WENTWORTH-SMITH MATHEMATICAL SERIES

SOLID GEOMETRY

BY

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AND

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PREFACE

Long after the death of Robert Recorde, England's first great writer of textbooks, the preface of a new edition of one of his works contained the appreciative statement that the book was "entail'd upon the People, ratified and sign'd by the approbation of Time." The language of this sentiment sounds quaint, but the noble tribute is as impressive to-day as when first put in print two hundred and fifty years ago.

With equal truth these words may be applied to the Geometry written by George A. Wentworth. For a generation it has been the leading textbook on the subject in America. It set a standard for usability that every subsequent writer upon geometry has tried to follow, and the number of pupils who have testified to its excellence has run well into the millions.

In undertaking to prepare a work to take the place of the Wentworth Geometry the authors have been guided by certain well-defined principles, based upon an extended investigation of the needs of the schools and upon a study of all that is best in the recent literature of the subject. The effects of these principles they feel should be summarized for the purpose of calling the attention of the wide circle of friends of the Wentworth-Smith series to the points of similarity and of difference in the two works.

1. Every effort has been made not only to preserve but to improve upon the simplicity of treatment, the clearness of expression, and the symmetry of page that characterized the successive editions of the Wentworth Geometry. It has been the purpose to prepare a book that should do even more than maintain the traditions this work has fostered.

2. The proofs have been given substantially in full, to the end that the pupil may always have before him a model for his independent treatment of the exercises.

3. To meet a general demand, the number of propositions has been decreased so as to include only the great basal theorems and problems. A little of the less important material has been placed in the Appendix, to be used or not as circumstances demand.

4. The exercises, in some respects the most important part of a course in geometry, have been rendered more dignified in appearance and have been improved in content. The number of simple exercises has been greatly increased, while the difficult puzzle is much less in evidence than in most American textbooks. The exercises are systematically grouped, appearing in general in full pages, in large type, and at frequent intervals. They are not all intended for one class, but are so numerous as to allow the teacher to make selections from year to year.

5. The work throughout has been made as concrete as is reasonable. Definitions have been postponed until they are actually needed, only well-recognized terms have been employed, the pupil is led to apply his geometry to practical cases in mensuration, and correlation is made with the algebra already studied.

6. All the references to Plane Geometry that are directly made in the proof of Solid Geometry have been prefixed to this edition so as to be easily accessible.

The authors are indebted to many friends of the Wentworth-Smith series for assistance and encouragement in the labor of preparing this edition, and they will welcome any further suggestions for improvement from any of their readers.

GEORGE WENTWORTH
DAVID EUGENE SMITH

CONTENTS

SYMBOLS AND ABBREVIATIONS

=	equals, equal, equal to, is equal to, or is equivalent to.	Adj.	adjacent.
>	is greater than.	Alt.	alternate.
<	is less than.	Ax.	axiom.
∥	parallel.	Const.	construction.
⊥	perpendicular.	Cor.	corollary.
∠	angle.	Def.	definition.
△	triangle.	Ex.	exercise.
▱	parallelogram.	Ext.	exterior.
□	rectangle.	Fig.	figure.
⊙	circle.	Hyp.	hypothesis.
st.	straight.	Iden.	identity.
rt.	right.	Int.	interior.
∵	since.	Post.	postulate.
∴	therefore.	Prob.	problem.
		Prop.	proposition.
		Sup.	supplementary.

These symbols take the plural form when necessary, as in the case of ∥s, ∠s, △s, ⊙s.

The symbols +, −, ×, ÷ are used as in algebra.

There is no generally accepted symbol for "is congruent to," and the words are used in this book. Some teachers use ≙ or ≅, and some use ≡, but the sign of equality is more commonly employed, the context telling whether equality, equivalence, or congruence is to be understood.

Q. E. D. is an abbreviation that has long been used in geometry for the Latin words *quod erat demonstrandum*, "which was to be proved."

Q. E. F. stands for *quod erat faciendum*, "which was to be done."

REFERENCES TO PLANE GEOMETRY

28. A portion of a plane bounded by three straight lines is called a triangle.

41. The whole angular space in a plane about a point is called a perigon.

52. The following are the most important axioms used in geometry:

1. If equals are added to equals, the sums are equal.
2. If equals are subtracted from equals, the remainders are equal.
3. If equals are multiplied by equals, the products are equal.
4. If equals are divided by equals, the quotients are equal.

In division the divisor is never zero.

5. Like powers and like positive roots of equals are equal.
6. If unequals are operated on by positive equals in the same way, the results are unequal in the same order.
7. If unequals are added to unequals in the same order, the sums are unequal in the same order; if unequals are subtracted from equals, the remainders are unequal in the reverse order.
8. Quantities that are equal to the same quantity or to equal quantities are equal to each other.
9. A quantity may be substituted for its equal in an equation or in an inequality.
10. If the first of three quantities is greater than the second, and the second is greater than the third, then the first is greater than the third.
11. The whole is greater than any of its parts, and is equal to the sum of all its parts.

53. Postulate 5. Any figure may be moved *from one* place to another without altering its size or shape.

56. All right angles are equal.

57. From a given point in a given line only one perpendicular can be drawn to the line.

60. If two lines intersect, the vertical angles are equal.

66. Definition of congruent figures.

67. Corresponding parts of congruent figures are equal.

68. Two triangles are congruent, if two sides and the included angle of the one are equal respectively to two sides and the included angle of the other.

69. Two right triangles are congruent, if the sides of the right angles are equal respectively.

72. Two triangles are congruent, if two angles and the included side of the one are equal respectively to ⋯.

80. Two triangles are congruent, if the three sides of the one are equal respectively to the three sides of the other.

82. Only one perpendicular can be drawn to a given line from a given external point.

84. Of two lines drawn from a point in a perpendicular to a given line, cutting off on the given line unequal segments from the foot of the perpendicular, the more remote is the greater.

89. Two right triangles are congruent, if the hypotenuse and a side of the one are equal respectively to the hypotenuse and a side of the other.

93. Lines that lie in the same plane and cannot meet however far produced are called parallel lines.

94. Through a given point only one line can be drawn parallel to a given line.

95. Two lines in the same plane perpendicular to the same line are parallel.

97. If a line is perpendicular to one of two parallel lines, it is perpendicular to the other also.

112. The sum of any two sides of a triangle is greater than the third side, and the difference between any two sides is less than the third side.

116. If two triangles have two sides of the one equal respectively to two sides of the other, but the third side of the first triangle greater than the third side of the second, then the angle opposite the third side of the first is greater than the angle opposite the third side of the second.

118. A quadrilateral may be a trapezoid, having two sides parallel; a parallelogram, having the opposite sides parallel; or it may have no sides parallel.

125. The opposite sides of a parallelogram are equal.

126. A diagonal divides a parallelogram into two congruent triangles.

127. Segments of parallel lines cut off by parallel lines are equal.

130. If two sides of a quadrilateral are equal and parallel, then the other two sides are equal and parallel, and the figure is a parallelogram.

131. The diagonals of a parallelogram bisect each other.

132. Two parallelograms are congruent, if two sides and the included angle of the one are equal respectively to two sides and the included angle of the other.

133. Two rectangles having equal bases and equal altitudes are congruent.

136. The line which joins the mid-points of two sides of a triangle is parallel to the third side, and is equal to half the third side.

142. Two polygons are

mutually equiangular, if the angles of the one are equal to the angles of the other respectively, taken in the same order;

mutually equilateral, if the sides of the one are equal to the sides of the other respectively, taken in the same order;

congruent, if mutually equiangular and mutually equilateral, since they then can be made to coincide.

145. Each angle of a regular polygon of n sides is equal to $\frac{2(n-2)}{n}$ right angles.

146. The sum of the exterior angles of a polygon, made by producing each of its sides in succession, is equal to four right angles.

148. To prove that a certain line or group of lines is the locus of a point that fulfills a given condition, it is necessary and sufficient to prove two things:

1. That any point in the supposed locus satisfies the condition.
2. That any point outside the supposed locus does not satisfy the given condition.

150. The locus of a point equidistant from the extremities of a given line is the perpendicular bisector of that line.

151. Two points each equidistant from the extremities of a line determine the perpendicular bisector of the line.

152. The locus of a point equidistant from two given intersecting lines is a pair of lines bisecting the angles formed by those lines.

