
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	5
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	5
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	5
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	5
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	5
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	5
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	5
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	4

PROBLEMS IN MENSURATION

Solve, using logarithms (obtain results to four figures):

1. The circumference of a circle is $2\pi R$. ($\pi = 3.1416$, $R =$ radius.)

(*a*) Find the circumference of a circle whose radius is 42 inches.

(*b*) Find the radius of a circle whose circumference is 6843 centimeters.

2. The area of a circle is πR^2.

(*a*) Find the area of a circle whose radius is 3.672 feet.

(*b*) Find the radius of a circle whose area is 64.37 feet.

3. The area of the surface of a sphere is $4\pi R^2$.

(*a*) The radius of the earth is 3958.79 miles. Find its surface.

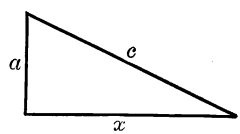
(*b*) Find the length of the equator.

4. The volume of a sphere is $\frac{4\pi R^3}{3}$.

(*a*) Find the radius of a sphere whose volume is 25 cubic feet.

(*b*) Find the diameter of a sphere whose volume is 85 cubic inches.

5. If the hypotenuse and one leg of a right triangle are given, the other leg can always be computed by logarithms.



In the adjacent figure let a and c be given and x required.

Then $x = \sqrt{c^2 - a^2} = \sqrt{(c+a)(c-a)}$.

Whence $\log x = \frac{1}{2}\log(c+a) + \frac{1}{2}\log(c-a)$.

(*a*) The hypotenuse of a right triangle is 377 and one leg is 288. Find the other leg.

(*b*) The hypotenuse of a right triangle is 1493 and one leg is 532. Find the other leg.

6. The area of an equilateral triangle whose side is s is $\frac{s^2}{4}\sqrt{3}$.

(*a*) Find in square feet the area of an equilateral triangle whose side is 11.47 inches.

(*b*) Find the side of an equilateral triangle whose area is 60 square centimeters.

7. The area of a triangle $= \sqrt{s(s-a)(s-b)(s-c)}$. Here a, b, and c are the sides of the triangle and $s = \frac{a+b+c}{2}$.

(*a*) Find the area of a triangle whose sides are 12 inches, 15 inches, and 19 inches respectively.

(*b*) Find the area of a triangle whose sides are 557, 840, and 1009.

112. Exponential equations. An **exponential** equation is an equation in which the unknown occurs as an exponent or in an exponent.

Many exponential equations are readily solved by means of logarithms, since $\log a^x = x \log a$. Thus let $a^x = c$. Then $x \log a = \log c$. Whence $x = \log c \div \log a$.

MISCELLANEOUS EXERCISES

Solve for x:

1. $8^x = 324$.

Solution: $\log 8^x = \log 324,$

or $x \cdot \log 8 = \log 324.$

Whence $$x = \frac{\log 324}{\log 8} = \frac{2.5105}{.9031} = 2.75+.$$

2. $3^x = 25$.

3. $64^x = 4$.

4. $16^x = 1024$.

5. $(-2)^x = 64$.

6. $3 = (1.04)^x$.

7. $2^x = 64$.

8. $4^{2x+1} = 84$.

9. $\frac{81^{\frac{1}{2}}}{3^{-x-1}} = 27^{\frac{2x-1}{3}}$.

10. $3^{x+7} = 5^x$.

11. In how many years will one dollar double itself at 3%, interest compounded annually?

Solution: At the end of one year the amount of $1 at 3% is $1.03; at the end of two years it is $(1.03)(1.03) or $$(1.03)^2$; at the end of three years it is $$(1.03)^3$; and at the end of x years it is $$(1.03)^x$.

If x is the number of years required, $(1.03)^x = 2$.

Taking the logarithms of both members of the equation,

$$x \log 1.03 = \log 2.$$

Solving, $$x = \frac{\log 2}{\log 1.03} = \frac{.3010}{.0128} = 23.5+.$$

12. In how many years will \$1 double itself at 5%, interest compounded annually?

13. In how many years will any sum of money treble itself at 4%, interest compounded annually?

14. In how many years will \$265 double itself at $3\frac{1}{2}$%, interest compounded annually?

15. In how many years will \$4000 amount to \$7360.80 at 5%, interest compounded annually?

16. About 300 years ago the Dutch paid \$24 for the island of Manhattan. At 4% compound interest, what would this payment amount to at the present time?

17. In how many years will \$12 double itself at 4%, interest compounded semiannually?

18. Show that the amount of P dollars in t years at r%, interest compounded annually, is $P(1+r)^t$; compounded semiannually is $P\left(1+\frac{r}{2}\right)^{2t}$; compounded quarterly is $P\left(1+\frac{r}{4}\right)^{?}$; and compounded monthly is $P\left(1+\frac{r}{?}\right)^{12t}$.

19. Find the amount of \$5000 at the end of four years, interest at 8% compounded (*a*) annually; (*b*) semiannually; (*c*) quarterly.

20. A man borrows \$6000 to build a house, agreeing to pay \$50 monthly until the principal and interest at 6% is paid. Find the number of full payments required.

21. If each payment in Exercise 20 is at once lent at 6%, compounded annually, what will they all amount to by the time the final payment of \$50 is made?

22. From Exercises 20 and 21 determine the total interest received by the money lender up to the time of the last payment. What rate per cent on the original $6000 is this?

Find the number of digits in:

23. (*a*) $2^9 \cdot 3^8 \cdot 5^7$; (*b*) 3^{52}; (*c*) 2^{340}.

24. Can the base of a system of logarithms be negative? Explain.

Find (without reference to the table) the numerical values of:

25. $\log_3 9$.

26. $\log_2 8$.

27. $\log_8 2$.

28. $\log_9 27$.

29. $\log_{27} 9$.

30. $\log_4 8 + \log_8 4$.

31. $\log_{27} 81 - \log_{81} 27$.

32. $\log_{25} 125 + \log_5 25 - \log_{125} 5$.

33. $\log_3 (\frac{1}{3}) - \log_9 (\frac{1}{27}) + \log_{27} 9$.

Simplify:

34. $\log \frac{5}{8} + \log \frac{24}{25}$.

35. $\log \frac{7}{32} - \log \frac{35}{64}$.

36. $\log \frac{25}{4} + \log \frac{36}{50} - \log \frac{3}{4}$.

37. $2 \log 3 + 3 \log 2$.

38. $3 \log 4 + 4 \log 3 - 2 \log 6$.

Solve for x:

39. $a^x = c^{x-1}$.

40. $a^{x-1} = c^{x-2}$.

41. $a^{x-1} \cdot b^x = c^{2x}$.

42. $3^x \cdot 2^{\frac{1}{x}} = 6$.

43. $e^x = e^{-x}$.

44. $a^{x+1} = b^{2x} \div c^{x-1}$.

45. $a^{4x} + 8a^{2x} = 6a^{3x}$.

46. $a^{5x} + a^{4x} = 6a^{4x} - 6a^{3x}$.

Solve for x and y:

47. $2^x = 3^y$, $3^{x-1} = 4^y$.

48. $2x - y = 5$, $3^x \cdot 9^{3y} = 9^{11}$.

49. $3x + y = 9$, $2^x \cdot 8^{2y} = 4^{10}$.

50. $8^x \cdot 5^y = 50$, $2^{6x} \cdot 3^{2y} = 328$.

51. $3^x - 6^y = 0$, $3^{x+1} - 6^x = 0$.

52. $x^y - y^x = 0$, $y - x^2 = 0$.

Note. It is not a little remarkable that just at the time when Galileo and Kepler were turning their attention to the laborious computation of the orbits of planets, Napier should be devising a method which simplifies these processes. It was said a hundred years ago, before astronomical computations became so complex as they now are, that the invention of logarithms, by shortening the labors, doubled the effective life of the astronomer. To-day the remark is well inside the truth.

In the presentation of the subject in modern textbooks a logarithm is defined as an exponent. But it was not from this point of view that they were first considered by Napier. In fact it was not till long after his time that the theory of exponents was understood clearly enough to admit of such application. This relation was noticed by the mathematician Euler, about one hundred fifty years after logarithms were invented.

It was by a comparison of the terms of certain arithmetical and geometrical progressions that Napier derived his logarithms. They were not exactly like those used commonly to-day, for the base which Napier used was not 10. Soon after the publication (1614) of Napier's work, Henry Briggs, an English professor, was so much impressed with its importance that he journeyed to Scotland to confer with Napier about the discovery. It is probable that they both saw the necessity of constructing a table for the base 10, and to this enormous task Briggs applied himself. With the exception of one gap, which was filled in by another computer, Briggs's tables form the basis for all the common logarithms which have appeared from that day to this.

CHAPTER XV

RATIO, PROPORTION, AND VARIATION

113. Ratio. The **ratio** of one number a to a second number b is the quotient obtained by dividing the first by the second, or $\frac{a}{b}$.

The ratio of a to b is also written $a : b$.

It follows from the above that all ratios of numbers are fractions and all fractions may be regarded as ratios.

Thus $\frac{3}{2}$, $\frac{c}{2x}$, $\frac{a+b}{a-b}$, and $\frac{\sqrt{2}}{\sqrt[3]{5}}$ are ratios.

The dividend, or numerator, in a ratio is called the **antecedent,** and the divisor, or denominator, is called the **consequent.**

114. Proportion. Four numbers, a, b, c, and d, are in **proportion** if the ratio of the first pair equals the ratio of the second pair.

This proportion is written $a : b = c : d$, or $\frac{a}{b} = \frac{c}{d}$.

Note. By the earlier mathematicians ratios were not treated as if they were numbers, and the equality of two ratios which we know as a proportion was not denoted by the same symbol as other kinds of equality. The usual sign of equality for ratios was : :, a notation which was introduced by the Englishman, Oughtred, in 1631, and brought into common use by John Wallis about 1686. The sign = was used in this connection by Leibnitz (1646–1716) in Germany, and by the continental writers generally, while the English clung to Oughtred's notation.

The first and fourth terms (a, d) are called the **extremes,** and the second and third terms (b, c) are called the **means.**

A **mean proportional** between two numbers, a and b, is the number m, if $\frac{a}{m} = \frac{m}{b}$. This means that $m^2 = ab$, or $m = \pm\sqrt{ab}$.

A **third proportional** to two numbers, a and b, is the number t, if $\frac{a}{b}=\frac{b}{t}\cdot$

A **fourth proportional** to three numbers, a, b, and c, is the number f, if $\frac{a}{b}=\frac{c}{f}\cdot$

Since a proportion is an equation, the axioms, subject to the limitations explained on pages 43–44, apply to any proportion.

Then in the proportion $\frac{a}{b}=\frac{c}{d}$ both members may be multiplied by bd, giving $\boldsymbol{ad = bc}$.

Therefore, *In any proportion the product of the means equals the product of the extremes.*

If $\boldsymbol{ps = qr}$ is divided by qs, we obtain

$$\frac{p\not{s}}{q\not{s}}=\frac{\not{q}r}{\not{q}s}, \quad \text{or} \quad \frac{p}{q}=\frac{r}{s}\cdot \tag{1}$$

Also $ps = qr$ divided by rs gives

$$\frac{p}{r}=\frac{q}{s}\cdot \tag{2}$$

And $qr = ps$ divided by pr gives

$$\frac{q}{p}=\frac{s}{r}\cdot \tag{3}$$

Therefore, *If the product of any two numbers* ($\boldsymbol{ps}$) *equals the product of two other numbers* ($\boldsymbol{qr}$), *one pair may be made the means and the other pair the extremes of a proportion.*

If $\frac{a}{b}=\frac{c}{d}$, then from (1) and (2), $\boldsymbol{\frac{a}{c}=\frac{b}{d}}\cdot$ Here $\frac{a}{c}=\frac{b}{d}$ is said to be obtained from $\frac{a}{b}=\frac{c}{d}$ by **alternation.**

If $\frac{a}{b}=\frac{c}{d}$, then from (1) and (3), $\boldsymbol{\frac{b}{a}=\frac{d}{c}}\cdot$ Here $\frac{b}{a}=\frac{d}{c}$ is said to be obtained from $\frac{a}{b}=\frac{c}{d}$ by **inversion.**

EXERCISES

Simplify the following ratios by writing them as fractions and reducing the fractions to lowest terms:

1. $42 : 28$.
2. $24a^3 : 56a^2$.
3. $(x^2 - y^2) : (x - y)$.
4. $(x^3 + 8y^3) : (x + 2y)$.
5. $\left(1 - \frac{4}{a^2}\right) : \left(1 - \frac{2}{a}\right)$.
6. $\left(a - \frac{16}{a}\right) : \left(\frac{24}{a^4} + \frac{10}{a^3} + \frac{1}{a^2}\right)$.

7. (*a*) 4 weeks : 12 hours; (*b*) 480 square inches : 2 square yards.

8. 1 mile : 1 kilometer. (1 meter = 39.37 inches.)

9. Separate 150 into three parts in the ratio $4 : 6 : 2$.

10. If a is a positive number, which is the greater ratio:

(*a*) $\frac{5 + 3a}{5 + 4a}$ or $\frac{5 + 4a}{5 + 5a}$? (*b*) $\frac{7 - 2a}{7 - 3a}$ or $\frac{7 - 3a}{7 - 4a}$?

(*c*) If a positive number is added to or subtracted from both terms of a proper fraction, what change is produced in the numerical value of the fraction?

If $a : b = c : d$, prove the following and state the corresponding theorem in words:

11. $a : c = b : d$.
12. $b : a = d : c$.
13. $a^n : b^n = c^n : d^n$.
14. $\sqrt[n]{a} : \sqrt[n]{b} = \sqrt[n]{c} : \sqrt[n]{d}$.
15. $(a + b) : b = (c + d) : d$.
16. $(a + b) : a = (c + d) : c$.
17. $(a - b) : b = (c - d) : d$.
18. $(a - b) : a = (c - d) : c$.
19. $(a + b) : (a - b) = (c + d) : (c - d)$.

The results in Exercises 15, 17, and 19 are said to be derived from $a : b = c : d$ by **addition, subtraction,** and **addition and subtraction** respectively.

20. If $a : b = c : d = e : f$, prove that $(a + c + e) : (b + d + f) = a : b$ and state the theorem in words.

21. Find a mean proportional between 1.44 and .0256.

22. Find a third proportional to 15 and 125.

23. Find a fourth proportional to $16\frac{1}{4}$, $8\frac{1}{3}$, and $62\frac{1}{2}$.

24. Write $5:15 = 8:24$ by addition, subtraction, addition and subtraction, alternation, and inversion.

Solve:

25. $8:12 = (3 - x):7.$

26. $4:x = x:169.$

27. $3:5 = \frac{1}{x}:2.$

28. $8:x = 12:(10 - x).$

29. $3:5 = (x - 3):(2x + 18).$

30. $20:x = x:(10 - x).$

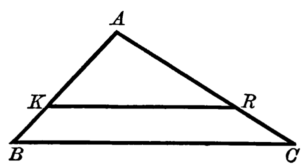
31. The surface of a sphere is $4\pi R^2$. Prove that for any two spheres $\frac{S_1}{S_2} = \frac{R_1^2}{R_2^2} = \frac{D_1^2}{D_2^2}$, S denoting the surface and D the diameter.