159. A closed curve lying in a plane, and such that all of its points are equally distant from a fixed point in the plane, is called a circle.

162. All radii of the same circle or of equal circles are equal; and all circles of equal radii are equal.

167. In the same circle or in equal circles equal arcs subtend equal central angles; and of two unequal arcs the greater subtends the greater central angle.

172. In the same circle or in equal circles, if two chords are equal, they subtend equal arcs; and if two chords are unequal, the greater subtends the greater arc.

174. A line through the center of a circle perpendicular to a chord bisects the chord and the arcs subtended by it.

178. In the same circle or in equal circles equal chords are equidistant from the center, and chords equidistant from the center are equal.

185. A tangent to a circle is perpendicular to the radius drawn to the point of contact.

195. If two circles intersect, the line of centers is the perpendicular bisector of their common chord.

204. When a variable approaches a constant in such a way that the difference between the two may become and remain less than any assigned positive quantity, however small, the constant is called the limit of the variable.

207. If, while approaching their respective limits, two variables are always equal, their limits are equal.

212. In the same circle or in equal circles two central angles have the same ratio as their intercepted arcs.

213. A central angle is measured by the intercepted arc.

261. In any proportion the product of the extremes is equal to the product of the means.

262. The mean proportional between two quantities is equal to the square root of their product.

269. In a series of equal ratios, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.

270. Like powers of the terms of a proportion are in proportion.

273. If a line is drawn through two sides of a triangle parallel to the third side, it divides the two sides proportionally.

274. One side of a triangle is to either of its segments cut off by a line parallel to the base as the third side is to its corresponding segment.

275. Three or more parallel lines cut off proportional intercepts on any two transversals.

282. Polygons that have their corresponding angles equal, and their corresponding sides proportional, are called similar polygons.

285. Two mutually equiangular triangles are similar.

288. If two triangles have an angle of the one equal to an angle of the other, and the including sides proportional, they are similar.

289. If two triangles have their sides respectively proportional, they are similar.

290. Two triangles which have their sides respectively parallel, or respectively perpendicular, are similar.

292. If two polygons are similar, they can be separated into the same number of triangles, similar each to each, and similarly placed.

298. If a perpendicular is drawn from any point on a circle to a diameter, the chord from that point to either extremity of the diameter is the mean proportional between the diameter and the segment adjacent to that chord.

322. The area of a parallelogram is equal to the product of its base by its altitude.

323. Parallelograms having equal bases and equal altitudes are equivalent.

325. The area of a triangle is equal to half the product of its base by its altitude.

326. Triangles having equal bases and equal altitudes are equivalent.

327. Triangles having equal bases are to each other as their altitudes; triangles having equal altitudes are to each other as their bases; any two triangles are to each other as the products of their bases by their altitudes.

329. The area of a trapezoid is equal to half the product of the sum of its bases by its altitude.

332. The areas of two triangles that have an angle of the one equal to an angle of the other are to each other as the products of the sides including the equal angles.

334. The areas of two similar polygons are to each other as the squares on any two corresponding sides.

377. If the number of sides of a regular inscribed polygon is indefinitely increased, the apothem of the polygon approaches the radius of the circle as its limit.

381. The circle is the limit which the perimeters of regular inscribed polygons and of similar circumscribed polygons approach, if the number of sides of the polygons is indefinitely increased.

The area of the circle is the limit which the areas of the inscribed and circumscribed polygons approach.

382. Two circumferences have the same ratio as their radii.

385. The circumference of a circle equals $2\pi r$.

389. The area of a circle $= \pi r^2$.

SOLID GEOMETRY

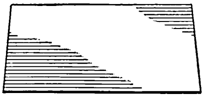
BOOK VI

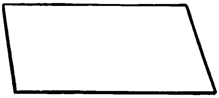
LINES AND PLANES IN SPACE

421. The Nature of Solid Geometry. In plane geometry we deal with figures lying in a flat surface, studying their properties and relations and measuring the figures. In solid geometry we shall deal with figures not only of two dimensions but of three dimensions, also studying their properties and relations and measuring the figures.

422. Plane. A surface such that a straight line joining any two of its points lies wholly in the surface is called a *plane.*

A plane is understood to be indefinite in extent, but it is conveniently represented by a rectangle seen obliquely, as here shown.





423. Determining a Plane. A plane is said to be *determined* by certain lines or points if it contains the given lines or points, and no other plane can contain them.

When we suppose a plane to be drawn to include given points or lines, we are said to *pass* the plane through these points or lines.

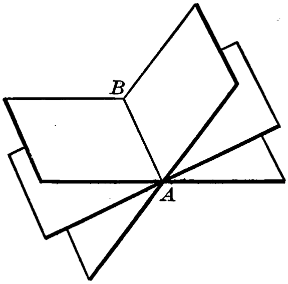
When a straight line is drawn from an external point to a plane, its point of contact with the plane is called its *foot.*

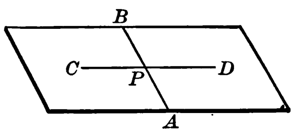
424. Intersection of Planes. The line that contains all the points common to two planes is called their *intersection.*

425. Postulate of Planes. Corresponding to the postulate that one straight line, and only one, can be drawn through two given points, the following postulate is assumed for planes:

One plane, and only one, can be passed through two given intersecting straight lines.

For it is apparent from the first figure that a plane may be made to turn about any single straight line *AB*, thus assuming different positions. But if *CD* intersects *AB* at *P*, as in the second figure, then when the plane through *AB* turns until it includes *C*, it must include *D*, since it includes two points, *C* and *P*, of the line (§ 422). If it turns any more, it will no longer contain *C*.





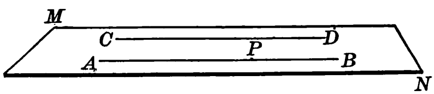
426. Corollary 1. *A straight line and a point not in the line determine a plane.*

For example, line *AB* and point *C* in the above figure.

427. Corollary 2. *Three points not in a straight line determine a plane.*

For by joining any one of them with the other two we have two intersecting lines (§ 425).

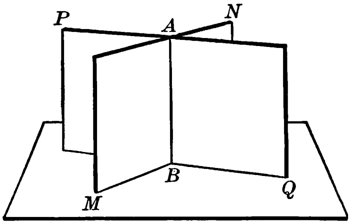
428. Corollary 3. *Two parallel lines determine a plane.*



For two parallel lines lie in a plane (§ 93), and a plane containing either parallel and a point *P* in the other is determined (§ 426).

Proposition I. Theorem

429. *If two planes cut each other, their intersection is a straight line.*



Given *MN* and *PQ*, two planes which cut each other.

To prove that the planes MN and PQ intersect in a straight line.

Proof. Let A and B be two points common to the two planes.

Draw a straight line through the points A and B.

Then the straight line AB lies in both planes. § 422

(*For it has two points in each plane.*)

No point not in the line AB can be in both planes; for one plane, and only one, can contain a straight line and a point without the line. § 426

Therefore the straight line through A and B contains all the points common to the two planes, and is consequently the intersection of the planes, by § 424. Q.E.D.

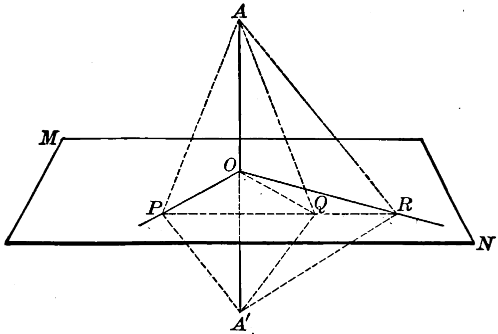
Discussion. What is the corresponding statement in plane geometry?

430. Perpendicular to a Plane. If a straight line drawn to a plane is perpendicular to every straight line that passes through its foot and lies in the plane, it is said to be *perpendicular* to the plane.

When a line is perpendicular to a plane, the plane is also said to be perpendicular to the line.

Proposition II. Theorem

431. *If a line is perpendicular to each of two other lines at their point of intersection, it is perpendicular to the plane of the two lines.*



Given the line *AO* perpendicular to the lines *OP* and *OR* at *O*.

To prove that AO is ⊥ to the plane MN of these lines.

Proof. Through O draw in MN any other line OQ, and draw PR cutting OP, OQ, OR, at P, Q, and R.

Produce AO to A', making OA' equal to OA, and join A and A' to each of the points P, Q, and R.

Then OP and OR are each ⊥ to AA' at its mid-point.

$\therefore AP = A'P$, and $AR = A'R$. § 150

$\therefore \triangle APR$ is congruent to $\triangle A'PR$. § 80

$\therefore \angle RPA = \angle A'PR$. § 67

That is, $\angle QPA = \angle A'PQ$.

$\therefore \triangle PQA$ is congruent to $\triangle PQA'$. § 68

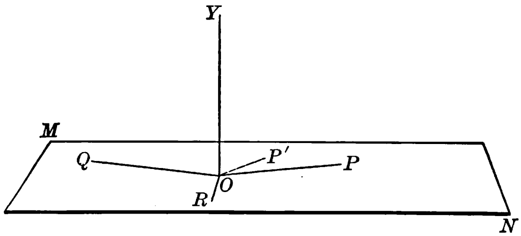
$\therefore AQ = A'Q$ (§ 67); and OQ is ⊥ to AA' at O. § 151

$\therefore AO$ is ⊥ to any and hence to every line in MN through O.

$\therefore AO$ is ⊥ to the plane MN, by § 430. Q.E.D.

Proposition III. Theorem

432. *All the perpendiculars that can be drawn to a given line at a given point lie in a plane which is perpendicular to the given line at the given point.*



Given the plane *MN* perpendicular to the line *OY* at *O*.

To prove that OP, any line ⊥ to OY at O, lies in MN.

Proof. Let the plane containing OY and OP intersect the plane MN in the line OP'; then OY is ⊥ to OP'. § 430

In the plane POY only one ⊥ can be drawn to OY at O. § 57

Therefore OP and OP' coincide, and OP lies in MN.

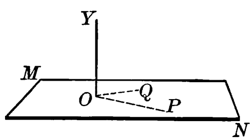
Hence every ⊥ to OY at O, as OQ, OR, lies in MN. Q.E.D.

433. Corollary 1. *Through a given point in a given line one plane, and only one, can be passed perpendicular to the line.*

434. Corollary 2. *Through a given external point one plane, and only one, can be passed perpendicular to a given line.*

Given the line OY and the point P.

Draw PO ⊥ to OY, and OQ ⊥ to OY. Then OQ and OP determine a plane through P ⊥ to OY.

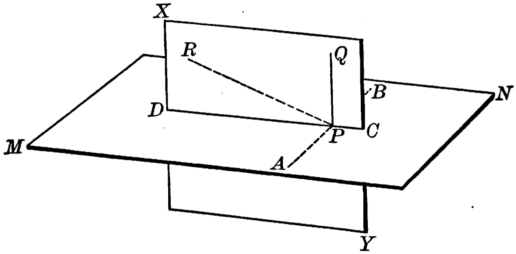


Only one such plane can be drawn; for only one ⊥ can be drawn to OY from the point P (§ 82).

435. Oblique Line. A line that meets a plane but is not perpendicular to it is said to be *oblique* to the plane.

Proposition IV. Theorem

436. *Through a given point in a plane there can be drawn one line perpendicular to the plane, and only one.*



Given the point *P* in the plane *MN*.

To prove that there can be drawn one line perpendicular to the plane MN at P, and only one.

Proof. Through the point P draw in the plane MN any line AB, and pass through P a plane $XY \perp$ to AB, cutting the plane MN in CD. § 433

At P erect in the plane XY the line $PQ \perp$ to CD.

The line AB, being $\perp$ to the plane XY by construction, is $\perp$ to PQ, which passes through its foot in the plane. § 430

That is, PQ is $\perp$ to AB; and as it is $\perp$ to CD by construction, it is $\perp$ to the plane MN. § 431

Moreover, any other line PR drawn from P is oblique to MN. For PQ and PR intersecting in P determine a plane.

To avoid drawing another plane, use XY again to represent the plane of PQ and PR, letting it cut MN in the line CD.

Then since PQ is $\perp$ to MN, it is $\perp$ to CD. § 430

Therefore PR is oblique to CD. § 57

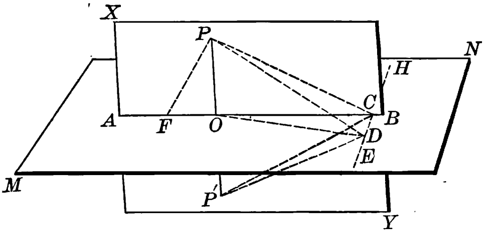
Therefore PR is oblique to MN. § 435

Therefore PQ is the only $\perp$ to MN at the point P. Q.E.D.

Discussion. What is the corresponding proposition in plane geometry?

PROPOSITION V. THEOREM

437. *Through a given external point there can be drawn one line perpendicular to a given plane, and only one.*



Given the plane *MN* and the external point *P*.

To prove that there can be drawn one line from P perpendicular to the plane MN, and only one.

Proof. In MN draw any line EH, and let XY be a plane through $P \perp$ to EH, cutting MN in AB, and EH in C.

Draw $PO \perp$ to AB, and in MN draw any line OD from O to EH.

Produce PO, making $OP' = OP$, and draw PC, PD, $P'C$, $P'D$.

Since DC is $\perp$ to XY, $\angle$s PCD and $P'CD$ are right angles. § 430

Since the side DC is common, and $PC = P'C$, § 150

$\therefore$ rt. $\triangle PCD$ is congruent to rt. $\triangle P'CD$. § 69

$\therefore PD = P'D$. § 67

$\therefore OD$ is $\perp$ to PP' at O. § 151

$\therefore PO$ is $\perp$ to MN, being $\perp$ to OD and AB. § 431

Moreover, every other line PF from P to MN is oblique to MN. (The proof is left for the student.)

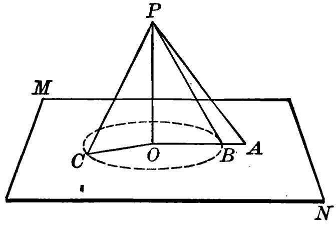
$\therefore PO$ is the only $\perp$ from P to MN. Q.E.D.

438. COROLLARY. *The perpendicular is the shortest line from a point to a plane.*

The length of this $\perp$ is called the *distance* from the point to the plane.

Proposition VI. Theorem

439. *Oblique lines drawn from a point to a plane, meeting the plane at equal distances from the foot of the perpendicular, are equal; and of two oblique lines, meeting the plane at unequal distances from the foot of the perpendicular, the more remote is the greater.*



Given the plane *MN*, the perpendicular line *PO*, the oblique lines *PA*, *PB*, *PC*, the equal distances *OB*, *OC*, and the unequal distances *OA*, *OC*, with *OA* greater than *OC*.

To prove that $PB = PC$, *and* $PA > PC$.

Proof. In the ⨻ OBP and OCP,

$OP = OP$, Iden.

$OB = OC$, Given

and $\angle BOP = \angle POC$. § 56

$\therefore \triangle OBP$ is congruent to $\triangle OCP$. § 69

$\therefore PB = PC$. § 67

Let A, B, and O lie in the same straight line.

Then $OA > OC$. Given

$\therefore OA > OB$. Ax. 9

$\therefore PA > PB$. § 84

$\therefore PA > PC$, by Ax. 9. Q.E.D.

Discussion. Compare the corresponding case in plane geometry.

440. Corollary 1. *Equal oblique lines drawn from a point to a plane meet the plane at equal distances from the foot of the perpendicular; and of two unequal oblique lines the greater meets the plane at the greater distance from the foot of the perpendicular.*

In the figure on page 280, if PB is given equal to PC, then since $PO = PO$, and the angles at O are right angles, what follows with respect to the ⚠ OBP and OCP? with respect to OB and OC?

Furthermore, if $PA > PC$, how does PA compare with PB?

Then how does OA compare with OB? Why?

Then how does OA compare with OC?

441. Corollary 2. *The locus of a point equidistant from all points on a circle is a line through the center, perpendicular to the plane of the circle.*

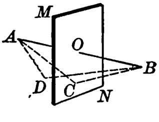
In the figure on page 280, in order to prove that PO is the required locus what must be proved for any point on PO (§ 148)? for any point not on PO? Prove both of these facts.

442. Corollary 3. *The locus of a point equidistant from the vertices of a triangle is a line through the center of the circumscribed circle, perpendicular to the plane of the triangle.*

How does this follow from Corollary 2?