32. The volume of a sphere is $\frac{4\pi R^3}{3}$. Prove that for any two spheres $\frac{V_1}{V_2} = \frac{R_1^3}{R_2^3} = \frac{D_1^3}{D_2^3}$, V denoting the volume and D the diameter.

33. Find the ratio of the surfaces of the earth and the moon, their diameters being 7920 miles and 2160 miles respectively.

34. Find the ratio of the volumes of the earth and the moon.

35. If ABC is any triangle, and KR is a line parallel to BC meeting AB at K and AC at R, then the area of ABC is to the area of AKR as $\overline{AB}^2:\overline{AK}^2$, or as $\overline{AC}^2:\overline{AR}^2$, or as $\overline{BC}^2:\overline{KR}^2$.



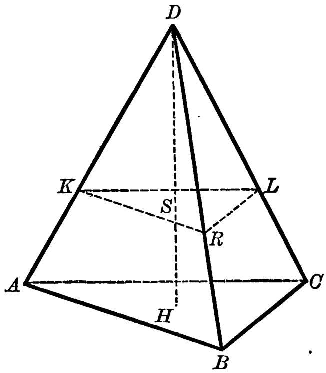
If in the adjacent figure the area of ABC is 100 square inches, that of AKR is 25 square inches, and AB equals 10 inches, find AK.

36. If in the figure of Exercise 35 $AB = 12$, and triangle AKR is $\frac{1}{2}$ triangle ABC, find AK.

37. If in Exercise 35 the trapezoid $KRCB$ is eight times as large as triangle AKR, and $AC = 40$, find AR.

38. If AB equals 32, and two parallels to BC separate triangle ABC into three parts of equal area, find to two decimals the lengths of the three parts into which AB is divided.

39. If a plane be passed parallel to the base of a pyramid (or cone) cutting it in KRL, then pyramid $D\text{-}ABC$: pyramid $D\text{-}KRL = \overline{DH}^3 : \overline{DS}^3$, etc.



If in the adjacent figure the volumes of the pyramids are 4 and 32 cubic inches respectively, and the altitude DH equals 18 inches, find DS.

40. If DH is 12 inches and the volume of one pyramid is one half the volume of the other, find DS to two decimals.

41. If the volume of the frustum is $\frac{19}{27}$ of the whole pyramid, and DH equals 36, find DS.

42. If two planes parallel to the base divide the whole pyramid into three parts having equal volumes, and $DH = 100$, find, using page 262, the parts into which the planes divide DH.

If $a : b = c : d$, prove:

43. $\dfrac{a + 3b}{a - 3b} = \dfrac{c + 3d}{c - 3d}$.

44. $\dfrac{a^2 + 2b^2}{a^2} = \dfrac{c^2 + 2d^2}{c^2}$.

45. $\dfrac{a^2 + b^2}{2ab} = \dfrac{c^2 + d^2}{2cd}$.

46. $\dfrac{5a^3 - b^3}{b^3} = \dfrac{5c^3 - d^3}{d^3}$.

47. $\dfrac{a^3 + b^3}{3a^2b + 3ab^2} = \dfrac{c^3 + d^3}{3c^2d + 3cd^2}$.

48. $\dfrac{a^2 - ab + b^2}{3ab} = \dfrac{c^2 - cd + d^2}{3cd}$.

49. In a right triangle h is the hypotenuse and a and b are the legs. The corresponding sides of another right triangle are H, A, and B. If $h : H = a : A$, prove $a : A = b : B$. Are the triangles similar?

115. Variation. The word **quantity** denotes anything which is measurable, such as distance, rate, and area.

Many operations and problems in mathematics deal with numerical measures of quantities, some of which are fixed and others constantly changing.

An abstract number, or the numerical measure of a fixed quantity, is called a **constant.**

Thus the abstract numbers 1, 3, and $-\frac{5}{7}$ are constants. Any definite quantities, as the area of a square whose side is 2, the circumference of a circle divided by its diameter (3.1416 nearly), the time of one revolution of the earth on its axis (23^h, 56^m, 4.09^s), and the velocity of light through space (186,330 miles per second) are constants.

The numerical measure of a changing quantity is called a **variable.**

For example, the distance (measured in any unit of length) between a passenger on a moving car and a point on the track either ahead of or behind him is a variable, decreasing in the first instance, increasing in the second. Other examples of variables are one's weight, the height of the mercury in the thermometer, and the distance to the sun.

The equation $x = 3y$ may refer to no physical quantities whatever, yet it is possible to imagine y as taking on in succession every possible numerical value, and the value of x as accompanying every change, and consequently always being three times as great as the corresponding value of y. In this sense, which is strictly mathematical, x and y are variables.

Problems in variation deal with at least two variables so related that any change in one is accompanied by a change in the other. Frequently one variable depends on several others.

For instance, the number of lines of printing on a page depends on the distance between the lines, the size of the type, and one dimension of the page.

The symbol for variation is $\propto$, and $x \propto y$ is read *x varies directly as y*, or *x varies as y.*

116. Direct variation. One hundred feet of copper wire of a certain size weighs 32 pounds. Obviously a piece of the same kind 200 feet long would weigh 64 pounds; a piece 300 feet long would weigh 96 pounds, and so on.

Here we have two variables W (weight) and L (length) so related that the value of W depends on the value of L, and in such a way that W increases proportionately as L increases. That is, W is directly proportional, or merely proportional, to L. Hence, if W_1 and W_2 are *any* two weights corresponding to the lengths L_1 and L_2 respectively,

$$W_1 : W_2 = L_1 : L_2. \tag{1}$$

In the form of a variation (1) becomes

$$W \propto L.$$

In general, if $x \propto y$, and x and y denote *any* two corresponding values of the variables, and x_1 and y_1 a *particular* pair of corresponding values,

$$\frac{x}{x_1} = \frac{y}{y_1}. \tag{2}$$

From (2),

$$x = \left(\frac{x_1}{y_1}\right)y. \tag{3}$$

But $\frac{x_1}{y_1}$ is a constant, being the quotient of two definite numbers. Call this constant K and (3) may be written

$$\boldsymbol{x = Ky.}$$

That is, if one variable varies as a second, the first equals the second multiplied by a constant.

Thus for the copper wire just mentioned, $W = \frac{32}{100} L$, or $\frac{8}{25} L$. Here, though W varies as L varies, W is always equal to L multiplied by the constant $\frac{8}{25}$.

The phrase **varies with** is often incorrectly used in place of **varies as.** The latter should be used to denote a *proportional* change in one variable with respect to a second; the former should not be so used. A boy's height varies *with* his age, but does not vary *as* his age. At 3 years the average boy is about 3 feet tall; at 12 years he is about 5 feet. At the latter time, if his height varied as his age from 3 years up to 12 years, he would be 12 feet tall.

117. Inverse variation. If a tank full of water is emptied in 24 minutes through a "smooth" outlet in which the area of the opening, A, is 1 square inch, an outlet in which A is 2 square inches would empty the tank twice as quickly, or in 12 minutes. And an outlet in which A is 3 square inches would empty the tank in 8 minutes.

Suppose it possible to increase or decrease A at will. We then have in t, the time required to empty the tank, and in A, the area of the opening, two related variables such that if A increases, t will decrease proportionally; while if A decreases, t will increase proportionally. That is, t and A are inversely proportional. This means that when A is doubled, t is halved; when A is trebled, t is divided by 3, and so on. The relation existing between the numerical values of A and t given in the preceding paragraph illustrates the truth of the last statement and of (1) which follows.

Now let t_1 and t_2 be *any* two times corresponding to the areas A_1 and A_2 respectively; then

$$t_1 : t_2 = A_2 : A_1. \tag{1}$$

The letters and the subscripts in (1) say: *The first time is to the second time as the second area is to the first area.*

The proportion (1) may be put in another form.

First, $$t_1 \cdot A_1 = t_2 \cdot A_2. \tag{2}$$

Dividing (2) by A_1A_2, $$\frac{t_1}{A_2} = \frac{t_2}{A_1}, \tag{3}$$

or $$t_1\left(\frac{1}{A_2}\right) = t_2\left(\frac{1}{A_1}\right). \tag{4}$$

Whence $$t_1 : t_2 = \frac{1}{A_1} : \frac{1}{A_2}. \tag{5}$$

Here the subscripts on the t's and those on the A's come in the same order.

In the form of a variation (5) becomes $t \propto \frac{1}{A}$.

In general x varies **inversely** as y when x varies as the **reciprocal** of y; that is,

$$x \propto \frac{1}{y}. \tag{6}$$

And if x and y denote any two corresponding values of the variable, and x_1 and y_1 a particular pair of corresponding values,

$$x : x_1 = \frac{1}{y} : \frac{1}{y_1}. \tag{7}$$

Whence $$\frac{x}{y_1} = \frac{x_1}{y}, \quad \text{or} \quad xy = x_1 y_1. \tag{8}$$

But $x_1 y_1$ is a constant, being the product of two definite numbers. Call this constant K.

Then (8) becomes $$\boldsymbol{xy = K.}$$

That is, if one variable varies inversely as another, the product of the two is a constant.

118. Joint variation. If the base of a triangle remains constant while the altitude varies, the area will vary as the altitude. Similarly, if the base varies while the altitude remains constant, the area will vary as the base. If *both* base and altitude vary, the area varies as the product of the two; that is, the area of the triangle varies **jointly** as the base and altitude. Further, if at *any* time A_1 denotes the area of a variable triangle, and h_1 and b_1 the corresponding altitude and base, and if A_2 denotes the area at *any other* time, and h_2 and b_2 the corresponding altitude and base, then $A_1 : A_2 = h_1 b_1 : h_2 b_2$.

In the form of a variation this last becomes

$$A \propto hb.$$

In general, any variable x varies jointly as two others, y and z, if

$$\boldsymbol{x \propto yz.} \tag{1}$$

If x varies jointly as y and z, and if x, y, and z denote *any* corresponding values of the variables, while x_1, y_1, and z_1 denote a *particular* set of such values, then

$$\frac{\boldsymbol{x}}{\boldsymbol{x_1}} = \frac{\boldsymbol{yz}}{\boldsymbol{y_1 z_1}}. \tag{2}$$

From (2), $$x = \left(\frac{x_1}{y_1 z_1}\right) yz. \tag{3}$$

But in (3) the fraction $\frac{x_1}{y_1 z_1}$ is a constant, since x_1, y_1, and z_1 are particular values of the variables x, y, and z. Calling this constant K, we may write $x \propto yz$ as the equation

$$x = Kyz.$$

One variable may vary directly as one variable (or several variables) and inversely as another (or several others). Also one variable may vary as the square, or the cube, or the square root, or the reciprocal, or as any algebraic expression whatever involving the other variable (or variables).

The theory of variation is really involved in proportion, but this fact is not obvious to the beginner. Hence it is necessary to make clear the meaning of the terms used in variation, and to show how proportion is applied to the solution of problems in variation. It is doubly necessary that the student himself make this application in many cases, otherwise he will not readily grasp numerous relations in physics, in chemistry, and in astronomy; for many important laws of these sciences are often stated in the form of a variation. In connection with these laws many problems arise which require for their solution clear notions of the principles of variation. With a knowledge of proportion only, the student would often find the laws vague and the problems difficult.

PROBLEMS

1. If $x \propto y$, and $x = 4$ when $y = 6$, find x when $y = 8$.

Solution: The variation is direct. Therefore

$$\frac{x_1}{x_2} = \frac{y_1}{y_2}. \qquad (1)$$

Substituting in (1), $$\frac{4}{x_2} = \frac{6}{8}. \qquad (2)$$

Solving (2), $$x_2 = 5\tfrac{1}{3}.$$

2. If $x \propto y$, and $x = 6$ when $y = 10$, find y when $x = 15$.

3. If $x \propto y$, and $x = h$ when $y = k$, find y when $x = m$.

4. If x varies inversely as y, and $x = 6$ when $y = 7$, find x when $y = 21$.

Solution: The variation is inverse. Hence

$$x_1 : x_2 = \frac{1}{y_1} : \frac{1}{y_2}. \qquad (1)$$

Substituting in (1), $\quad 6 : x_2 = \frac{1}{7} : \frac{1}{21}. \qquad (2)$

Solving (2), $\quad x_2 = 2.$

5. If $x \propto \frac{1}{y}$, and $x = 4$ when $y = 100$, find x when $y = 10$.

6. If $y \propto \frac{1}{z}$, and $y = h$ when $z = k$, find y when $z = m$.

7. If x varies jointly as y and z, and $x = 24$ when $y = 6$ and $z = 8$, find x when $y = 9$ and $z = 4$.

Solution: The variation is joint. Therefore

$$\frac{x_1}{x_2} = \frac{y_1 z_1}{y_2 z_2}. \qquad (1)$$

Substituting in (1), $\quad \frac{24}{x_2} = \frac{6 \cdot 8}{9 \cdot 4}. \qquad (2)$

Solving (2), $\quad x_2 = 18.$

8. If x varies jointly as y and z, and $x = 3$ when $y = 4$ and $z = 5$, find x when $y = 20$ and $z = 2$.

9. If x varies directly as y and inversely as z, and $x = 10$ when $y = 4$ and $z = 9$, find x when $y = 2$ and $z = 6$.

HINT. Here $\quad x_1 : x_2 = \frac{y_1}{z_1} : \frac{y_2}{z_2}.$

10. If d varies directly as t^2, and $d = 64$ when $t = 2$, find d when $t = 4$.

HINT. Here $\quad \frac{d_1}{d_2} = \frac{t_1^2}{t_2^2}.$

11. If V varies directly as T and inversely as P, and $V = 80$ when $P = 15$ and $T = 400$, find P when $T = 450$ and $V = 45$.

12. The weight of any object below the surface of the earth varies directly as its distance from the center of the earth. An object weighs 100 pounds at the surface of the earth. What would be its weight (*a*) 1000 miles below the surface (radius of the earth = 4000 miles)? (*b*) 2000 miles below the surface? (*c*) at the center of the earth?

13. If a wagon wheel 4 feet 8 inches in diameter makes 360 revolutions in going a certain distance, how many revolutions will a wheel 5 feet in diameter make in going the same distance?

14. The distance which sound travels varies directly as the time. A man measures with a stop watch the time elapsing between the sight of the smoke from a hunter's gun and the sound of its report. When the hunter was 1 mile distant, the time was $4\frac{4}{5}$ seconds. How far off was the hunter when the observed time was 2 seconds?

15. When the volume of air in a bicycle pump is 24 cubic inches, the pressure on the handle is 30 pounds. Later, when the volume of air is 20 cubic inches, the pressure is 36 pounds. Assume that a proportion exists here, determine whether it is direct or inverse, and find the volume of the air when the pressure is 48 pounds.

16. The distance (in feet) through which a body falls from rest varies as the square of the time in seconds. If a body falls 16 feet in 1 second, how far will it fall in 6 seconds?

17. The intensity (brightness) of light varies inversely as the square of the distance from the source of the light. A reader holds his book 4 feet from a lamp, and later 6 feet distant. At which distance does the page appear brighter? how many times as bright?

18. A lamp shines on the page of a book 9 feet distant. Where must the book be held so that the page will receive four times as much light? twice as much light?