What locus is the line through the center of the inscribed circle, perpendicular to the plane of the triangle?

443. Corollary 4. *The locus of a point equidistant from two given points is the plane perpendicular to the line joining them, at its mid-point.*



For any point C in this plane lies in a $\perp$ to AB at O, its mid-point (§ 430).

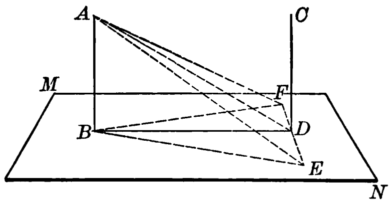
Hence how do CA and CB compare (§ 150)?

And any point D outside the plane MN cannot lie in a $\perp$ to AB at O. What may therefore be said as to the distances from D to A and B (§ 150)?

What is the proposition in plane geometry corresponding to Corollary 4? In what respect do the two proofs differ?

Proposition VII. Theorem

444. *Two lines perpendicular to the same plane are parallel.*



Given the lines *AB* and *CD*, perpendicular to the plane *MN*.

To prove that AB and CD are parallel.

Proof. Draw *AD* and *BD*, and in *MN* draw through *D* *EF* ⊥ to *BD*, making *DE* = *DF*. Draw *BE*, *AE*, *BF*, *AF*.

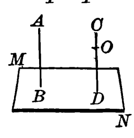
Now prove that △s *BDE* and *BDF* are congruent (§ 69), that ∠s *ADE* and *ADF* are right angles (§ 80), and that *BD*, *CD*, and *AD* lie in the same plane (§ 432).

But *AB* also lies in this plane, § 422

and *AB* and *CD* are both ⊥ to *BD*. § 430

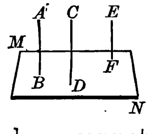
∴ *AB* is ∥ to *CD*, by § 95. Q.E.D.

445. Corollary 1. *If one of two parallel lines is perpendicular to a plane, the other is also perpendicular to the plane.*



For if through any point *O* of *CD* a line is drawn ⊥ to *MN*, how is it related to *AB* (§ 444)? Now apply § 94.

446. Corollary 2. *If two lines are parallel to a third line, they are parallel to each other.*



For a plane *MN* ⊥ to *CD* is ⊥ to *AB* and *EF* (§ 445).

447. Line and Plane Parallel. If a line and plane cannot meet, however far produced, they are said to be *parallel*.

EXERCISE 74

1. Why does folding a sheet of paper give a straight edge?

2. If equal oblique lines are drawn from a given external point to a plane, they make equal angles with lines drawn from the points where the oblique lines meet the plane to the foot of the perpendicular from the given point.

3. If from the foot of a perpendicular to a plane a line is drawn at right angles to any line in the plane, the line drawn from its intersection with the line in the plane to any point of the perpendicular is perpendicular to the line of the plane.

4. If two perpendiculars are drawn from a point to a plane and to a line in that plane respectively, the line joining the feet of the perpendiculars is perpendicular to the given line.

5. From two vertices of a triangle perpendiculars are let fall on the opposite sides. From the intersection of these perpendiculars a perpendicular is drawn to the plane of the triangle. Prove that a line drawn to any vertex of the triangle, from any point on this perpendicular, is perpendicular to the line drawn through that vertex parallel to the opposite side.

6. Find the point in a plane to which lines may be drawn from two given external points on the same side of the plane so that their sum shall be the least possible.

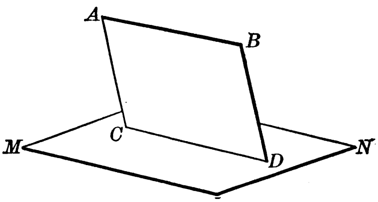
From one point A suppose a $\perp$ AO drawn to the plane and produced to A', making $OA' = OA$. Connect A' and the other point B by a line cutting the plane at P. Then BPA is the shortest line.

7. If three equal oblique lines are drawn from an external point to a plane, the perpendicular from the point to the plane meets the plane at the center of the circle circumscribed about the triangle having for its vertices the feet of the oblique lines.

8. State and prove the propositions of plane geometry corresponding to §§ 444, 445, and 446. Why do not the proofs of those propositions apply to these sections?

Proposition VIII. Theorem

448. *If two lines are parallel, every plane containing one of the lines, and only one, is parallel to the other line.*



Given the parallel lines *AB* and *CD*, and the plane *MN* containing *CD* but not *AB*.

To prove that the plane MN is parallel to AB.

Proof. *AB* and *CD* are in the same plane, *AD*. § 93

This plane *AD* intersects the plane *MN* in *CD*. Given

Now *AB* lies in the plane *AD*, however far produced. § 422

Therefore, if *AB* meets the plane *MN* at all, the point of meeting must be in the line *CD*. § 422

But since *AB* is ∥ to *CD*, Given

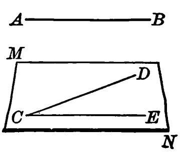
∴ *AB* cannot meet *CD*. § 93

∴ *AB* cannot meet the plane *MN*.

∴ *MN* is ∥ to *AB*, by § 447. Q.E.D.

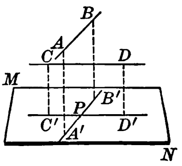
449. Corollary 1. *Through either of two lines not in the same plane one plane, and only one, can be passed parallel to the other.*

For if *AB* and *CD* are the lines, and we pass a plane through *CD* and a line *CE* which is drawn parallel to *AB*, what can be said of the plane *MN* determined by *CD* and *CE*, with respect to the line *AB*? Why can there be only one such plane?



450. Corollary 2. *Through a given point one plane, and only one, can be passed parallel to any two given lines in space.*

Suppose P the given point and AB and CD the given lines. If, now, we draw through P the line $A'B'$ parallel to AB, and the line $C'D'$ parallel to CD, these lines will determine the plane MN (§ 425). Then what may be said of the plane MN with respect to the lines AB and CD? Why can only one plane be so passed through P?



Discussion. Proposition VIII might of course be made more general by allowing both of the parallels to lie in the plane MN. That is, *If two lines are parallel, a plane containing one of the lines cannot intersect the other*, although the other line might lie in it.

In the figure of Corollary 2 the $\angle D'PB'$ is sometimes spoken of as the angle between the nonintersecting lines AB and CD, although this is not commonly done in elementary geometry.

451. Parallel Planes. Two planes which cannot meet, however far produced, are said to be *parallel.*

EXERCISE 75

1. What is the locus of a point in a plane equidistant from two parallel lines? What is the corresponding locus in space, given two parallel planes instead of two parallel lines? Draw the figure, without proof.

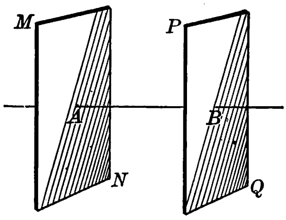
2. Find the locus in a plane of a point at a given distance from a given external point. What is the corresponding case of plane geometry?

3. If a given line is parallel to a given plane, the intersection of the plane with any plane passed through the given line is parallel to that line.

4. If a given line is parallel to a given plane, a line parallel to the given line drawn through any point of the plane lies in the plane.

Proposition IX. Theorem

452. *Two planes perpendicular to the same line are parallel.*



Given the planes *MN* and *PQ* perpendicular to the line *AB*.

To prove that the planes MN and PQ are parallel.

Proof. If *MN* and *PQ* are not parallel, they must meet.

If they could meet, we should have two planes from a point of their intersection ⊥ to the same straight line.

But this is impossible. § 434

∴ *MN* and *PQ* are parallel, by § 451. Q.E.D.

EXERCISE 76

1. What is the locus of a point equidistant from two given points *A*, *B*, and also equidistant from two other given points *C*, *D*?

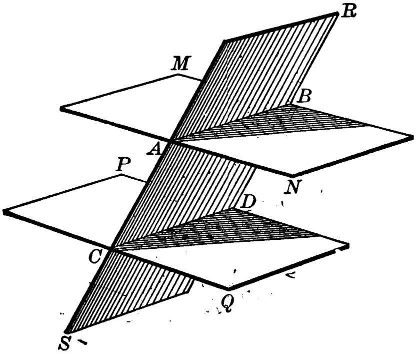
2. What is the locus of a point at the distance d from a given plane P, and at the distance d' from a given plane P'?

3. What is the locus of a point at the distance d from a given plane P, and equidistant from two given points *A*, *B*?

4. Find a point at the distance d from a given plane P, at the distance d' from a given plane P', and equidistant from two given points *A*, *B*. Can there be more than one such point? Draw the figure, without proof.

Proposition X. Theorem

453. *The intersections of two parallel planes by a third plane are parallel lines.*



Given the parallel planes *MN* and *PQ*, cut by the plane *RS* in *AB* and *CD* respectively.

To prove that the intersections AB and CD are parallel.

Proof. *AB* and *CD* are in the same plane *RS*. Given

If *AB* and *CD* meet, the planes *MN* and *PQ* must meet, since *AB* is always in *MN* and *CD* is always in *PQ*. § 422

But *MN* and *PQ* cannot meet. § 451

∴ *AB* is ∥ to *CD*, by § 93. Q.E.D.

454. Corollary 1. *Parallel lines included between parallel planes are equal.*

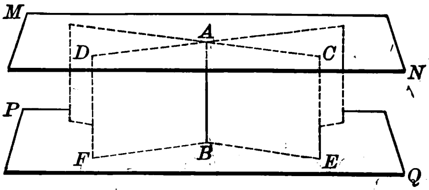
In the above figure, suppose *AC* ∥ to *BD*. Then the plane of *AC* and *BD* will intersect *MN* and *PQ* in lines that are how related to each other? Then what kind of a figure is *ACDB*?

455. Corollary 2. *Two parallel planes are everywhere equidistant from each other.*

Drop perpendiculars from any points in *MN* to *PQ*. Prove that these perpendiculars are parallel and hence (§ 454) that they are equal.

PROPOSITION XI. THEOREM

456. *A line perpendicular to one of two parallel planes is perpendicular to the other also.*



Given the line *AB* perpendicular to the plane *MN*, and the plane *PQ* parallel to the plane *MN*.

To prove that AB is perpendicular to the plane PQ.

Proof. Pass through AB two planes AE, AF, intersecting MN in AC, AD, and intersecting PQ in BE, BF, respectively.

Then AC is $\parallel$ to BE, and AD is $\parallel$ to BF. § 453

But AB is $\perp$ to AC and AD. § 430

$\therefore$ AB is $\perp$ to BE and BF. § 97

$\therefore$ AB is $\perp$ to the plane PQ, by § 431. Q.E.D.

457. COROLLARY 1. *Through a given point one plane, and only one, can be passed parallel to a given plane.*

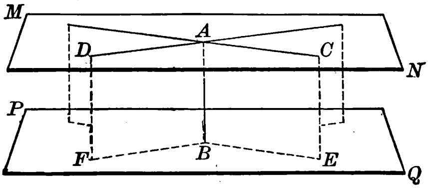
How is a plane through A, $\perp$ to AB, related to PQ? Now use § 433.

458. COROLLARY 2. *The locus of a point equidistant from two parallel planes is a plane perpendicular to a line which is perpendicular to the planes and which bisects the segment cut off by them.*

459. COROLLARY 3. *The locus of a point equidistant from two parallel lines is a plane perpendicular to a line which is perpendicular to the given lines and which bisects the segment cut off by them.*

Proposition XII. Theorem

460. *If two intersecting lines are each parallel to a plane, the plane of these lines is parallel to that plane.*



Given the intersecting lines *AC*, *AD*, each parallel to the plane *PQ*, and let *MN* be the plane determined by *AC* and *AD*.

To prove that *MN is parallel to PQ.*

Proof. Draw $AB \perp$ to PQ.

Pass a plane through AB and AC intersecting PQ in BE, and a plane through AB and AD intersecting PQ in BF.

Then AB is $\perp$ to BE and BF. § 430

But AC and BE lie in the same plane, Const.

and AC cannot meet BE without meeting the plane PQ, which is impossible. § 447

$\therefore BE$ is $\parallel$ to AC. § 93

Similarly BF is $\parallel$ to AD.

$\therefore AB$ is $\perp$ to AC and to AD. § 97

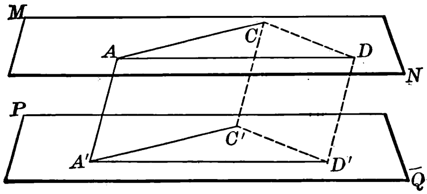
$\therefore AB$ is $\perp$ to the plane MN. § 431

$\therefore MN$ is $\parallel$ to PQ, by § 452. Q.E.D.

Discussion. It is evident that this proposition does not depend upon the position of A. For example, C and D might remain where they are and A might recede a long distance, AC and AD becoming more nearly parallel. So long as the lines intersect, and only so long, are we certain that the planes are parallel.

Proposition XIII. Theorem

461. *If two angles not in the same plane have their sides respectively parallel and lying on the same side of the straight line joining their vertices, the angles are equal, and their planes are parallel.*



Given the angles A and A', in the planes MN and PQ respectively, and their corresponding sides parallel and lying on the same side of AA'.

To prove that $\angle A = \angle A'$, *and that* MN *is* $\parallel$ *to* PQ.

Proof. Take AD and $A'D'$ equal, also AC and $A'C'$ equal.

Draw DD', CC', CD, $C'D'$.

Since AD is equal and $\parallel$ to $A'D'$,

$\therefore AA'$ is equal and $\parallel$ to DD'. § 130

In like manner AA' is equal and $\parallel$ to CC'.

$\therefore DD'$ and CC' are equal, Ax. 8

and DD' and CC' are parallel. § 446

$\therefore CD = C'D'$. § 130

$\therefore \triangle ADC$ is congruent to $\triangle A'D'C'$. § 80

$\therefore \angle A = \angle A'$. § 67

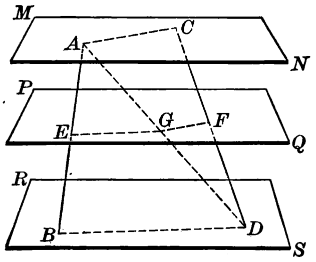
But MN is $\parallel$ to each of the lines $A'C'$ and $A'D'$. § 448

$\therefore MN$ is $\parallel$ to PQ, by § 460. Q.E.D.

Discussion. Why does not the proof of the corresponding proposition in plane geometry apply here?

PROPOSITION XIV. THEOREM

462. *If two lines are cut by three parallel planes, their corresponding segments are proportional.*



Given the lines *AB* and *CD*, cut by the parallel planes *MN*, *PQ*, *RS*, in the points *A*, *E*, *B*, and *C*, *F*, *D*, respectively.

To prove that $AE : EB = CF : FD$.

Proof. Draw AD cutting the plane PQ in G.

Pass a plane through AB and AD, intersecting PQ in the line EG, and intersecting RS in the line BD.

Also pass a plane through AD and CD, intersecting PQ in the line GF, and intersecting MN in the line AC.

Then EG is $\parallel$ to BD,

and GF is $\parallel$ to AC. § 453

$\therefore AE : EB = AG : GD$,

and $CF : FD = AG : GD$. § 273

$\therefore AE : EB = CF : FD$, by Ax. 8. Q.E.D.

Discussion. This is a generalization of § 275. It may be stated still more generally, *If two lines are cut by any number of parallel planes, their corresponding segments are proportional.* In particular, the case might be considered in which AB and CD intersect between the planes.

Why does not the proof of the corresponding case (§ 275) in plane geometry apply here?

EXERCISE 77

1. Find the locus of a line drawn through a given point, parallel to a given plane.

2. Find the locus of a point in a given plane that is equidistant from two given points not in the plane.

3. Find the locus of a point equidistant from three given points not in a straight line.

4. Find the locus of a point equidistant from two given parallel planes and also equidistant from two given points.

5. What is the locus of a point in a plane at a given distance from a given line in the plane? What is the locus of a point at a given distance from a given plane?

6. The line AB cuts three parallel planes in the points A, E, B; and the line CD cuts these planes in the points C, F, D. If $AE = 6$ in., $EB = 8$ in., and $CD = 12$ in., compute CF and FD.

7. The line AB cuts three parallel planes in the points A, E, B; and the line CD cuts these planes in the points C, F, D. If $AB = 8$ in., $CF = 5$ in., and $CD = 9$ in., compute AE and EB.

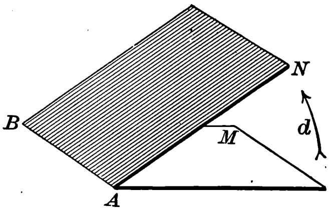
8. To draw a perpendicular to a given plane from a given point without the plane.