19. The weight of an object above the surface of the earth varies inversely as the square of its distance from the center of the earth. An object weighs 100 pounds at the surface of the earth. What would it weigh (*a*) 1000 miles above the surface? (*b*) 2000 miles above the surface? (*c*) 4000 miles?

20. The area of a circle varies as the square of its radius. The area of a certain circle is 154 square inches and its radius is 7 inches. Find the radius of a circle whose area is 594 square inches.

21. The weight of a sphere of given material varies directly as the cube of its radius. Two spheres of the same material have radii 2 inches and 6 inches respectively. The first weighs 6 pounds. Find the weight of the second.

22. The time required by a pendulum to make one vibration varies directly as the square root of its length. If a pendulum 100 centimeters long vibrates once in 1 second, find the time of one vibration of a pendulum 64 centimeters long.

23. Find the length of a pendulum which vibrates once in 2 seconds; once in 5 seconds.

24. The pressure of wind on a plane surface varies jointly as the area of the surface and the square of the wind's velocity. The pressure on 1 square foot is .9 pounds when the rate of the wind is 15 miles per hour. Find the velocity of the wind when the pressure on 1 square yard is 18 pounds.

25. The pressure of water on the bottom of a containing vessel varies jointly with the area of the bottom and the depth of the water. When the water is 1 foot deep, the pressure on 1 square foot of the bottom is 62.5 pounds. (*a*) Find the pressure on the bottom of a tank 12 feet long and 8 feet wide in which the water is 6 feet deep. (*b*) Find the total pressure on one end and on one side.

26. The cost of ties for a railroad varies directly as the length of the road and inversely as the distance between the ties. The cost of ties for a certain piece of road, the ties being 2 feet apart, was $1320. Find the cost of ties for a piece twenty times as long as the first, if the ties are $2\frac{1}{2}$ feet apart.

27. The force of gravitation between the earth and the sun is 36×10^{17} tons. Imagine this force replaced by a gigantic cable the ends of which are tied one to the earth and the other to the sun. Then compute in miles the diameter of the cross section of the cable which would just stand the strain, knowing that a cable whose cross section is 1 square inch will support 60,000 pounds without breaking.

28. It has been shown that if one variable varies as another, the second multiplied by a constant number equals the first. It is often desirable to determine this constant. Suppose such to be the case in Problem 16.

Solution: Since $d \propto t^2$, (1)

then $d = Kt^2$ (K being some constant). (2)

But when $t = 2$ and $d = 64$, substituting in (2) gives

$$64 = K(2)^2 = 4K; \text{ whence } K = 16.$$

29. In Problem 21 $W \propto r^3$; hence $W = Kr^3$. Find K.

30. In Problem 22 $t \propto \sqrt{l}$. Find the constant which multiplied by $\sqrt{l}$ gives t.

31. In Problem 14 find the constant connecting d and t in the equation $d = Kt$ and determine its practical meaning.

32. It has been shown that if one variable varies inversely as a second, the product of the two is a constant. Find this constant in Problem 15.

33. The area of a triangle varies jointly as its base and altitude. What is the constant involved?

34. The area of a circle varies as the square of the diameter. What is the constant involved?

35. The volume of a sphere varies as the cube of the diameter. What is the constant involved?

36. Give concrete illustrations of direct, inverse, and joint variation different from those given in this book.

37. If $x^2 + y^2 \propto x^2 - y^2$, prove $x + y \propto x - y$.

38. If $x^3 + y^3 \propto x^3 - y^3$, prove $x + y \propto x - y$.

39. If $x + y \propto x - y$, prove $x^2 - xy + y^2 \propto x^2 + xy + y^2$.

40. In the equation $x = Ky$ assign some numerical value to K and graph the resulting equation.

Direct variation between two variables is represented by what sort of a curve?

41. Assign some numerical value to K in $x = \frac{K}{y}$ and then graph the resulting equation.

Inverse variation between two variables is represented by what sort of a curve?

42. Construct a graph showing the relation between two variables one of which varies as the square of the other.

43. Construct a graph showing the relation between two variables if one varies inversely as the square of the other.

44. The cost of a certain kind of nails is 5 cents per pound. If c represents the cost of n pounds, what is the equation which connects c and n? What is the graph of this equation?

45. If a quantity of gas is under a changing pressure, the volume decreases as the pressure increases, and vice versa. Let P denote the pressure and V the volume; then state an equation expressing the relation between P and V and determine the kind of a curve which will represent this relation.

May negative pressures or negative volumes be considered here?

CHAPTER XVI

IMAGINARIES

119. Definitions. When the square root of a negative number arose in our previous work, it was called an **imaginary**, and no attempt was then made to use it or to explain its meaning. The treatment of imaginaries was deferred because there were so many topics of more importance to the beginner. It must not be supposed, however, that imaginaries are not of great value in mathematics. They are also of much use in certain branches of applied science; and it is unfortunate that symbols which can be used in numerical computations to obtain practical results should ever have been called imaginary. By such a name something unreal and fanciful is suggested. To obviate this it has been proposed to call imaginary numbers *orthotomic* numbers, but this name has been little used.

The equation $x^2 + 1 = 0$, or $x^2 = -1$, asks the question, "What is the number whose square is -1?" By defining a new number, $\sqrt{-1}$, as a number whose square is -1, we obtain one root for the equation $x^2 + 1 = 0$. Similarly, $\sqrt{-5}$ is a number whose square is -5. And, in general, $\sqrt{-n}$ is a number whose square is $-n$. Obviously $\sqrt{-5}$ means something very different from $\sqrt{5}$.

The positive numbers are all multiples of the unit $+1$, and the negative numbers are all multiples of the unit -1. Similarly, **pure imaginary** numbers are real multiples of the imaginary unit $\sqrt{-1}$.

Thus $\sqrt{-1} + \sqrt{-1} = 2\sqrt{-1}$, and $\sqrt{-1} + 2\sqrt{-1} = 3\sqrt{-1}$, etc.
Further, $\sqrt{-4} = 2\sqrt{-1}$; $\sqrt{-a^2} = a\sqrt{-1}$; $\sqrt{-5} = \sqrt{5}\sqrt{-1}$.

The imaginary unit $\sqrt{-1}$ is often denoted by the letter i; that is, $3\sqrt{-1} = 3\,i$.

If a real number be united to a pure imaginary by a plus sign or a minus sign, the expression is called a **complex number.**

Thus $-2+\sqrt{-1}$ and $3-2\sqrt{-4}$ are *complex numbers.* The general form of a complex number is $\boldsymbol{a+bi}$, in which a and b may be any real numbers.

Note. Up to the time of Gauss (1777–1855) complex numbers were not clearly understood, and were usually thought of as absurd. The situation reminds one of the time when negative numbers were similarly regarded, and the veil was removed from both in about the same way. It was found that negative numbers really had a significance; that they could be used in problems that involve debt, opposite directions, and many other everyday relations. The interpretation of imaginary numbers is not quite so obvious, but none the less actual and simple. As soon as it was seen that they could be represented with real numbers as points on a plane (see page 231) the ice was broken, and it needed only the insight and authority of a man like Gauss to give them their proper place in mathematics.

120. Addition and subtraction of imaginaries. The fundamental operations of addition and subtraction are performed on imaginary and complex numbers as they are performed on real numbers and ordinary radicals of the same form.

Thus $$2\sqrt{-1}+4\sqrt{-1}=6\sqrt{-1},$$
and $$5\sqrt{-1}-3\sqrt{-1}=2\sqrt{-1}.$$
Also $$3+5\sqrt{-1}+4-2\sqrt{-1}=7+3\sqrt{-1}.$$
Similarly, $$a+bi+c+di=a+c+(b+d)i.$$

EXERCISES

Simplify:

1. $3\sqrt{-1}+4\sqrt{-1}-2\sqrt{-1}$.

2. $\sqrt{-4}+\sqrt{-9}$.

3. $\sqrt{-25}-\sqrt{-16}$.

4. $5\sqrt{-1}+\sqrt{-9}$.

5. $\sqrt{-4}+\sqrt{-16}$.

6. $5\sqrt{-36x^2}-2\sqrt{-49x^2}$.

7. $\sqrt{-18}+\sqrt{-8}$.

8. $(-12)^{\frac{1}{2}}+(-27)^{\frac{1}{2}}$.

9. $3+2\sqrt{-1}+5-6\sqrt{-1}$.

10. $5\sqrt{-x^2}-7a-3\sqrt{-x^2}$.

11. $4-8\sqrt{-1}+16-3\sqrt{-4}$.

12. $6-2\sqrt{-64x^2}-3\sqrt{-25x^2}+8$.

13. $18 - 3(-1)^{\frac{1}{2}} + 6(-2)^{\frac{1}{2}} + (-100)^{\frac{1}{2}} + 4.$

14. $5\sqrt{-3} + 3\sqrt{-2} - \sqrt{-27} + 2\sqrt{-8}.$

15. $6\sqrt{-4a^4} - 7a^2\sqrt{-9} + 3\sqrt{-6} - 5\sqrt{-24}.$

16. $(12 - 6\sqrt{-9}) - (15 + 2\sqrt{-36}).$

17. $3a - 2x - (2a\sqrt{-a^2} - 5ra^2\sqrt{-1}).$

18. $(x - iy) - (n - iv).$

Write as a multiple of $\sqrt{-1}$:

19. $\sqrt{-10}.$	**21.** $2\sqrt{-3}.$	**23.** $a\sqrt{-b}.$
20. $\sqrt{-6}.$	**22.** $\sqrt{-a}.$	**24.** $\sqrt{-a-b}.$

121. Multiplication of imaginaries. By the definition of square root, the square of $\sqrt{-n}$ is $-n$.

Therefore

$$(\sqrt{-1})^2 = -1.$$
$$(\sqrt{-1})^3 = (\sqrt{-1})^2\sqrt{-1} = -1\sqrt{-1}.$$
$$(\sqrt{-1})^4 = (\sqrt{-1})^2(\sqrt{-1})^2 = (-1)(-1) = 1.$$

To multiply $\sqrt{-2}$ by $\sqrt{-3}$ we write $\sqrt{-2}$ as $\sqrt{2}\cdot\sqrt{-1}$, and $\sqrt{-3}$ as $\sqrt{3}\cdot\sqrt{-1}$.

Then
$$\sqrt{-2}\cdot\sqrt{-3} = (\sqrt{2}\cdot\sqrt{-1})(\sqrt{3}\cdot\sqrt{-1})$$
$$= \sqrt{6}\cdot\sqrt{-1}\cdot\sqrt{-1} = -\sqrt{6}.$$

Similarly,

$$2\sqrt{-5}(-3\sqrt{-2}) = 2\sqrt{5}\cdot\sqrt{-1}(-3\sqrt{2}\cdot\sqrt{-1}) = 6\sqrt{10}.$$

In general, if the $\sqrt{-a}$ and the $\sqrt{-b}$ are two imaginaries whose product (or quotient) is desired, they should first be written in the form $\sqrt{a}\cdot\sqrt{-1}$ and $\sqrt{b}\cdot\sqrt{-1}$, and the multiplication (or division) should then be performed. This method will prevent many errors.

In this connection it must be clearly understood that one rule followed in multiplication of radicals (see page 98) does not apply to imaginary numbers.

Thus $\sqrt{2}\cdot\sqrt{3} = \sqrt{2\cdot 3} = \sqrt{6}.$
But $\sqrt{-2}\cdot\sqrt{-3}$ **does not equal** $\sqrt{(-2)(-3)}$, which equals $\sqrt{6}.$

In multiplying two complex numbers, write each expression in the form $a + bi$ and proceed as in the following

EXAMPLE

Multiply $2+\sqrt{-3}$ by $3-\sqrt{-7}$.

Solution:

$$2+\sqrt{-3} = 2+\sqrt{3}\cdot\sqrt{-1}$$
$$3-\sqrt{-7} = 3-\sqrt{7}\cdot\sqrt{-1}$$

Multiplying, $\quad 6+3\sqrt{3}\sqrt{-1}-2\sqrt{7}\sqrt{-1}+\sqrt{21}$

Rewriting, $\quad 6+3\sqrt{-3}-2\sqrt{-7}+\sqrt{21}$

EXERCISES

Perform the indicated multiplications and simplify results:

1. $(\sqrt{-1})^5$.

2. $(\sqrt{-1})^6$.

3. $(\sqrt{-1})^7$.

4. $(\sqrt{-1})^8$.

5. $2\sqrt{-1}\cdot 3\sqrt{-1}$.

6. $\sqrt{-9}\cdot\sqrt{-16}$.

7. $\sqrt{-5}\,(-\sqrt{-6})$.

8. $\sqrt{-25}\cdot\sqrt{3}$.

9. $2\sqrt{-3}\cdot 3\sqrt{-2}$.

10. $\sqrt{-m}\cdot\sqrt{-n}$.

11. $4\sqrt{-5}\,(-3\sqrt{-6})$.

12. $\sqrt{a+b}\cdot\sqrt{-a-b}$.

13. $(2+\sqrt{-1})(2-\sqrt{-1})$.

14. $(3+\sqrt{-2})(3-\sqrt{-2})$.

15. $(4-2\sqrt{3}\,i)(4+2\sqrt{3}\,i)$.

16. $(3+\sqrt{-1})(6-\sqrt{-2})$.

17. $(4-2\,i)(3-2\sqrt{3}\,i)$.

18. $(a+ib)(c+id)$.

19. $(a+ib)(a+ib)$.

20. $(a+bi)(a-bi)$.

21. $(-\frac{1}{2}+\frac{1}{2}\sqrt{-3})^2$.

22. $(-\frac{1}{2}-\frac{1}{2}\sqrt{-3})^2$.

23. $(x-iy)^2-(x+iy)^2$.

24. $(-\frac{1}{2}+\frac{1}{2}\sqrt{-3})^3-(-\frac{1}{2}-\frac{1}{2}\sqrt{-3})^3$.

25. $(a+i\sqrt{1-x^2})(a-i\sqrt{1-x^2})$.

Note. Long before the time of Gauss mathematicians had performed the operations of multiplication and division on complex numbers by the same rules that they used for real numbers. As early as 1545 Cardan stated that the product of $5+\sqrt{-15}$ and $5-\sqrt{-15}$ was 40. However, he was not always equally fortunate in obtaining correct results, for in another place he sets $\frac{1}{4}\left(-\sqrt{-\frac{1}{4}}\right)=\frac{1}{\sqrt{64}}=\frac{1}{8}$.

Even the rather complicated formula for extracting any root of a complex number was discovered in the early part of the eighteenth century. But all of these operations were purely formal, and seemed to most mathematicians a mere juggling with symbols until Gauss showed clearly the place and usefulness of such numbers.

122. Division of imaginaries. One complex number is the conjugate of another if their product is *real.* Thus $a + bi$ and $a - bi$ are **conjugates.** Conjugate complex numbers are used in division of imaginary expressions as conjugate radicals are used in division of radicals.

Division by an imaginary is performed by writing the dividend over the divisor as a fraction and then multiplying both numerator and denominator by the simplest imaginary expression which will make the resulting denominator real and rational.

EXAMPLES

1. $\sqrt{-6} \div \sqrt{-2}$.

Solution:

$$\sqrt{-6} \div \sqrt{-2} = \frac{\sqrt{-6}}{\sqrt{-2}} = \frac{\sqrt{-6} \cdot \sqrt{-2}}{\sqrt{-2} \cdot \sqrt{-2}}$$

$$= \frac{\sqrt{6} \cdot \sqrt{-1} \cdot \sqrt{2} \cdot \sqrt{-1}}{\sqrt{2} \cdot \sqrt{-1} \cdot \sqrt{2}\sqrt{-1}} = \frac{-\sqrt{12}}{-2} = \sqrt{3}.$$

2. $3 \div (2 + \sqrt{-3})$.

Solution:

$$3 \div (2 + \sqrt{-3}) = \frac{3}{2 + \sqrt{-3}} = \frac{3(2 - \sqrt{-3})}{(2 + \sqrt{-3})(2 - \sqrt{-3})}$$

$$= \frac{6 - 3\sqrt{-3}}{4 + 3} = \frac{6 - 3\sqrt{-3}}{7}.$$

EXERCISES

Perform the indicated division:

1. $\sqrt{-8} \div \sqrt{-2}$.

2. $\sqrt{-6} \div \sqrt{-3}$.

3. $2\sqrt{-3} \div 3\sqrt{-1}$.

4. $\sqrt[4]{-1} \div \sqrt[4]{-4}$.

5. $\sqrt{6} \div \sqrt{-2}$.

6. $(-25)^{\frac{1}{4}} \div (-81)^{\frac{1}{4}}$.

7. $\sqrt{ax} \div \sqrt{-x}$.

8. $\sqrt{-a} \div \sqrt{-b}$.

9. $(-5ax)^{\frac{1}{2}} \div (-2x)^{\frac{1}{2}}$.

10. $\left[(-a^6)^{\frac{1}{4}} - (-a^2)^{\frac{1}{4}}\right] \div (-a)^{\frac{1}{4}}$.

11. $2 \div (1 - \sqrt{-1})$.

12. $3 \div (2 - \sqrt{-2})$.

13. $2\sqrt{-1} \div (\sqrt{-1} + 3)$.

14. $3\sqrt{-2} \div (2\sqrt{-3} + 2)$.

15. $\dfrac{-1 + \sqrt{-3}}{-1 - \sqrt{-3}}$.

16. $\dfrac{1+i}{1-i}$.

17. $\dfrac{a}{a+bi}$.

18. $\dfrac{a+ib}{c+id}$.

19. $(2 + 3i) \div (2i - 1)(5i - 3) = ?$

20. Is $i\sqrt{3} - 1$ a cube root of 8?

21. Is $1 - \sqrt{-3}$ a cube root of -8?

22. Does $x^2 - 4x + 7 = 0$, if $x = 2 \pm \sqrt{-3}$?

23. Does $x = \frac{8}{5}\sqrt{-10}$, $y = -\frac{3}{5}\sqrt{-10}$ satisfy the system $x^2 - xy - 12y^2 = 8$, $x^2 + xy - 10y^2 = 20$?

24. Determine whether the sum and the product of $2 + 3\sqrt{-1}$ and $2 - 3\sqrt{-1}$ are real numbers.

25. Show that the *sum* and the *product* of any two conjugate complex numbers is *real.*

26. Show that the *quotient* of two conjugate complex numbers is *complex.*

27. Point out the error in the following:

The equation $\sqrt{x-y} = i\sqrt{y-x}$ is an identity. (1)

Let $x = a$ and $y = b$, and (1) becomes

$$\sqrt{a-b} = i\sqrt{b-a}. \qquad (2)$$

Now let $x = b$ and $y = a$, and (1) becomes

$$\sqrt{b-a} = i\sqrt{a-b}. \qquad (3)$$

From (2) and (3),

$$\sqrt{a-b} \cdot \sqrt{b-a} = i^2(\sqrt{b-a} \cdot \sqrt{a-b}). \qquad (4)$$

Whence $1 = i^2$, or $1 = -1$.

123. Equations with imaginary roots. The student should now be able to solve and check equations which have imaginary roots.

EXERCISES

Solve and check the equations which follow:

1. $x^2 + 4x + 12 = 0.$
2. $x^2 - 6x + 36 = 0.$
3. $x^2 + 5x + 7 = 0.$
4. $x^2 - 3x + 10 = 0.$
5. $2x^2 + 6x + 5 = 0.$
6. $3x^2 - 7x + 6 = 0.$
7. $x^3 = 1.$

   HINT. If $x^3 = 1$, $x^3 - 1 = 0$.
   Hence $(x-1)(x^2 + x + 1) = 0$.
   Then $x - 1 = 0,$
   and $x^2 + x + 1 = 0$, etc.
8. $x^3 = 8.$
9. $x^3 = 27.$
10. $x^3 = -8.$
11. $x^4 = 1.$
12. $x^4 = 16.$
13. $x^6 = 1.$
14. $x^6 = 64.$
15. How many square roots has any real number? cube roots? fourth roots? sixth roots?
16. What do Exercises 7–14 indicate regarding the number of nth roots which any real number may have?
17. $8x^3 - 27 = 0.$
18. $125x^3 + 64 = 0.$
19. $(x^2+5)(x^2-7)+27=0.$
20. $x^3 - x^2 + 2x - 2 = 0.$
21. $x^6 + 7x^3 - 8 = 0.$
22. $4x^4 + 20x^2 + 21 = 0.$
23. $64x^4 - 12x^2 - 27 = 0.$
24. $9x^4 + 18x^2 + 8 = 0.$
25. $50x^4 + 135x^2 + 36 = 0.$
26. $(x^2 + 9)(x^2 + 2x + 8) = 0.$
27. $(x^2 + x)^2 + 13(x^2 + x) + 36 = 0.$
28. $(x^2 + 5x)^2 + 17(x^2 + 5x) + 66 = 0.$
29. Solve $x + y = 4$, $x^2 - 3xy - y^2 = -39$ and check.
30. Solve $z^2 + x^2 = 130$, $z + x + 2\sqrt{z + x} = 2$ and check.
31. Solve Exercise 4, page 146, and check.

124. Factors involving imaginaries. After studying radicals we enlarged our previous notion of a factor, and, with certain limitations, employed radicals among the terms of a factor. Now in a similar manner, with like restrictions, we extend our notion of a factor still farther and use imaginary numbers as coefficients or as terms in a factor. Consequently $x^2 + 1$ may hereafter be regarded as factorable.

For $x^2+1 = x^2-(-1) = (x+\sqrt{-1})(x-\sqrt{-1})$.

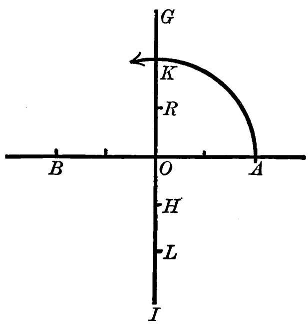
Similarly, $4x^2+9 = 4x^2-(-9) = (2x+3\sqrt{-1})(2x-3\sqrt{-1})$,

and $x^2+6 = x^2-(-6) = (x+\sqrt{-6})(x-\sqrt{-6})$.

Further, $x^3-1=(x-1)(x^2+x+1)$. Hitherto the trinomial x^2+x+1 has been regarded as prime; but the student can easily prove that $x^2+x+1=(x+\frac{1}{2}+\frac{1}{2}\sqrt{-3})(x+\frac{1}{2}-\frac{1}{2}\sqrt{-3})$. Therefore x^3-1 has three factors, $x-1$, $x+\frac{1}{2}+\frac{1}{2}\sqrt{-3}$, and $x+\frac{1}{2}-\frac{1}{2}\sqrt{-3}$.

If the student is curious as to the way in which the factors of x^2+x+1 were found, he may discover the method for himself by studying the results of Exercise 7, page 228.

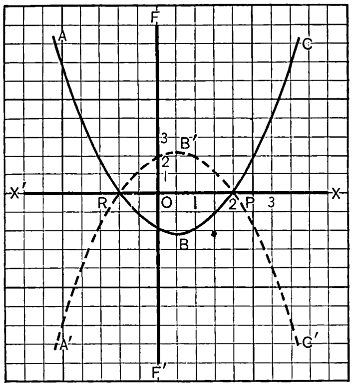
125. Graphical interpretation of pure imaginaries. In our previous graphical work a positive number and a numerically equal negative number, as $+2$ and -2, were represented by equal distances measured in opposite directions, such as *OA* and *OB* of the adjacent figure. Now multiplying $+2$ by -1 gives -2. Hence, if we choose to do so, we may regard -1 as an operator (rotor) which turns *OA* in the direction of the arrow into the position *OB*, or through two right angles (180 degrees).



To make this point clearer note the two curves of the figure on page 230. Curve *ABC* is the graph of the function x^2-x-2. Curve *A'B'C'* is the graph of the function $2+x-x^2$. The latter function was obtained by multiplying x^2-x-2 by -1. The graphical effect of this multiplication is to turn the whole curve *ABC* about *X'X* as an axis through *two right angles* to the position *A'B'C'*.

The preceding illustrations indicate a method of interpreting the $\sqrt{-1}$, which is in strict conformity with our previous graphical work.

First $\sqrt{-1} \cdot \sqrt{-1} = -1$. Now multiplying a number by -1 produces the same effect as multiplying twice in succession by $\sqrt{-1}$. Therefore multiplying by $\sqrt{-1}$ once may be regarded as producing a rotation of one right angle (90 degrees), or one half as much rotation as multiplying by -1.

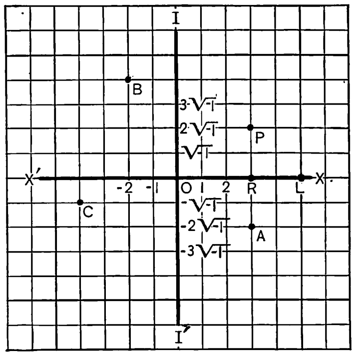


Returning to the figure on page 229, OA, or $+2$, multiplied by $\sqrt{-1}$ would be turned to the position OK. Hence the point K is said to correspond to the number $2\sqrt{-1}$. Similarly, point R corresponds to $\sqrt{-1}$ and point G to $3\sqrt{-1}$. And OH being measured in a direction opposite to OR, OK, and OG, would correspond to $-\sqrt{-1}$. In like manner OL corresponds to $-2\sqrt{-1}$. This last result, however, may be reached differently. Multiplying 2 by $\sqrt{-1}$ three times gives $-2\sqrt{-1}$. These successive multiplications by $\sqrt{-1}$ may be regarded as producing a counterclockwise rotation through three right angles which would locate the point corresponding to $-2\sqrt{-1}$ on OI at L as before.

Therefore the graphical representation of a *pure imaginary* number $b\sqrt{-1}$ is by a point on an axis perpendicular (at right angles) to the axis of real numbers, b units in the direction of OG if b is positive, b units in the direction of OI if b is negative. This new axis will be called the imaginary or I-axis.

126. Graphical representation of a complex number. The complex number $x = 3 + 2\sqrt{-1}$ consists of a real part 3 and the imaginary part $2\sqrt{-1}$. To represent such a number we measure in the adjacent figure 3 units along OX from O to R, and then 2 units parallel to the imaginary axis II' from R. This gives the point P, which is the graphical representation of the complex number

$$x = 3 + 2\sqrt{-1}.$$



If the student pays proper attention to signs, he should now see that the point A corresponds to $x = 3 - 2\sqrt{-1}$, B to $x = -2 + 4\sqrt{-1}$, and C to $x = -4 - \sqrt{-1}$.

In general, if x is a complex number $a + bi$, x is represented by a point a units from the imaginary axis and b units from the real axis, the positive and negative directions being as indicated in the adjacent figure.

Note. It was the discovery of a graphical interpretation for the imaginary numbers which did more than anything else to make them mean something to students of mathematics. Until this discovery they were tolerated because their appearance as the roots of equations was a constant reminder of their existence. But they were usually regarded as meaningless, and the less one had to do with them the better he liked it. About 1800 a Norwegian by the name of Wessel, and the Frenchman, Argand, gave practically the same graphical interpretation as that found in the text, but their work was little noticed till Gauss adopted the method and, by his influence and ability, placed the imaginary number on a firm basis.

EXERCISES

Locate the point x if:

1. $x = 2 + i.$	**6.** $x = -3 - 2i.$	**11.** $x = 2 \pm \sqrt{-3}.$
2. $x = 3 + 3i.$	**7.** $x = 4 - i.$	**12.** $x = 3 \pm 3\sqrt{-2}.$
3. $x = 4 - 2i.$	**8.** $x = 4 + 4i.$	**13.** $x = 2 + \sqrt{-11}.$
4. $x = 1 - 3i.$	**9.** $x = -1 - i.$	**14.** $x = 2 - \sqrt{-11}.$
5. $x = -2 + i.$	**10.** $x = -3 - 5i.$	**15.** $x = \sqrt{-12} - 5.$

The graphical interpretation for the definition of equality of complex numbers is that equal complex numbers are represented by the same point in the plane.

The student should not conclude from the preceding exercises that the imaginary axis has merely replaced the y-axis of our previous graphical work, for such is by no means the case. We have simply used our device of rectangular axes to represent complex numbers. The fact that we have previously made use of the same means to graph a function should not embarrass us any more than it would disturb us to play croquet on a baseball ground. The fact that both take place on the same field does not make them the same game.

For some purposes it is convenient to relate the xy-plane, in which real number relations are represented, and the complex plane more closely to each other. In fact, we may proceed as if we had merely added a third axis, the axis of imaginary numbers, to the two axes of real numbers which we had before. This axis may be regarded as passing through the origin at right angles to both the x-axis and the y-axis. By means of it we may locate such a point as $y = 3$, $x = 2 + 4\sqrt{-1}$, or any point, in fact, in which one coördinate is real and the other imaginary or complex. Then we may go on and construct the imaginary branches of many of those curves, the real graphs of which the student has already drawn.

Note on use of imaginaries. We have explained the laws of addition, subtraction, multiplication, and division for imaginary (and complex) numbers and have made some use of them. It is largely because imaginaries obey these laws that we call them numbers, for it must be admitted that we cannot count objects with imaginary numbers. Nor can we state by means of them our age, our weight, or the area of the earth's surface. It should be remembered, however, that we can do none of these things with negative numbers. We may have a group of objects — books, for example — whose number is 5; but no group of *objects* exists whose number is -5, or -3, or any negative number whatever. If it be asked, How, then, can negative numbers and imaginary numbers have any practical use? the answer is this: They have a practical use because when they enter into our calculations and we have performed the necessary operations upon them and obtained our final result, that result can frequently be interpreted as a concrete number such as is dealt with in ordinary arithmetic. Moreover, if the result cannot be so interpreted, it is, in applied mathematics at least, finally rejected.

In that part of electrical engineering where the theory and measurement of alternating currents of electricity are treated, complex numbers have had extensive use. Their employment in the difficult problems which there arise has given a briefer, a more direct, and a more general treatment than the earlier ones where such numbers are not used.