9. To erect a perpendicular to a given plane at a given point in the plane.

10. It is proved in plane geometry that if three or more parallels intercept equal segments on one transversal, they intercept equal segments on every transversal. State and prove a corresponding proposition in solid geometry.

11. It is proved in plane geometry that the line joining the mid-points of two sides of a triangle is parallel to the third side. State and prove a corresponding proposition in solid geometry, referring to a plane passing through the mid-points of two sides of a triangle.

463. Dihedral Angle. The opening between two intersecting planes is called a *dihedral angle.*



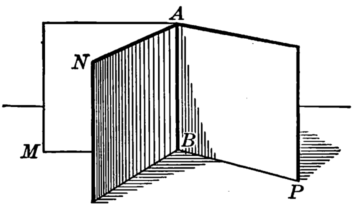
In this figure the two planes *AM* and *BN* are called the *faces* of the dihedral angle, and the line of intersection *AB* is called the *edge* of the angle.

A dihedral angle is read by naming the letters designating its edge, or its faces and edge, or by a small letter within. Thus the dihedral angle here shown may be designated by *AB*, *M-AB-N*, or *d*.

464. Size of a Dihedral Angle. The size of a dihedral angle depends upon the amount of turning necessary to bring one face into the position of the other.

The analogy to the plane angle is apparent, and is still further seen as we proceed.

465. Adjacent Dihedral Angles. If two dihedral angles have a common edge, and a common face between them, they are said to be *adjacent dihedral angles.*



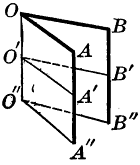
For example, *M-AB-N* and *N-BA-P* are adjacent dihedral angles.

466. Right Dihedral Angle. If one plane meets another plane and makes the adjacent dihedral angles equal, each of these angles is called a *right dihedral angle.*

Dihedral angles are said to be *straight*, *acute*, *obtuse*, *reflex*, *complementary*, *supplementary*, *conjugate*, and *vertical*, under conditions similar to those obtaining with plane angles. There is little occasion, however, to use any of these terms in connection with dihedral angles.

467. Perpendicular Planes. If two planes intersect and form a right dihedral angle, each of the planes is said to be *perpendicular* to the other plane.

468. Plane Angle of a Dihedral Angle. The plane angle formed by two straight lines, one in each plane, perpendicular to the edge at the same point, is called the *plane angle of the dihedral angle.*



For example, $\angle AOB$ is the plane angle of the dihedral angle OO'', if AO and BO are each $\perp$ to OO''.

469. Corollary. *The plane angle of a dihedral angle has the same magnitude from whatever point in the edge the perpendiculars are drawn.*

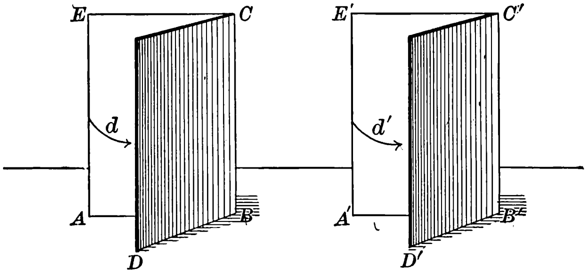
How is $O'B'$ related to OB, and $O'A'$ to OA (§ 95)? Then how is $\angle A'O'B'$ related to $\angle AOB$ (§ 461)?

470. Relation of Dihedral Angles to Plane Angles. It is apparent that the demonstrations of many properties of dihedral angles are identically the same as the demonstrations of analogous properties of plane angles. A few of the more important propositions will be proved, but the following may be assumed or may be taken as exercises:

1. If a plane meets another plane, it forms with it two adjacent dihedral angles whose sum is equal to two right dihedral angles.
2. If the sum of two adjacent dihedral angles is equal to two right dihedral angles, their exterior faces are in the same plane.
3. If two planes intersect each other, their vertical dihedral angles are equal.
4. If a plane intersects two parallel planes, the alternate-interior dihedral angles are equal; the exterior-interior dihedral angles are equal; and the two interior dihedral angles on the same side of the transverse plane are supplementary.
5. When two planes are cut by a third plane, if the alternate-interior dihedral angles are equal, or the exterior-interior dihedral angles are equal, and the edges of the dihedral angles thus formed are parallel, the two planes are parallel.
6. Two dihedral angles whose faces are parallel each to each are either equal or supplementary.
7. Two dihedral angles whose faces are perpendicular each to each, and whose edges are parallel, are either equal or supplementary.

Proposition XV. Theorem

471. *Two dihedral angles are equal if their plane angles are equal.*



Given two equal plane angles ABD and $A'B'D'$ of the two dihedral angles d and d'.

To prove that the dihedral angles d and d' are equal.

Proof. Apply dihedral angle d' to dihedral angle d, making the plane $\angle A'B'D'$ coincide with its equal $\angle ABD$.

Then since $B'C'$ is $\perp$ to $A'B'$ and $D'B'$, § 468

$\therefore B'C'$ is $\perp$ to the plane $A'B'D'$. § 431

$\therefore B'C'$ will also be $\perp$ to the plane ABD at B. Post. 5

$\therefore B'C'$ will fall on BC. § 436

Then the planes $A'B'C'$ and ABC, having in common the two intersecting lines AB and BC, coincide. § 425

In the same way it may be shown that the planes $D'B'C'$ and DBC coincide.

Therefore the two dihedral angles d and d' coincide and are equal. Q.E.D.

Discussion. May we have equal straight dihedral angles? equal reflex dihedral angles? What is the authority for saying that right dihedral angles are equal?

Proposition XVI. Theorem

472. *Two dihedral angles have the same ratio as their plane angles.*

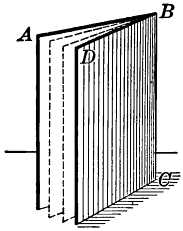


Fig. 1

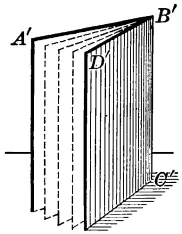


Fig. 2

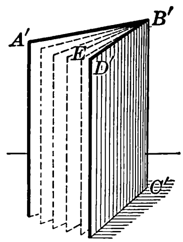


Fig. 3

Given two dihedral angles BC and $B'C'$, and let their plane angles be ABD and $A'B'D'$ respectively.

To prove that $\angle B'C' : \angle BC = \angle A'B'D' : \angle ABD.$

Case 1. *When the plane angles are commensurable.*

Proof. Suppose the $\angle$s ABD and $A'B'D'$ (Figs. 1 and 2) have a common measure, which is contained m times in $\angle ABD$ and n times in $\angle A'B'D'$.

Then $$\angle A'B'D' : \angle ABD = n : m.$$

Apply this measure to $\angle ABD$ and $\angle A'B'D'$, and through the lines of division and the edges BC and $B'C'$ pass planes.

These planes divide $\angle BC$ into m parts, and $\angle B'C'$ into n parts, equal each to each. § 471

$$\therefore \angle B'C' : \angle BC = n : m.$$

$$\therefore \angle B'C' : \angle BC = \angle A'B'D' : \angle ABD, \text{ by Ax. 8.}$$ Q.E.D.

As with plane angles, there is also the case of incommensurables. Since the common measure may be taken as small as we please, it is evident that for practical purposes the above proof is sufficient. The proof for the incommensurable case, p. 297, may be omitted at the discretion of the teacher without destroying the sequence.

CASE 2. *When the plane angles are incommensurable.*

Proof. Divide the $\angle ABD$ into any number of equal parts, and apply one of these parts to the $\angle A'B'D'$ (Figs. 1 and 3) as a unit of measure.

Since $\angle ABD$ and $\angle A'B'D'$ are incommensurable, a certain number of these parts will form the $\angle A'B'E$, leaving a remainder $\angle EB'D'$, less than one of the parts.

Pass a plane through $B'E$ and $B'C'$.

Since the plane angles of the dihedral angles $A\text{-}BC\text{-}D$ and $A'\text{-}B'C'\text{-}E$ are commensurable,

$\therefore A'\text{-}B'C'\text{-}E : A\text{-}BC\text{-}D = \angle A'B'E : \angle ABD.$ Case 1

By increasing the number of equal parts into which $\angle ABD$ is divided we can diminish the magnitude of each part, and therefore can make the $\angle EB'D'$ less than any assigned positive value, however small.