In theoretical mathematics complex numbers have been of great value in many ways. For example, numerous important theorems about functions are more easily proved under the assumption that the variable is complex. Then, by letting the imaginary part of the complex number become zero, we obtain the proof of the theorem for real values of the variable. Indeed, the student need not go very far beyond this point in his mathematical work to learn that, if e is 2.7182 + (see page 189), $e^{\sqrt{-1}} + e^{-\sqrt{-1}}$ is equal to the real number 1.082 +. At the same time he will learn also how such a form arises, and something of its importance. In a way which we cannot now explain, even so involved an expression as $(a + ib)^{c+id}$ has in higher work a meaning and a use. If the student pursues his mathematical studies far enough, that meaning and use, and a multitude of other uses for complex numbers, will become familiar to him. But the numbers which we have learned in this book to use, namely fractions, negative numbers, irrational numbers, and complex numbers, complete the number system of ordinary algebra, for it can be proved that from the fundamental operations no other forms of number can arise.

CHAPTER XVII

THEORY OF QUADRATIC EQUATIONS

127. Character of the roots of a quadratic equation. It is often desirable to determine the character of the roots of a quadratic without actually solving it. To determine the character of the roots of an equation means to find out whether the roots are real or imaginary, rational or irrational, equal or unequal. These properties of the roots of a given quadratic depend on the three coefficients, which correspond to a, b, and c, in the general quadratic equation $ax^2 + bx + c = 0$. The solution of this equation gives the roots:

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a},$$

and

$$r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

This expression $b^2 - 4ac$ which occurs in each root is called the **discriminant** of the quadratic. If a, b, and c are rational numbers, it is evident from an inspection of the discriminant where it occurs in the values of r_1 and r_2, that the following statements are true:

I. *If $b^2 - 4ac$ is positive and not a perfect square, the roots are real, unequal, and irrational.*

II. *If $b^2 - 4ac$ is positive and a perfect square, the roots are real, unequal, and rational.*

III. *If $b^2 - 4ac$ is zero, the roots are equal.*

In this case there is really but one root, $\frac{-b}{2a}$.

IV. *If $b^2 - 4ac$ is negative, the roots are imaginary.*

EXERCISES

Determine the character of the roots of the following equations by the use of the discriminant:

1. $2x^2 + 5x - 6 = 0$.

Solution: $b^2 - 4ac = (5)^2 - 4 \cdot 2 \cdot (-6) = 25 + 48 = 73$.
Therefore the roots are real, unequal, and irrational.

2. $x^2 - 5x + 6 = 0$.
3. $5x^2 - 11x + 2 = 0$.
4. $4x^2 - 20x + 25 = 0$.
5. $5x^2 - 3x - 3 = 0$.
6. $7x^2 - 2x + 10 = 0$.
7. $x^2 - 6x + 6 = 0$.
8. $4x^2 = 9 - 9x$.
9. $5x = x^2 + 5$.
10. $x^2 - 5x + 7 = 0$.
11. $12x^2 - 7x + 6 = 0$.
12. $x(x - 5) = x - 16$.
13. $\frac{3}{x^2} + \frac{11}{x} = 20$.
14. $5x - \frac{11}{7} - \frac{6}{7x} = 0$.

Determine the values of K which will make the roots of the following equations equal. (To say the roots of a quadratic are equal is the usual mathematical way of stating that *the equation has but one root.*)

15. $x^2 - Kx + 16 = 0$.

Solution: $a = 1,\ b = -K,\ c = 16$.
Hence $b^2 - 4ac = K^2 - 64$.
In order that the roots be equal, $b^2 - 4ac$ must equal zero.
Therefore $K^2 - 64 = 0$.
Whence $K = \pm 8$.

Check: Substituting 8 for K in the original equation,
$x^2 - 8x + 16 = 0$.
Whence $x = 4$, only.
Similarly, substituting $K = -8$, $x^2 + 8x + 16 = 0$.
Whence $x = -4$, only.

16. $x^2 - Kx + 36 = 0$.
17. $x^2 - 3Kx + 81 = 0$.
18. $2x^2 + 4Kx + 98 = 0$.
19. $x^2 - 10x + K = 0$.
20. $2x^2 + 8x + K = 0$.
21. $9x^2 + 30x + K + 9 = 0$.

22. $4Kx^2 - 60x + 25 = 0.$

23. $9K^2x^2 - 84x + 49 = 0.$

24. $49x^2 - (K+3)x + 4 = 0.$

25. $(K^2+5)x^2 - 30x + 25 = 0.$

26. $(K^2 + 17)x^2 + (5K - 4)x + 4 = 0.$

Determine the relation between h and k which will make the roots of the following equations equal:

27. $k^2x^2 + 6hx + 9 = 0.$

28. $kx^2 - 2hx + 16 = 0.$

29. $x^2 + 4kx + 4h = 0.$

30. $kx^2 - 2hx + 6 = 0.$

Determine the values of a for which the following systems will have two sets of equal roots:

31. $y^2 = ax,$ $y = x + 1.$

32. $y^2 = 2x,$ $y = x + a.$

33. $x^2 + y^2 = a^2,$ $y = x + 1.$

34. $x^2 + y^2 = 2x,$ $y = x + a.$

128. Relations between the roots and the coefficients. The roots of $ax^2 + bx + c = 0$ are

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \tag{1}$$

$$r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \tag{2}$$

(1) + (2) gives $$r_1 + r_2 = \frac{-2b}{2a} = -\frac{b}{a}. \tag{3}$$

(1) × (2) gives $$r_1 r_2 = \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{c}{a}. \tag{4}$$

The general quadratic equation may be written

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0. \tag{5}$$

Then for any quadratic in which the coefficient of x^2 is 1:

I. *From* (3) *and* (5) *the coefficient of* x, $\frac{b}{a}$, *is the sum of the roots with the sign changed.*

II. *From* (4) *and* (5) *the constant term,* $\frac{c}{a}$, *is the product of the roots.*

I and II may be used to form a quadratic whose roots are given.

EXERCISES

Form the quadratic whose roots are:

1. $5, -3$.

Solution: $r_1 + r_2$ with the sign changed $= -2$.

$$r_1 \times r_2 = 5(-3) = -15.$$

Hence the required equation is $x^2 - 2x - 15 = 0$.

2. $2, 7$.
3. $-3, 10$.
4. $-4, -5$.
5. $-12, -1$.
6. $\frac{2}{3}, 5$.
7. $10, -\frac{1}{5}$.
8. $2+\sqrt{3}, 2-\sqrt{3}$.
9. $-3 \pm \sqrt{5}$.
10. $\frac{4}{3} \pm \sqrt{7}$.
11. $\frac{3}{2} \pm \frac{1}{2}\sqrt{6}$.
12. $\dfrac{-6 \pm 2\sqrt{3}}{5}$.
13. $\sqrt{5}, -3\sqrt{5}$.
14. $3-\sqrt{2}, 2+\sqrt{2}$.
15. a, c.
16. $a, \dfrac{1}{a}$.
17. $3a, \dfrac{2a}{3}$.
18. $a+1, \dfrac{1}{a-1}$.

Solve the following equations and check each by showing that the sum of the roots with its sign changed is the coefficient of x, and that the product of the roots is the constant term:

19. $x^2 - 12x - 13 = 0$.
20. $x^2 - 10x + 16 = 0$.
21. $x^2 + 3x + 3 = 0$.
22. $x^2 - 5x + 20 = 0$.
23. $x^2 + 2x + 2 = 0$.

24. One root of $x^2 - 4x - 12 = 0$ is -2. Find the other root.

Solution: Let r_2 be the required root.

Then $\quad -(r_1 + r_2) = -(-2 + r_2) = -4.$

Solving, $\quad r_2 = 6.$

Check: $\quad r_1 r_2 = (-2)(6) = -12.$

25. One root of $x^2 + 7x - 18 = 0$ is -9. Find the other root.

Find the value of the literal coefficient in the following:

26. $x^2 + 2x - c = 0$, if one root is 3.
27. $x^2 - x - c = 0$, if one root is 10.
28. $x^2 + 8x - c = 0$, if one root is -2.
29. $x^2 - cx - 70 = 0$, if one root is 10.
30. $x^2 + 2bx + 25 = 0$, if one root is -5.

31. $x^2 - 3ax - 52 = 0$, if one root is 4.

32. $2x^2 - 11x + c = 0$, if one root is 5.

33. $ax^2 - 20x + 12 = 0$, if one root is $\frac{2}{3}$.

34. $ax^2 - 6x - 21 = 0$, if one root is -3.

35. $x^2 - 8x + c = 0$, if one root is three times the other.

36. $x^2 + 7x + c = 0$, if one root exceeds the other by 1.

37. $x^2 + 11x + b = 0$, if the difference between the roots is 9.

38. $x^2 - 5x - c = 0$, if the difference between the roots is 7.

39. $x^2 - 5x - a = 0$, if the difference between the roots is -13.

129. Number of roots of a quadratic. Up to this we have assumed that a quadratic equation has but two roots. This fact can be proved from the preceding work as follows:

If we write the equation $ax^2 + bx + c = 0$ in the form $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ and substitute therein from (3) and (4) on page 236, we get $x^2 - (r_1 + r_2)x + r_1r_2 = 0$. This can be factored and written as $(x - r_1)(x - r_2) = 0$. Now if any value of x different from r_1 and r_2, say r_3, be a root of this equation, such a value when substituted for x must satisfy the equation $(x - r_1)(x - r_2) = 0$.

Hence $(r_3 - r_1)(r_3 - r_2)$ must equal zero. By definition, however, r_3 is different from r_1 and r_2. Consequently neither the factor $(r_3 - r_1)$ nor $(r_3 - r_2)$ can equal zero, and therefore their product cannot equal zero. This proves that no additional value, r_3, can satisfy the equation $x^2 - (r_1 + r_2)x + r_1r_2 = 0$. As this equation is but another form of $ax^2 + bx + c = 0$, the latter has only two roots.

130. Formation of equations with given roots. The method of forming quadratic equations which was used in the preceding exercise applies to equations having two roots only. A reversal of the method of solving equations by factoring (page 31), however, enables us to build up an equation with any number of given roots.

The correctness of the method for three given roots will be clear from what follows:

Form the equation whose roots are a, b, and c.

Write $(x - a)(x - b)(x - c) = 0$. (1)

Now if a is put for x in (1), we obtain $0 = 0$.

Similarly, if b or c is put for x in (1), the result is $0 = 0$.

Therefore (1) is the equation whose roots are a, b, and c, and the expanded form of (1), $x^3 - (a + b + c)x^2 + (ab + ac + bc)x + abc = 0$, is the required equation.

The same reasoning applies if we form in this way an equation with any number of given roots.

Note. The relations between the roots and the coefficients of an equation were discovered at about the same time by Vieta in France, by Girard in Holland, and by Harriot in England. Vieta actually wrote the cubic equation in about the form given in the text so as to display these relations. The very important algebraical theories which result from these properties were developed in detail by Newton, and have been the subject of study by many of the most distinguished mathematicians since his time.

EXAMPLES

1. Form the equation whose roots are 3 and -5.

Solution: By the conditions, $x = 3$ and $x = -5$.
Therefore $x - 3 = 0$ and $x + 5 = 0$.
Then $(x - 3)(x + 5) = 0$. (1)
Expanding, $x^2 + 2x - 15 = 0$. (2)
Substitution shows that 3 and -5 are the roots of (1) and (2).

2. Form the equation whose roots are 1, 3, and -2.

Solution: As before, $x - 1 = 0$, $x - 3 = 0$, and $x + 2 = 0$.
Therefore $(x - 1)(x - 3)(x + 2) = 0$. (1)
Expanding, $x^3 - 2x^2 - 5x + 6 = 0$. (2)
Inspection shows that the given roots 1, 3, and -2 satisfy the equations (1) and (2).

EXERCISES

By the method used in the preceding examples form the equation whose roots are:

1. 3, 7.

2. 4, -5, 6.

3. $1 + \sqrt{3}$, $1 - \sqrt{3}$.

4. $2 \pm \sqrt{5}$.

5. $\dfrac{3 \pm \sqrt{7}}{2}$.

6. 3, -3, 8.

7. 1, $\frac{3}{2}$, -2.

8. $1 \pm \sqrt{3}$, 3.

9. $a + b$, $a - b$.

10. $\frac{1}{a}$, $5a$.

11. $3c \pm \sqrt{2a}$.

12. $\frac{4a \pm \sqrt{3c}}{2}$.

13. r_1, r_2, r_3.

14. 3, $2 \pm \sqrt{a}$.

15. -5, -7, 6, 8.

16. $2 \pm \sqrt{3}$, $3 \pm \sqrt{2}$.

17. 1, -2, $a \pm \sqrt{a}$.

131. Factors of quadratic expressions. Let r_1 and r_2 be the roots of $ax^2 + bx + c = 0$.

Then $x^2 + \frac{b}{a}x + \frac{c}{a} = x^2 - (r_1 + r_2)x + r_1r_2 = (x - r_1)(x - r_2)$,

or $$ax^2 + bx + c = a(x - r_1)(x - r_2).$$

Therefore the three factors a, $x - r_1$, and $x - r_2$ of any quadratic expression can be written if we first set the expression equal to zero (see § 17) and solve for r_1 and r_2 the equation thus formed. Obviously the character of the roots so obtained will determine the character of the factors. Hence by the use of the discriminant $b^2 - 4ac$ we can decide whether the factors of a quadratic *expression* are real or imaginary, rational or irrational, without factoring it.

EXERCISES

Determine which of the following expressions have rational factors:

1. $x^2 - 3x - 40$.
2. $2x^2 + 5x - 7$.
3. $7x^2 - 9x + 18$.
4. $24x^2 - x - 10$.
5. $72x^2 - 17x + 1$.
6. $5x^2 + 3x - 20$.
7. $3x^2 - 9x + 28$.
8. $33h^2 - 233h - 6$.
9. $x^2 - 2ax + (a^2 - b^2)$.
10. $abx^2 - (b^2 + a^2)x + ab$.

Separate into rational, irrational, or imaginary factors:

11. $2x^2 + 5x - 8$.

Solution: Let $2x^2 + 5x - 8 = 0$.

Solving by formula, $x = \frac{-5 \pm \sqrt{25 - (-64)}}{4} = \frac{-5 \pm \sqrt{89}}{4}$.

Then $r_1 = \dfrac{-5+\sqrt{89}}{4}$ and $r_2 = \dfrac{-5-\sqrt{89}}{4}$.

Therefore $2x^2 + 5x - 8 = 2\left[x - \dfrac{-5+\sqrt{89}}{4}\right]\left[x - \dfrac{-5-\sqrt{89}}{4}\right]$

$= \frac{1}{8}(4x + 5 - \sqrt{89})(4x + 5 + \sqrt{89})$.

12. $x^2 - 7x - 30$.

13. $x^2 - 4x - 1$.

14. $x^2 + 2x + 2$.

15. $x^2 + 4x - 9$.

16. $4x^2 - 12x - 9$.

17. $25x^2 + 20x + 4$.

18. $6x^2 + 14x - 40$.

19. $10 - 9x - 9x^2$.

20. $10x^2 + 12 - 26x$.

21. $x^2 + 7x + 8$.

22. $x^2 + x + 1$.

23. $x^2 + 1$.

24. $x^2 + 9$.

25. $x^2 - 2ax + a^2 - b$.

26. $x^2 + 6ax + 9a^2 - 4b$.

27. $4x^2 + 4ax + a^2 - 4c$.

28. $x^2 - 4ax + 4a^2 + c$.

29. $ax^2 + bx + c$.

30. $x^2 - xy + 5x - 2y + 6$.

Solution: Let $x^2 - xy + 5x - 2y + 6 = 0$.

Then $x^2 + (5 - y)x - 2y + 6 = 0$.

Solving for x in terms of y by the formula,

$$x = \frac{-(5-y) \pm \sqrt{(5-y)^2 - 4(-2y+6)}}{2}$$

$$= \frac{-5 + y \pm \sqrt{y^2 - 2y + 1}}{2}.$$

Whence $x = -2$ and $y - 3$.

Therefore $x^2 - xy + 5x - 2y + 6 = (x + 2)(x - y + 3)$.

31. $3x^2 - 6xy + 14x - 4y + 8$.

32. $x^2 - xy - 2y^2 + 3x - 6y$.

33. $x^2 - 4xy - y + 3y^2 - 2 - x$.

34. $x^2 - 2y^2 - xy + 2x + 5y - 3$.

35. $6x^2 + xy - 12y^2 + x + 10y - 2$.

CHAPTER XVIII

THE BINOMIAL THEOREM

132. Powers of binomials. The following identities are easily obtained by actual multiplication:

$$(a+b)^2 = a^2 + 2\,ab + b^2. \tag{1}$$

$$(a+b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3. \tag{2}$$

$$(a+b)^4 = a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4. \tag{3}$$

$$(a+b)^5 = a^5 + 5\,a^4b + 10\,a^3b^2 + 10\,a^2b^3 + 5\,ab^4 + b^5. \tag{4}$$

If $a+b$ is replaced by $a-b$, the even-numbered terms in each of the preceding expressions will then be negative and the odd-numbered terms will be positive.

133. The expansion of $(a+b)^n$. The form of the expansion for the general case will now be indicated:

The first term is $\boldsymbol{a^n}$ *and the last is* $\boldsymbol{b^n}$.

The second term is $\boldsymbol{na^{n-1}b}$.

The exponents of **a** *decrease by 1 in each term after the first.*

The exponents of **b** *increase by 1 in each term after the second.*

The product of the coefficient of any term and the exponent of **a** *in that term, divided by the exponent of* **b** *increased by 1, gives the coefficient of the next term.*

The sign of each term of the expansion is + if **a** *and* **b** *are positive; the sign of the odd-numbered terms is − if* **b** *only is negative.*

The number of terms in the expansion is $n+1$.

According to the rule, $(a+b)^n = a^n + \frac{n}{1}a^{n-1}b +$

$$\frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \cdots + b^n. \tag{1}$$

The preceding expansion expresses in symbols the law known as the **binomial theorem.** The theorem holds for all positive values of n and with certain limitations (see § 136) for negative values as well. This will be assumed without proof.

Note. The coefficients of the various terms in the binomial expansion are displayed in a most elegant form as follows:

$$\begin{array}{c} 1 \\ 1\ \ 1 \\ 1\ \ 2\ \ 1 \\ 1\ \ 3\ \ 3\ \ 1 \\ 1\ \ 4\ \ 6\ \ 4\ \ 1 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{array}$$

In this arrangement each row is derived from the one above it by observing that each number is equal to the sum of the two numbers, one to the right and the other to the left of it, in the line above. Thus $4 = 1 + 3$, $6 = 3 + 3$, etc. The next line is 1 5 10 10 5 1. The successive lines of this table give the coefficients for the expansions of $(a + b)^n$ for the various values of n. Thus the numbers in the last line of the triangle are seen to be the coefficients when $n = 4$; the next line would give those for $n = 5$. This arrangement is known as Pascal's Triangle, and was published in 1665. It was probably known to Tartaglia nearly a hundred years before its discovery by Pascal.

134. The factorial notation. The notation $5!$, or $\underline{5|}$, signifies $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$, or 120. Similarly, $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$.

In general, $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-2)(n-1)n$.

The sign $n!$, or $\underline{n|}$, is read **factorial *n*.**

With the factorial notation the denominators of the third and fourth terms in the expansion of $(a + b)^n$ in § 133 become $2!$ and $3!$ respectively.

EXERCISES

Expand by the rule:

1. $(a + b)^6$. **3.** $(a + 1)^7$. **5.** $(a + 3)^7$.

2. $(a - 1)^6$. **4.** $(a + 2)^6$. **6.** $(2 - a)^6$.

Obtain the first four terms of:

7. $(a + b)^{20}$. **8.** $(a + b)^{30}$. **9.** $(a + 1)^{40}$. **10.** $(a - 2)^{20}$.

Expand:

11. $(a^2 + 2b)^5$. HINT. To avoid confusion of exponents first write

$$(a^2)^5 + \quad (a^2)^4(2b)^1 + \quad (a^2)^3(2b)^2 + \quad (a^2)^2(2b)^3 + \quad (a^2)^1(2b)^4 + (2b)^5.$$

Then in the spaces left for them put in the coefficients according to the rule of § 133.

Finally, expand, and simplify each term.

12. $(a^2 - 2)^6$.

13. $(a^2 + 2b)^7$.

14. $\left(a^2 + \frac{1}{b}\right)^5$.

15. $\left(a^2 - \frac{1}{a^3}\right)^6$.

Obtain in simplest form the first four terms of:

16. $(a^2 + 2b)^{20}$.

17. $\left(a^2 - \frac{2}{a}\right)^{30}$.

18. $\left(\frac{a}{b} + \frac{3b}{a}\right)^{20}$.

19. $\left(\frac{2x}{y^3} - \frac{y^4}{6x^5}\right)^7$.

20. $(a^2 - 3b^2)^{10}$.

21. $\left(\frac{3x^5}{y^3} - \frac{2y^{15}}{9x^{12}}\right)^6$.

22. $\left(\frac{a^2}{b^3} - \frac{2b^2}{a^4}\right)^{12}$.

23. $\left(\frac{\sqrt{x}}{y} + \frac{\sqrt{y}}{x}\right)^{12}$.

24. Write the first six terms of the expansion of $(a + b)^n$ and test it for $n = 1$, $n = 2$, $n = 3$, and $n = 4$. How does the number of terms compare with n? What is the value of each coefficient after the $(n + 1)$st? Why does not the expansion extend to more than six terms when $n = 5$?

25. Write the first four terms of $\left(1 + \frac{1}{n}\right)^n$.

Compute the following, correct to two decimal places:

26. $(1.1)^{10}$. HINT. $(1.1)^{10} = (1 + .1)^{10}$, etc.

27. $(.98)^{11}$. HINT. $(.98)^{11} = (1 - .02)^{11}$, etc.

28. $(2.9)^8$.

29. $(1.06)^6$.

30. $6!$.

31. $2! \cdot 4!$.

32. $6! \div 3!$.

33. $4! - 3! \cdot 2!\, 2!$.

34. $2 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!}$.

35. $\frac{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{7!}$.

135. Extraction of roots by the binomial theorem. By reference to the expansion (1), page 242, it can be seen that none of the factors n, $n-1$, $n-2$, $n-3$, etc. become zero for fractional or negative values of n. Hence for such exponents the development of $(a+b)^n$ becomes an infinite series. If a is numerically greater than b, and n has any one of the values $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, etc., the resulting series has a limiting value. In those expansions where a is considerably greater than b, this value can be readily approximated by finding the sum of the first few terms. Therefore the square root, cube root, and all other roots can be obtained approximately by the aid of the binomial theorem.

Note. The process of extracting the square root and even the cube root by means of the binomial expansion was familiar to the Hindus more than a thousand years ago. The German, Stifel (1486–1567), stated the binomial theorem for all powers up to the seventeenth, and also extracted roots of numbers by this method.

EXAMPLES

Find to three decimals by the binomial theorem:

1. $(27)^{\frac{1}{2}}$.

Solution:
$$\begin{aligned}(27)^{\frac{1}{2}} &= (25+2)^{\frac{1}{2}}\\ &= 25^{\frac{1}{2}} + \tfrac{1}{2}\cdot 25^{-\frac{1}{2}}\cdot 2 - \tfrac{1}{8}\cdot 25^{-\frac{3}{2}}\cdot 2^2 + \tfrac{1}{16}\cdot 25^{-\frac{5}{2}}\cdot 2^3 \cdots\\ &= 5 + \tfrac{1}{5} - \tfrac{1}{250} + \tfrac{1}{6250}\cdots\\ &= 5 + .2 - .004 + .00016 = 5.196+.\end{aligned}$$

2. $(67)^{\frac{1}{3}}$.

Solution:
$$\begin{aligned}(67)^{\frac{1}{3}} &= (64+3)^{\frac{1}{3}}\\ &= 64^{\frac{1}{3}} + \tfrac{1}{3}\cdot 64^{-\frac{2}{3}}\cdot 3 - \tfrac{1}{9}\cdot 64^{-\frac{5}{3}}\cdot 3^2 + \tfrac{5}{81}\cdot 64^{-\frac{8}{3}}\cdot 3^3 \cdots\\ &= 4 + \tfrac{1}{16} - \tfrac{1}{1024} +\\ &= 4 + .0625 - .00097 = 4.0615.\end{aligned}$$

Here three terms give the result correct to five figures.

3. $(79)^{\frac{1}{2}}$.

HINT. $(79)^{\frac{1}{2}} = (81-2)^{\frac{1}{2}} = 81^{\frac{1}{2}} - \frac{1}{2}\cdot 81^{-\frac{1}{2}}\cdot 2 + \frac{1}{8}\cdot 81^{-\frac{3}{2}}\cdot 2^2 + \cdots$.

Here $(81-2)^{\frac{1}{2}}$ yields more accurate results with fewer terms than does $(64+15)^{\frac{1}{2}}$.

EXERCISES

Find to two decimals by the binomial theorem:

1. $(26)^{\frac{1}{2}}$.
2. $(38)^{\frac{1}{2}}$.
3. $(79)^{\frac{1}{2}}$.
4. $(120)^{\frac{1}{2}}$.
5. $(28)^{\frac{1}{3}}$.
6. $(66)^{\frac{1}{3}}$.
7. $(25)^{\frac{1}{3}}$.
8. $(720)^{\frac{1}{3}}$.

Find the first four terms of:

9. $(1+x)^{\frac{1}{2}}$.
10. $(2+x)^{\frac{1}{2}}$.
11. $(3-x)^{\frac{1}{2}}$.
12. $(1+x)^{\frac{1}{3}}$.
13. $(2+x)^{\frac{1}{3}}$.
14. $(3-x)^{\frac{1}{3}}$.

136. Limitations on a and b in $(a+b)^n$. The expansion $(a+b)^n$ has a meaning for all values of n, only if a and b are properly chosen. To illustrate the truth of this statement we shall consider the expansion of $(1+x)^{-1}$ for various values of x. By the theorem,

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - \cdots. \quad (1)$$

Now $(1+x)^{-1} = \dfrac{1}{1+x}\cdot$ Hence the left member of (1) has a meaning for all values of x except -1. The right member of (1) is an infinite geometrical series whose ratio is $-x$. This series has a limiting value only when x is numerically less than 1 (see page 173); that is, if x be a positive or negative proper fraction. For positive or negative values of x numerically greater than 1 the series has no definite value. Therefore the expansion has a meaning only when x is numerically less than 1. Here 1 corresponds to a and x to b in $(a+b)^n$, and the preceding discussion *indicates* but does not prove the truth of the following statement:

The expansion $(a+b)^n$ has a definite value if n is positive or negative, integral or fractional, provided a is greater than b.

A proof of this last statement is beyond the scope of this book.

Note. The binomial theorem occupies a remarkable place in the history of mathematics. By means of it Napier was led to the discovery of logarithms, and its use was of the greatest assistance to

Newton in making his most wonderful mathematical discoveries. But to-day the results of Napier and of Newton are explained without even so much as a mention of the binomial theorem, for simpler methods of obtaining these results have been discovered.

It was Newton who first recognized the truth of the theorem, not only for the case where n is a positive integer, which had long been familiar, but for fractional and negative values as well. He did not give a demonstration of the general validity of the binomial development, and none even passably satisfactory was given until that of Euler (1707–1783). The first entirely satisfactory proof of this difficult theorem was given by the brilliant young Norwegian, Abel (1802–1829).

137. The rth term of $(a+b)^n$. According to the binomial theorem the fifth term of the expansion (1) on page 242 is

$$\frac{n(n-1)(n-2)(n-3)a^{n-4}b^4}{1\cdot 2\cdot 3\cdot 4}.$$

If we note particularly this term and those on page 242, we can write down, from the considerations which follow, any required term without writing other terms of the expansion.

The *denominator of the coefficient* of the fifth term is $4!$. From the law of formation the denominator in the sixth term would be $5!$, in the seventh term $6!$, etc. Consequently in the rth term the denominator of the coefficient would be $(r-1)!$.

The *numerator of the coefficient* of the fifth term contains the product of the four factors $n(n-1)(n-2)(n-3)$. The sixth term would contain these four and the factor $n-4$. Similarly, the last factor in the seventh term would be $n-5$, etc. Hence the last factor in the rth term would be $n-(r-2)$. Therefore the numerator of the coefficient of the rth term is $n(n-1)(n-2)(n-3)\cdots(n-r+2)$.

The *exponent of a* in the fifth term is $n-4$, in the sixth term it would be $n-5$, etc. Therefore in the rth term the exponent of a would be $n-(r-1)$ or $n-r+1$.

The *exponent of b* in the fifth term is 4, in the sixth term it would be 5, etc. Therefore in the rth term the exponent of b would be $r-1$.

The *sign* of any term of the expansion (if n is a positive integer) is plus if the binomial is $a+b$. If the binomial is $a-b$, the terms containing the odd powers of b will be negative; the sign of the rth term being minus if $r-1$ is odd.

Therefore the rth term $(r \neq 1)$ of $(a+b)^n$ equals plus *or* minus

$$\frac{n(n-1)(n-2)(n-3)\cdots(n-r+2)}{(r-1)!}a^{n-r+1}b^{r-1}. \quad (1)$$

The formula for the $(r+1)$st term is more simple and more easily applied. It is plus *or* minus

$$\frac{n(n-1)(n-2)(n-3)\cdots(n-r+1)}{r!}a^{n-r}b^{r}. \quad (2)$$

If we wanted the 12th term, we would in using (1) substitute 12 for r, and in using (2) we would substitute 11 for r.

EXERCISES

Write the:

1. 5th term of $(a+b)^{10}$.

Solution: Substituting 10 for n and 5 for r in the formula (1) gives

$$\frac{10\cdot 9\cdot 8\cdot 7}{4!}a^6b^4 = \frac{10\cdot 9\cdot 8\cdot 7}{4\cdot 3\cdot 2}a^6b^4 = 210\,a^6b^4.$$

2. 6th term of $(a+b)^9$.

3. 4th term of $(a+b)^{20}$.

4. 7th term of $(a-b)^{10}$.

5. 8th term of $(a-b)^{15}$.

6. 4th term of $\left(a+\frac{1}{a}\right)^{30}$.

7. 5th term of $(a^2-b)^{20}$.

8. 6th term of $\left(\frac{a}{b}-\frac{b^2}{a}\right)^{18}$.

9. 7th term of $\left(\frac{a^2}{b}-\frac{2b^2}{a}\right)^{14}$.

10. middle term of $(x^2-x)^{16}$.

11. 5th term of $\left(\sqrt{x}-\sqrt{\frac{y}{x}}\right)^{15}$.