Hence the $\angle EB'D'$ approaches zero as a limit, as the number of parts is indefinitely increased, and at the same time the corresponding dihedral $\angle E\text{-}B'C'\text{-}D'$ approaches zero as a limit. § 204

Therefore the $\angle A'B'E$ approaches the $\angle A'B'D'$ as a limit, and the $\angle A'\text{-}B'C'\text{-}E$ approaches the $\angle A'\text{-}B'C'\text{-}D'$ as a limit.

$\therefore$ the variable $\dfrac{\angle A'B'E}{\angle ABD}$ approaches $\dfrac{\angle A'B'D'}{\angle ABD}$ as a limit,

and the variable $\dfrac{\angle A'\text{-}B'C'\text{-}E}{\angle A\text{-}BC\text{-}D}$ approaches $\dfrac{\angle A'\text{-}B'C'\text{-}D'}{\angle A\text{-}BC\text{-}D}$ as a limit.

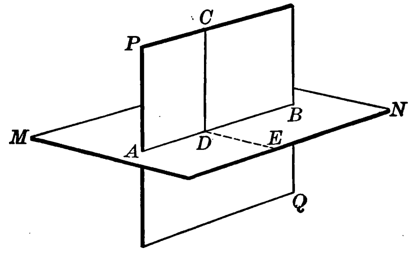
But $\dfrac{\angle A'B'E}{\angle ABD}$ is always equal to $\dfrac{\angle A'\text{-}B'C'\text{-}E}{\angle A\text{-}BC\text{-}D}$, as $\angle A'B'E$ varies in value and approaches $\angle A'B'D'$ as a limit. Case 1

$$\therefore \frac{\angle A'\text{-}B'C'\text{-}D'}{\angle A\text{-}BC\text{-}D} = \frac{\angle A'B'D'}{\angle ABD}, \text{ by § 207.}$$ Q.E.D.

473. COROLLARY. *The plane angle of a dihedral angle may be taken as the measure of the dihedral angle.*

PROPOSITION XVII. THEOREM

474. *If two planes are perpendicular to each other, a line drawn in one of them perpendicular to their intersection is perpendicular to the other.*



Given the planes *MN* and *PQ* perpendicular to each other, and the line *CD* in *PQ* perpendicular to their intersection *AB*.

To prove that CD is perpendicular to the plane MN.

Proof. In the plane *MN* draw *DE* ⊥ to *AB* at *D*.

Then $\angle EDC$ is a right angle, § 473

and $\angle CDA$ is also a right angle. Given

∴ *CD* is ⊥ to the plane *MN*, by § 431. Q.E.D.

475. COROLLARY 1. *If two planes are perpendicular to each other, a perpendicular to one of them at any point of their intersection will lie in the other.*

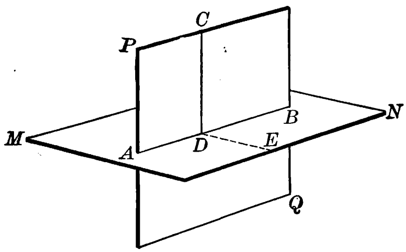
Will a line *CD* drawn in the plane *PQ* ⊥ to *AB* at *D* be ⊥ to the plane *MN*? How many ⊥s can be drawn from *D* to the plane *MN*?

476. COROLLARY 2. *If two planes are perpendicular to each other, a perpendicular to the first from any point in the second will lie in the second.*

Will a line *CD* drawn in the plane *PQ* from *C* ⊥ to *AB* be ⊥ to the plane *MN*? How many ⊥s can be drawn from *C* to the plane *MN*?

PROPOSITION XVIII. THEOREM

477. *If a line is perpendicular to a plane, every plane passed through this line is perpendicular to the plane.*



Given the line *CD* perpendicular to the plane *MN* at the point *D*, and *PQ* any plane passed through *CD* intersecting *MN* in *AB*.

To prove that the plane PQ is perpendicular to the plane MN.

Proof. Draw DE in the plane $MN \perp$ to AB.

Since CD is $\perp$ to MN, Given

$\therefore CD$ is $\perp$ to AB. § 430

$\therefore \angle EDC$ measures $\angle N\text{-}AB\text{-}P$. § 473

But $\angle EDC$ is a right angle. § 430

$\therefore PQ$ is $\perp$ to MN, by § 467. Q.E.D.

EXERCISE 78

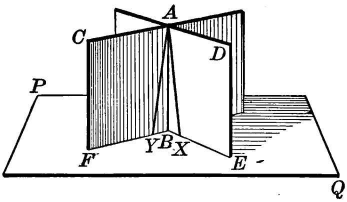
1. A plane perpendicular to the edge of a dihedral angle is perpendicular to each of its faces.

2. If one line is perpendicular to another, is any plane passed through the first line perpendicular to the second? Prove it.

3. If three lines are perpendicular to one another at a common point, what is the relation to one another of the three planes determined by the three pairs of lines? Prove it.

Proposition XIX. Theorem

478. *If two intersecting planes are each perpendicular to a third plane, their intersection is also perpendicular to that plane.*



Given two planes *BC* and *BD*, intersecting in *AB*, and each perpendicular to the plane *PQ*.

To prove that AB is perpendicular to the plane PQ.

Proof. Let the plane *BC* intersect the plane *PQ* in *BF*,

and let the plane *BD* intersect the plane *PQ* in *BE*.

From any point *A* on *AB* draw *AX* ⊥ to *BE*,

and from *A* draw *AY* ⊥ to *BF*.

Then *AX* and *AY* are both ⊥ to the plane *PQ*. § 474

But it is impossible to draw two ⊥s to the plane *PQ*

from a point outside the plane *PQ*, § 437

or from a point in the plane *PQ*. § 436

∴ *AX* and *AY* must coincide.

But *AX* and *AY* can coincide only if they lie in both planes.

And all points common to both planes lie in *AB*. § 429

∴ *AX* and *AY* coincide with *AB*.

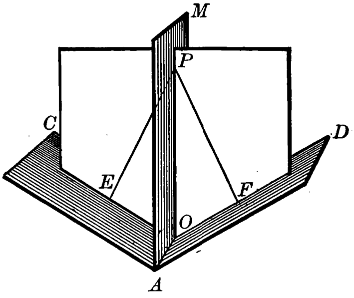
∴ *AB* is ⊥ to the plane *PQ*. Q.E.D.

Discussion. How does it appear from this proof that *AB* cannot be parallel to *PQ*?

The proposition is illustrated in the intersection of two walls of a room with the floor or the ceiling.

Proposition XX. Theorem

479. *The locus of a point equidistant from the faces of a dihedral angle is the plane bisecting the angle.*



Given the plane *AM* bisecting the dihedral angle formed by the planes *AD* and *AC*.

To prove that the plane AM is the locus of a point equidistant from the planes AD and AC.

Proof. Let *EOF* be a plane ⊥ to *AO*, the intersection of the planes *AD* and *AC*, at *O*.

Since *AO* is ⊥ to the plane *EOF*,

∴ the planes *AD*, *AM*, and *AC* are ⊥ to the plane *EOF*. § 477

From any point *P*, in the intersection of the planes *AM* and *EOF*, draw *PF* ⊥ to *OF*, and *PE* ⊥ to *OE*.

Then *PF* is ⊥ to *AD*, and *PE* is ⊥ to *AC*. § 474

∴ *PF* and *PE* measure the distances from the point *P* to the planes *AD* and *AC*. § 438

Since *AO* is ⊥ to *OF*, *OP*, and *OE*, § 430

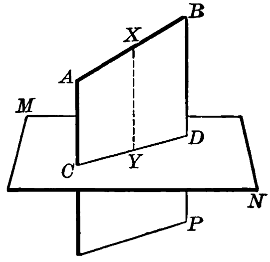
∴ *OP* bisects ∠ *FOE*. § 473

∴ *OP* is the locus of a point equidistant from *OF* and *OE*. § 152

∴ *AM*, which contains all points *P*, is the locus of a point equidistant from the planes *AD* and *AC*. Q.E.D.

Proposition XXI. Theorem

480. *Through a given line not perpendicular to a given plane, one plane and only one can be passed perpendicular to the plane.*



Given the line *AB* not perpendicular to the plane *MN*.