Find the coefficient of:

12. x^5 in $(1+x)^{10}$.

13. x^8 in $(1+x^2)^{16}$.

14. x^{15} in $(x^3+1)^{15}$.

15. x^{10} in $(x^2-x^{-1})^{14}$.

16. Expand $(3+1)^{-1}$ and $(1+3)^{-1}$ by the binomial theorem, and, if possible, find the sum of each series thus obtained.

17. Treat $(2+1)^{-1}$ and $(1+2)^{-1}$ as in Exercise 16.

CHAPTER XIX

SUPPLEMENTARY TOPICS

138. Mathematical induction. The truth of many theorems which may be stated as formulas involving a certain letter can, for integral values of this letter, be established with certainty and elegance by a method of proof known as **mathematical induction.** This process will best be understood by an illustrative

Example: Let us suppose that some one has discovered the remarkable relation that *the sum of the cubes of the consecutive integers 1, 2, 3, ⋯, n is equal to the square of the sum of the integers.* Further, let us assume that he had expressed this in the formula

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2. \qquad (1)$$

Lastly, let us suppose that he has tested the truth of this formula for all the positive integral values of n up to 15 and has become *convinced* that the formula is true for *any* positive integral value of n. He knows, however, that he has not proved the formula for $n = 16$, nor for any one of the infinite number of integers greater than 15. The labor of verifying by actual substitution, however, has become too great and he desires a general proof of (1).

He could arrive at such a proof in the following manner:

The right member of (1) is an arithmetical series of n terms; the first term is 1 and the common difference is 1. Substituting in $S = \frac{n}{2}[2a + (n-1)d]$ gives $S = \frac{n(n+1)}{2}$. Therefore, if k replaces n, (1) may be written

$$1^3 + 2^3 + 3^3 + \cdots + k^3 = (1 + 2 + 3 + \cdots + k)^2 = \left[\frac{k(k+1)}{2}\right]^2. \qquad (2)$$

Now (2) is known to be true for any value of k between 1 and 15. Consequently we can *obtain the correct formula* for $k+1$, or 16, terms by actually adding the $(k+1)$st term to each member. This gives

$$1^3+2^3+3^3+\cdots+k^3+(k+1)^3=\frac{k^2(k+1)^2}{4}+(k+1)^3 \qquad (3)$$

$$=(k+1)^2\left(\frac{k^2}{4}+k+1\right) \qquad (4)$$

$$=(k+1)^2\left(\frac{k^2+4k+4}{4}\right) \qquad (5)$$

$$=\frac{(k+1)^2(k+2)^2}{4}. \qquad (6)$$

Now (2) is the correct formula for all the numbers tried, while (6) is the correct formula for all the numbers tried and for one not tried, the number 16. Another important fact about (6) is that, though derived through equations (3), (4), and (5), it can also be obtained by merely substituting $k+1$ for k in (2). This is obvious on inspection. We have proved, then, that (1) is true for any integral value of n from 1 to 16, but without actually substituting 16 for n. This, however, is equivalent to proving that *if* (1) *is true for n terms, it is true for $n+1$ terms.* But (1) is true for 15 terms; therefore it is true for 16 terms. Again, if true for 16 terms, (1) is true for 17 terms, etc. Therefore (1) is true for any positive integral value of n.

The steps in a proof by mathematical induction are:

1. Test the given or derived formula for integral values of n until its general truth seems probable.

2. Write the formula with k in place of n, k denoting *any* number *within the range of trial.*

3. Derive (or establish the truth of) the *correct* formula for a value of k greater by 1. (In the case of a series this can frequently be done by actually adding the $(k+1)$st term to both members of the equation; for many formulas some other appropriate device must be thought out.) Then, if the formula

thus obtained is the same as, or can be reduced to, the original one, with $k+1$ put for k, we have established the truth of the formula for one integer *beyond the range of trial.*

4. Reason thus: If the formula is true for $n = k$, it is true for $n = k + 1$. But it is true (let us say) for $k = 5$; therefore it is true for $k = 6$. But if true for $k = 6$, it is true for $k = 7$, and so on. Therefore the formula is true for all positive integral values of n.

Observe that a proof of a formula or theorem by mathematical induction is possible only for integral values of n. (A series, for example, cannot contain $3\frac{1}{2}$ terms.)

Frequently a formula is true only when n is even or when n is odd. In such cases the next higher value of n is greater by 2.

Note. It is astonishing how far certain formulas will stand numerical verification, yet ultimately fail. It can be proved that no rational, algebraic formula for the determination of prime numbers exists, yet many formulas have been discovered which give prime numbers for many consecutive values of n. Those who discovered such formulas often believed them true for all integral values of n. For example, the Chinese, as early as the time of Confucius, believed that n is prime if $2^{n-1}-1$ was exactly divisible by n. The *least* value of n for which this theorem is not true is 341. Legendre (1752–1833) gave $2n^2 + 29$ as a formula for primes, and Euler (1707–1783) gave $n^2 + n + 41$. Both of these fail before n reaches 50. Of course each of these men knew that his formula is not true generally. If the student will determine the least value of n for which either formula fails, he will realize that a large number of numerical verifications of a theorem is very far indeed from a general proof of it. The necessity of following several numerical verifications by mathematical induction to establish the general truth of a formula should then be apparent.

EXAMPLES

1. It has been proved by the factor theorem that for all integral values of n, $a^n - b^n$ is exactly divisible by $a - b$. Nevertheless, a proof by induction will be given, since it *appears* to be slightly different from the example on pages 26–27, though it is essentially the same.

Proof. From the work in factoring we know that $a^n - b^n$ is divisible by $a - b$ for integral values of n up to 7 at least.

Therefore $a^k - b^k$ is exactly divisible by $a - b$ for all values of k from 1 to 7.

Hence for these values of k, $a - b$ will exactly divide

$$a(a^k - b^k) + b^k(a - b). \quad (1)$$

But (1) becomes $$a^{k+1} - ab^k + ab^k - b^{k+1}, \quad (2)$$

or $$a^{k+1} - b^{k+1}. \quad (3)$$

Therefore (3) is exactly divisible by $a - b$.

Hence since $a^7 - b^7$ is exactly divisible by $a - b$, $a^8 - b^8$ is also. But if $a^8 - b^8$ is exactly divisible by $a - b$, so is $a^9 - b^9$, etc. Therefore $a^n - b^n$ is exactly divisible by $a - b$ for all positive integral values of n.

2. An illustration somewhat different from either of those given, and *apparently* very difficult, is the following:

Prove that $(9^{n+1} - 8n - 9) \div 64$ gives an integral quotient for all values of n.

Proof. If $n = 1, 2, 3,$ and 4, $(9^{n+1} - 8n - 9) \div 64$ becomes respectively $\frac{64}{64}$, $\frac{704}{64}$, $\frac{6528}{64}$, and $\frac{59008}{64}$, the respective quotients being 1, 11, 102, and 922.

We know, therefore, that (if c is a properly chosen integer) for every value of k from 1 to 4

$$9^{k+1} - 8k - 9 = 64 \cdot c. \quad (1)$$

Then $$9(9^{k+1} - 8k - 9) = 9 \cdot 64 \cdot c, \quad (2)$$

and $$9(9^{k+1} - 8k - 9) + 64k + 64 = 9 \cdot 64 \cdot c + 64k + 64. \quad (3)$$

Whence

$$9^{k+2} - 72k - 81 + 64k + 64 = 64(9c + k + 1), \quad (4)$$

$$9^{k+2} - 8k - 8 - 9 = 64(9c + k + 1), \quad (5)$$

or $$9^{k+2} - 8(k + 1) - 9 = 64(9c + k + 1). \quad (6)$$

Since the right member of (6) is divisible by 64 the left member is also.

But the left member of (6) is the same as $9^{k+1} - 8k - 9$, with $k + 1$ put for k.

Hence $(9^{k+1} - 8k - 9) \div 64$ is an integer when k is replaced by $k + 1$. Consequently (1) is true when $k = 5$. Then it follows that (1) is true when $k = 6$, and so on.

Therefore $(9^{n+1} - 8n - 9) \div 64$ gives an integer for all positive integral values of n.

EXERCISES

Prove by mathematical induction that

1. The sum of the first n integers is $\frac{n}{2}(n + 1)$.

2. The sum of the first n odd integers is n^2.

3. The sum of the squares of the first n integers is $\frac{n}{6}(n + 1)(2n + 1)$.

4. The sum of the squares of the first n odd integers is $\frac{n}{3}(2n + 1)(2n - 1)$.

5. The sum of the first n integral powers of the number 2 is $2(2^n - 1)$.

6. $a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}$.

7. $3 + 6 + 9 + \cdots + 3n = \frac{3n}{2}(n + 1)$.

8. $x^{2n} - y^{2n}$ is divisible by $x + y$ if n is a positive integer.

9. $x^n + y^n$ is divisible by $x + y$ if n is odd.

10. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n + 1) = \frac{n}{3}(n + 1)(n + 2)$.

11. $2 \cdot 5 + 3 \cdot 6 + 4 \cdot 7 + \cdots + (n+1)(n+4) = \frac{n}{3}(n+4)(n+5)$.

12. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n + 1)} = \frac{n}{n + 1}$.

13. $4^2 + 7^2 + 10^2 + \cdots + (3n + 1)^2 = \frac{n}{2}(6n^2 + 15n + 11)$.

14. $9^{n+1} - 1$ is exactly divisible by 4.

15. Prove $3^{2n} - 1$ is exactly divisible by 8.

16. Prove $3^{2n} - 8n - 1$ is exactly divisible by 64.

17. Prove $9^{n+1} - 1$ is exactly divisible by 8.

18. Prove $3n^2 + 15n + 6$ is exactly divisible by 6.

19. Prove $n(n+1)(n+5)$ is exactly divisible by 6.

20. Prove $(a+b)^n$

$$= a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots.$$

21. A pyramid of shot stands on a triangular base having m shot on a side. How many shot are in the pile?

22. From Exercises 3 and 4 derive a formula for the sum of the squares of the first n even integers.

139. Proof of remainder theorem. This theorem is stated on page 24 as follows: If any rational integral expression in x be divided by $x - n$, the remainder is the same as the original expression with n substituted for x. Let $f(x)$ denote any rational integral function of x as follows:

$$f(x) = ax^n + bx^{n-1} + cx^{n-2} + \cdots p. \qquad (1)$$

Then $$f(k) = ak^n + bk^{n-1} + ck^{n-2} + \cdots p. \qquad (2)$$

(1) – (2), $$f(x) - f(k) = ax^n + bx^{n-1} + cx^{n-2} + \cdots$$
$$- ak^n - bk^{n-1} - ck^{n-2} - \cdots \qquad (3)$$
$$= a(x^n - k^n) + b(x^{n-1} - k^{n-1})$$
$$+ c(x^{n-2} - k^{n-2}) + \cdots. \qquad (4)$$

By Example 1, page 251, each binomial in the right member of (4) is divisible by $x - k$. Denote the quotient by $Q(x)$, and the right member of (4) may be written $(x-k)Q(x)$.

Therefore $f(x) - f(k) = (x-k)Q(x)$. (5)

Transposing $f(k)$ and dividing by $x - k$, (5) becomes

$$\frac{f(x)}{x-k} = Q(x) - \frac{f(k)}{x-k}. \qquad (6)$$

That is, $f(k)$ is the remainder when $f(x)$ is divided by $(x-k)$.

140. Theorems on irrational numbers. In order to solve a linear equation or a system of linear equations having rational coefficients, we need use only the operations of addition, subtraction, multiplication, and division. When, however, we attempt to solve the equation of the second degree, $x^2 = 2$, we find that there is no rational number which satisfies it. This last fact can be proved if (as in Example 2, page 252) we assume that:

An integral factor of one member of an identity between integers is also a factor of the other member.

This is certainly true. For example, let $2a = b$, where a and b are integers. Then since 2 is a factor of the left member it is also a factor of the right.

THEOREM 1. *No rational number satisfies the equation* $x^2 = 2$.

Proof. Evidently no integer satisfies the equation. Let us make the supposition that a rational fraction in its lowest terms, $\frac{a}{b}$, satisfies it. Then

$$\left(\frac{a}{b}\right)^2 = 2, \tag{1}$$

or

$$a^2 = 2b^2. \tag{2}$$

From (2) it is seen that 2 is a divisor of the left member and therefore a divisor of a^2, and hence a divisor of a. Let us then suppose $a \div 2 = m$, or

$$a = 2m. \tag{3}$$

Then

$$a^2 = 4m^2. \tag{4}$$

From (2) and (4),

$$4m^2 = 2b^2, \tag{5}$$

or

$$2m^2 = b^2. \tag{6}$$

Hence 2 must be a divisor of b^2 and therefore of b. Then 2 is a divisor of both a and b, which contradicts the hypothesis that $\frac{a}{b}$ is a rational fraction in its lowest terms.

Therefore no rational number satisfies the equation $x^2 = 2$.

Note. This theorem, when stated in geometrical language, asserts that the hypotenuse of an isosceles right triangle is not commensurate with the legs of the triangle. In this form the theorem was stated, and perhaps proved, by Pythagoras, about 525 B.C. The proof given here is found in Euclid's "Geometry," and some historians think that it is the very demonstration given by Pythagoras himself, and was inserted by Euclid in his book for its historical interest.

THEOREM 2. *The square root of a rational number cannot be the sum of a rational number and a quadratic surd.*

Proof. Suppose x is a rational number and $\sqrt{a}$ and $\sqrt{b}$ are surds. Then, if possible, suppose

$$\sqrt{a} = \sqrt{b} \pm x. \tag{1}$$

Squaring each member of (1),

$$a = b + x^2 \pm 2x\sqrt{b}. \tag{2}$$

Solving (2), $$\sqrt{b} = \pm \frac{a - b - x^2}{2x}. \tag{3}$$

But (3) is impossible, for it asserts that a surd equals a rational number.

Therefore $\sqrt{a} \neq \sqrt{b} \pm x$ if $\sqrt{a}$ and $\sqrt{b}$ are surds.

THEOREM 3. *If each member of an equation consists of a rational number and a quadratic surd, then the rational parts are equal and the irrational parts are equal.*

Proof. Let $$a + \sqrt{b} = c + \sqrt{d}. \tag{1}$$

If possible, suppose $$c = a \pm x. \tag{2}$$

Then $$a + \sqrt{b} = a \pm x + \sqrt{d}, \tag{3}$$

or $$\sqrt{b} = \pm x + \sqrt{d}. \tag{4}$$

But (4) by the preceding theorem is impossible.

Consequently $a = c$, and hence from (1), $\sqrt{b} = \sqrt{d}$.

Therefore, if $a + \sqrt{b} = c + \sqrt{d}$, $a = c$ and $\sqrt{b} = \sqrt{d}$.

141. Cube root of algebraic expressions. Since by actual multiplication

$$(t + u)^3 = t^3 + 3t^2u + 3tu^2 + u^3,$$

a careful inspection of the expression

$$t^3 + 3t^2u + 3tu^2 + u^3$$

will enable one to extract the cube root of any polynomial which is a perfect cube; for the extraction of cube and other roots is not a mysterious, unreasonable process, but merely an intelligent undoing of the work of multiplication. We see that the first term of the result is the cube root of the first term of the polynomial t^3. The second term of the cube root, u, can be obtained by squaring t, multiplying it by 3, and dividing the result as a trial divisor into the second term of the polynomial, thus obtaining u. Since $t^3 + 3t^2u + 3tu^2 + u^3 = t^3 + (3t^2 + 3tu + u^2)u = (t + u)^3$, we may then form the complete divisor as indicated by the trinomial in parenthesis. A systematic arrangement of the work follows:

Example 1.

$$\begin{array}{l|l}
 & t^3 + 3t^2u + 3tu^2 + u^3 \;\underline{|\, t + u} \\
 & t^3 \\
\text{Trial divisor, } 3 \cdot t^2 = 3t^2 & 3t^2u + 3tu^2 + u^3 \\
\text{Second term of root,} & \\
\quad 3t^2u \div 3t^2 = u & \\
\text{Complete divisor, } 3t^2 + 3tu + u^2 & 3t^2u + 3tu^2 + u^3 = (3t^2 + 3tu + u^2)u
\end{array}$$

Example 2. Extract the cube root of $27a^3 - 8x^3 + 36ax^2 - 54a^2x$.

Solution:

$$\begin{array}{r|l}
 & 27a^3 - 54a^2x + 36ax^2 - 8x^3 \;\underline{|\, 3a - 2x} \\
t^3 = (3a)^3 = \quad 27a^3 & \\
\text{Trial divisor, } 3t^2 = 3 \cdot (3a)^2 = 27a^2 & -54a^2x + 36ax^2 - 8x^3 \\
\text{Second term of root, } u \text{, equals} & \\
-54a^2x \div 27a^2 = -2x & \\
3tu = 3 \cdot 3a(-2x) = -18ax & \\
u^2 = \quad (2x)^2 = 4x^2 & \\
\text{Complete divisor, } 27a^2 - 18ax + 4x^2 & \\
-2x & \\
\hline
-54a^2x + 36ax^2 - 8x^3 & -54a^2x + 36ax^2 - 8x^3
\end{array}$$

The student should note particularly the form of the trial divisor and of the complete divisor. They are very important in extracting the cube root of a polynomial or of an arithmetical number.

If t in the preceding example be replaced by the binomial $h + t$, we obtain

$$[(h + t) + u]^3 = (h + t)^3 + 3(h + t)^2u + 3(h + t)u^2 + u^3.$$

If this last were expanded fully, we would obtain a polynomial of ten terms which would be a perfect cube. Its cube root could be obtained as before, first obtaining h and then t. Then we could regard $h + t$ as a single term, form the trial divisor and the complete divisor as before, and obtain the third term of the root. Thus the method may be extended to any polynomial of more than four terms which is a perfect cube.

The method just illustrated may be stated in the

RULE. *Arrange the terms of the polynomial according to the powers of some letter in it.*

Extract the cube root of the first term. Write the result as the first term of the root, and subtract its cube from the given polynomial.

Square the part of the root already found and multiply the result by 3 for a trial divisor. Divide the first term of this product into the first term of the remainder, and write the quotient as the second term of the root.

Annex to the trial divisor three times the product of the first term and the second term of the root, and the square of the second term also, thus forming the complete divisor.

Multiply the complete divisor by the second term of the root, and subtract the result from the remainder.

If terms of the polynomial still remain, square the part of the root already found, and multiply the result by 3 for a trial divisor. Divide the first term of the trial divisor into the first term of the remainder, and write the quotient as the third term of the root, form the complete divisor, and proceed as before until the process ends, or until the required number of terms have been obtained.

EXERCISES

Extract the cube root of:

1. $x^3 + 3x^2 + 3x + 1.$

2. $8x^3 - 12x^2 + 6x - 1.$

3. $27x^3 + 27x^2y + 9xy^2 + y^3.$

4. $64a^3 - 144a^2c + 108ac^2 - 27c^3.$

5. $x^{12} - 15x^{10} + 75x^8 - 125x^6.$

6. $x^6 - \dfrac{3x^4}{2} + \dfrac{3x^2}{4} - \dfrac{1}{8}.$

7. $\dfrac{x^3}{27} - \dfrac{2}{3} + \dfrac{4}{x^3} - \dfrac{8}{x^6}.$

8. $\dfrac{a^3}{c^6} - \dfrac{3}{c^3} - \dfrac{c^3}{a^6} + \dfrac{3}{a^3}.$

9. $x^{\frac{3}{2}} + \dfrac{12x^{\frac{1}{2}}}{a^2} - \dfrac{6x}{a} - \dfrac{8}{a^3}.$

10. $a^3 + b^3 - 1 - 3a^2 - 3b^2 - 6ab + 3a^2b + 3ab^2 + 3a + 3b.$

11. $x^6 + 8 - 9x^5 + 66x^2 - 36x + 33x^4 - 63x^3$.

12. Find the sixth root of $x^6 - 12x^5 + 64 - 192x + 240x^2 + 60x^4 - 160x^3$.

13. Find the first three terms in the cube root of $1 + 3x$.

142. Cube root of arithmetical numbers. The process of extracting the cube root of an arithmetical number does not differ greatly from the method of extracting the cube root of any polynomial. The formula for the complete divisor, $3t^2 + 3tu + u^2$, can be used to guide the important steps in the work. The first step, however, is pointing off, the reason for which appears from a study of the following table:

$n =$	1	10	100	1000
$n^3 =$	1	1000	1,000,000	1,000,000,000

From this table it is obvious that the cube root of an integral number of three digits or less must contain only *one* digit on the left of the decimal point. Similarly, we see that the cube root of an integral number containing four, five, or six digits contains *two* digits on the left of the decimal point; and the cube root of an integral number of seven, eight, or nine digits contains *three* digits on the left of the decimal point. Hence in cube root we find it convenient to begin at the decimal point and point off the number in periods of three figures each, — to the left if the number is integral, to the right if it is decimal; to both the left and right if the number is part integral and part decimal. There may, of course, be an incomplete period on the left. Zeros should be used to complete any partial period on the right.

If we now imagine $(t + u)^3$ to be a number consisting of a tens and a units digit, we may translate $t^3 + 3t^2u + 3tu^2 + u^3$ thus: *the cube of the tens + 3 times the square of the tens times the units + 3 times the tens times the square of the units + the cube of the units.*

Now subtracting t^3 from the polynomial, we may write the other three terms thus: $(3t^2 + 3tu + u^2)u$. Here the trinomial in parenthesis is the complete divisor. The process of extracting the cube root of 50,653 follows:

		50′653	$30 + 7 = 37$
	$t^3 = (30)^3 =$	27 000	
Trial divisor,	$3t^2 = 3 \cdot (30)^2 = 2\,700$	23 653	
Second term of root, u,	$23653 \div 2700 = 7+$		
	$3tu = 3 \cdot 30 \cdot 7 = 630$		
	$u^2 = 7^2 = 49$		
Complete divisor,	$3t^2 + 3tu + u^2 = 3\,379$	$23\,653 = 3\,379 \times 7$	

To obtain the second term of the root, we divided 23,653 by 2700, which gave almost exactly the number 8. But since the trial divisor 2700 must be increased by $3tu$ and u^2 to form the complete divisor, a moment's thought showed that 8 was too great. This means that the trial divisor is really a *trial divisor*, and its use does not give us with certainty the next term of the root. A little experience will enable one to look ahead and decide mentally on the next root figure. If one decides on a root figure either too great or too small, the product of the complete divisor and this root digit will be too great or too small and the subsequent work will show the error.

Since 374, for example, may be regarded as 37 tens plus 4 units, the process just illustrated may be applied to a number whose cube root contains three digits, as follows:

	644′972′544	$800 + 60 + 4 = 864$
$t^3 = (800)^3 =$	512 000 000	
$3t^2 = 3(800)^2 = 1\,920\,000$	132 972 544	
Second term of root, u, equals $132\,972\,534 \div 1\,920\,000 = 60 +$		
$3tu = 3(800)(60) = 144\,000$		
$u^2 = (60)^2 = 3\,600$		
$3t^2 + 3tu + u^2 = 2\,067\,600$	$124\,056\,000 = 2\,067\,600 \times 60$	
$3t^2 = 3(860)^2 = 2\,218\,800$	8 916 544	
Third term of root, $8\,916\,534 \div 2\,218\,800 = 4 +$		
$3tu = 3(860)\,4 = 10\,320$		
$u^2 = 4^2 = 16$		
$3t^2 + 3tu + u^2 = 2\,229\,136$	$8\,916\,544 = 2\,229\,136 \times 4$	

EXERCISES

(Obtain roots in Exercises 6–9 correct to three decimals.)

Extract the cube root of:

1. 15625.	**4.** 13481272.	**7.** .0173.
2. 12167.	**5.** 41063.625.	**8.** .004913.
3. 1404928.	**6.** 1.0528.	**9.** .000062.

10. Find $\sqrt[3]{35}$ to three decimal places.

11. Find $\sqrt[3]{\frac{2}{3}}$ to three decimal places.

12. Find the edge of a cube whose volume is 5832 cubic inches.

13. Find the diagonal of a cube whose volume is 46656 cubic meters.

14. Find the sixth root of 46656000.

Note. Problems in cube root afford excellent drill if the time can be spared to study the subject thoroughly. Ability to extract cube root is not a real necessity, however, for in engineering practice, or in any work requiring cube (or higher) roots, it is customary to obtain them from a table of roots, or by means of a slide rule or a table of logarithms. The roots can be obtained in any of these ways far more rapidly than by the method explained in the text.

No.	Squares	Cubes	Square Roots	Cube Roots	No.	Squares	Cubes	Square Roots	Cube Roots
1	1	1	1.000	1.000	51	2,601	132,651	7.141	3.708
2	4	8	1.414	1.259	52	2,704	140,608	7.211	3.732
3	9	27	1.732	1.442	53	2,809	148,877	7.280	3.756
4	16	64	2.000	1.587	54	2,916	157,464	7.348	3.779
5	25	125	2.236	1.709	55	3,025	166,375	7.416	3.802
6	36	216	2.449	1.817	56	3,136	175,616	7.483	3.825
7	49	343	2.645	1.912	57	3,249	185,193	7.549	3.848
8	64	512	2.828	2.000	58	3,364	195,112	7.615	3.870
9	81	729	3.000	2.080	59	3,481	205,379	7.681	3.892
10	100	1,000	3.162	2.154	60	3,600	216,000	7.745	3.914
11	121	1,331	3.316	2.223	61	3,721	226,981	7.810	3.936
12	144	1,728	3.464	2.289	62	3,844	238,328	7.874	3.957
13	169	2,197	3.605	2.351	63	3,969	250,047	7.937	3.979
14	196	2,744	3.741	2.410	64	4,096	262,144	8.000	4.000
15	225	3,375	3.872	2.466	65	4,225	274,625	8.062	4.020
16	256	4,096	4.000	2.519	66	4,356	287,496	8.124	4.041
17	289	4,913	4.123	2.571	67	4,489	300,763	8.185	4.061
18	324	5,832	4.242	2.620	68	4,624	314,432	8.246	4.081
19	361	6,859	4.358	2.668	69	4,761	328,509	8.306	4.101
20	400	8,000	4.472	2.714	70	4,900	343,000	8.366	4.121
21	441	9,261	4.582	2.758	71	5,041	357,911	8.426	4.140
22	484	10,648	4.690	2.802	72	5,184	373,248	8.485	4.160
23	529	12,167	4.795	2.843	73	5,329	389,017	8.544	4.179
24	576	13,824	4.898	2.884	74	5,476	405,224	8.602	4.198
25	625	15,625	5.000	2.924	75	5,625	421,875	8.660	4.217
26	676	17,576	5.099	2.962	76	5,776	438,976	8.717	4.235
27	729	19,683	5.196	3.000	77	5,929	456,533	8.774	4.254
28	784	21,952	5.291	3.036	78	6,084	474,552	8.831	4.272
29	841	24,389	5.385	3.072	79	6,241	493,039	8.888	4.290
30	900	27,000	5.477	3.107	80	6,400	512,000	8.944	4.308
31	961	29,791	5.567	3.141	81	6,561	531,441	9.000	4.326
32	1,024	32,768	5.656	3.174	82	6,724	551,368	9.055	4.344
33	1,089	35,937	5.744	3.207	83	6,889	571,787	9.110	4.362
34	1,156	39,304	5.830	3.239	84	7,056	592,704	9.165	4.379
35	1,225	42,875	5.916	3.271	85	7,225	614,125	9.219	4.396
36	1,296	46,656	6.000	3.301	86	7,396	636,056	9.273	4.414
37	1,369	50,653	6.082	3.332	87	7,569	658,503	9.327	4.431
38	1,444	54,872	6.164	3.361	88	7,744	681,472	9.380	4.447
39	1,521	59,319	6.244	3.391	89	7,921	704,969	9.433	4.464
40	1,600	64,000	6.324	3.419	90	8,100	729,000	9.486	4.481
41	1,681	68,921	6.403	3.448	91	8,281	753,571	9.539	4.497
42	1,764	74,088	6.480	3.476	92	8,464	778,688	9.591	4.514
43	1,849	79,507	6.557	3.503	93	8,649	804,357	9.643	4.530
44	1,936	85,184	6.633	3.530	94	8,836	830,584	9.695	4.546
45	2,025	91,125	6.708	3.556	95	9,025	857,375	9.746	4.562
46	2,116	97,336	6.782	3.583	96	9,216	884,736	9.797	4.578
47	2,209	103,823	6.855	3.608	97	9,409	912,673	9.848	4.594
48	2,304	110,592	6.928	3.634	98	9,604	941,192	9.899	4.610
49	2,401	117,649	7.000	3.659	99	9,801	970,299	9.949	4.626
50	2,500	125,000	7.071	3.684	100	10,000	1,000,000	10,000	4.641

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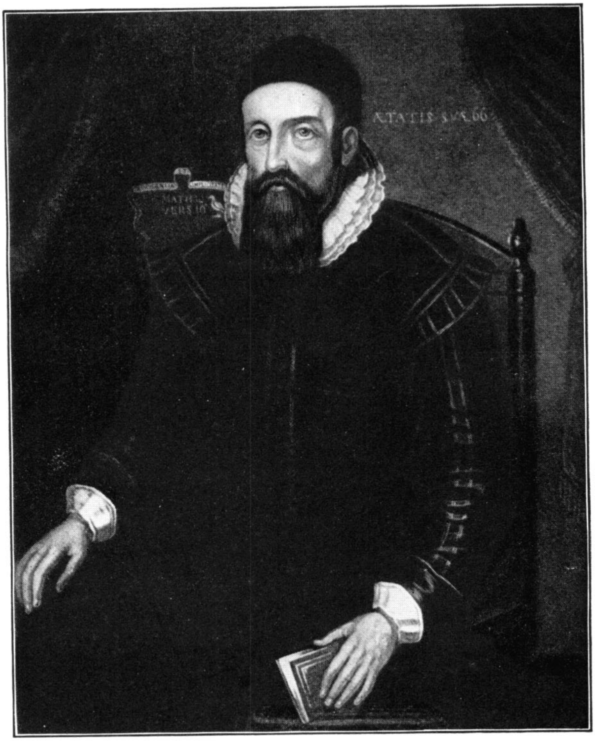
FELIX KLEIN



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