To prove that one plane can be passed through AB perpendicular to the plane MN, and only one.

Proof. From any point X of AB draw $XY \perp$ to the plane MN, and through AB and XY pass a plane AP. § 425

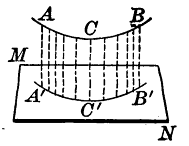
The plane AP is $\perp$ to the plane MN, since it passes through XY, a line $\perp$ to MN. § 477

Moreover, if two planes could be passed through $AB \perp$ to the plane MN, their intersection AB would be $\perp$ to MN. § 478

But this is impossible, since AB is not $\perp$ to MN. Given

Hence one plane can be passed through $AB \perp$ to the plane MN, and only one. Q.E.D.

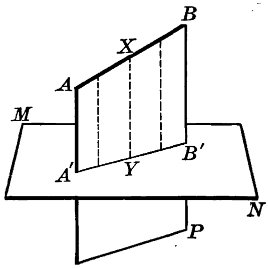
481. Projection of a Point. The foot of the line from a given point perpendicular to a plane is called the *projection of the point* on the plane.



482. Projection of a Line. The locus of the projections of the points of a line on a plane is called the *projection of the line* on the plane.

PROPOSITION XXII. THEOREM

483. *The projection of a straight line not perpendicular to a plane, upon that plane, is a straight line.*



Given the straight line *AB* not perpendicular to the plane *MN*, and *A′B′* the projection of *AB* upon *MN*.

To prove that *A′B′ is a straight line.*

Proof. From any point X of AB draw $XY \perp$ to MN,

and pass a plane AP through XY and AB. § 425

The plane AP is $\perp$ to the plane MN, § 477

and contains all the ⊥s drawn from AB to MN. § 476

Hence $A'B'$ must be the intersection of these two planes.

Therefore $A'B'$ is a straight line, by § 429. Q.E.D.

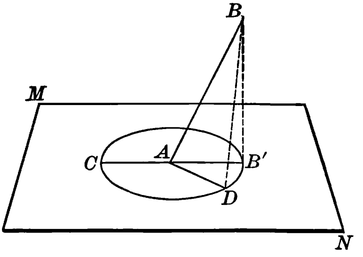
484. COROLLARY. *The projection of a straight line perpendicular to a plane, upon that plane, is a point.*

485. Inclination of a Line. The angle which a line makes with its projection on a plane is considered as the angle which it makes with the plane, and is called the *inclination of the line* to the plane.

Therefore a line ordinarily makes an acute angle with a plane, since it makes an acute angle with its projection on the plane. The cases of perpendicular and parallel lines have already been considered.

Proposition XXIII. Theorem

486. *The acute angle which a line makes with its projection upon a plane is the least angle which it makes with any line of the plane.*



Given the line AB meeting the plane MN at A, AB' being the projection of AB upon the plane MN, and AD being any other line drawn through A in the plane MN.

To prove that $\angle B'AB$ is less than $\angle DAB$.

Proof. Make AD equal to AB', and draw BB' and BD.

Then in △ BAB' and BAD,

$$AB = AB,$$ Iden.

$$AB' = AD,$$ Const.

and $$BB' < BD.$$ § 438

$\therefore \angle B'AB < \angle DAB$, by § 116. Q. E. D.

Discussion. Since $\angle B'AB$ is the least angle that AB makes with any line of the plane, how does $\angle BAC$ compare with the angles that AB makes with other lines of the plane? State the general proposition involved in the answer.

If AB is parallel to the plane, what interpretation may be given to the proposition?

If AB is perpendicular to the plane, what interpretation may be given to the proposition?

As AD swings around from the position AB' to the position AC, what kind of change takes place in the angle DAB?

EXERCISE 79

1. Describe the position of a segment of a line relative to a given plane if the projection of the segment on the plane is equal to its own length.

2. From a point A, 4 in. from a plane MN, an oblique line AC 5 in. long is drawn to the plane and made to turn around the perpendicular AB dropped from A to the plane. Find the area of the circle described by the point C.

3. From a point A, 8 in. from a plane MN, a perpendicular AB is drawn to the plane; with B as a center and a radius equal to 6 in., a circle is described in the plane; at any point C on this circle a tangent CD is drawn 24 in. in length. Find the distance from A to D.

4. Equal lines drawn from a given external point to a given plane are equally inclined to the plane.

5. If three equal lines are drawn to a plane from an external point, the perpendicular from the point to the plane determines the center of the circle circumscribed about the triangle determined by the planes of the three lines.

6. Three lines not in the same plane meet in a point. How shall a line be drawn so as to make equal angles with all three of these lines?

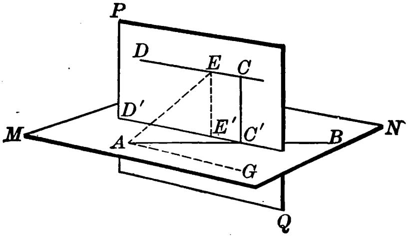
7. From a point P two perpendiculars PX and PY are drawn to two planes MN and AC which intersect in AB. From Y a perpendicular YZ is drawn to MN. Prove that the line XZ is perpendicular to AB.

8. If the length of the shadow of a tree standing on level ground exceeds the height of the tree, the angle made by the sun above the horizon must be less than what known angle?

9. Find the locus of a point at a given distance from a given plane and equidistant from two given points not in the plane.

PROPOSITION XXIV. THEOREM

487. *Between two lines not in the same plane there can be one common perpendicular, and only one.*



Given AB and CD, two lines not in the same plane.

To prove that there can be one common perpendicular, and only one, between AB and CD.

Proof. Through any point A of AB draw $AG \parallel$ to DC.

Let MN be the plane determined by AB and AG. § 425

Then the plane MN is $\parallel$ to DC. § 448

Through DC pass the plane $PQ \perp$ to the plane MN. § 480

Then DC cannot meet $D'C'$, since it is $\parallel$ to the plane MN and lies in the plane PQ. § 422

$\therefore DC$ is $\parallel$ to $D'C'$. § 93

$\therefore$ if AB is $\parallel$ to $D'C'$ it must be $\parallel$ to DC. § 446

But AB is not $\parallel$ to DC, for they are not in the same plane. Given

$\therefore AB$ must intersect $D'C'$ at some point as C'.

Draw $C'C \perp$ to the plane MN.

Then $C'C$ is $\perp$ to AB and to $D'C'$. § 430

Since $C'C$ is $\perp$ to $D'C'$, and lies in plane PQ, § 475

$\therefore C'C$ is $\perp$ to DC. § 97

Therefore one common perpendicular can be drawn.

It remains to be proved that no other can be drawn.

If it were possible that another common perpendicular could be drawn, we might suppose EA to be $\perp$ to both AB and CD.

Then EA would be $\perp$ to AG, § 97

and therefore EA would be $\perp$ to the plane MN. § 431

Draw $EE' \perp$ to $D'C'$.

Then EE' is $\perp$ to the plane MN. § 474

But this is impossible, if EA is also $\perp$ to the plane MN. § 437

Hence the supposition that there is a second common perpendicular, EA, leads to an absurdity.

Therefore there can be one common perpendicular, and only one, between AB and CD. Q. E. D.

488. COROLLARY. *The common perpendicular between two lines not in the same plane is the shortest line joining them.*

How does CC' compare in length with EE'? Why?
How does EE' compare in length with EA?

EXERCISE 80

1. Parallel lines have parallel projections on a plane.

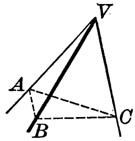
2. If two planes are perpendicular to each other, any line perpendicular to one of them is how related to the other?

3. If three lines passing through a given point P are cut by a fourth line that does not pass through P, the four lines all lie in the same plane.

4. Seven lines, no three of which lie in the same plane, pass through the same point. How many planes are determined by these lines?

5. A cubical tank 10 in. deep contains water to a depth of 7 in. A foot rule is placed obliquely on the bottom so as just to reach the top edge of the tank. Make a sketch of the tank, and compute the length of the rule covered by water.

489. Polyhedral Angle. The opening of three or more planes which meet at a common point is called a *polyhedral angle.*



The common point V is called the *vertex* of the angle; the intersections VA, VB, etc., of the planes are called the *edges;* the portions of the planes lying between the edges are called the *faces;* and the angles formed by adjacent edges are called the *face angles.*

Every two adjacent edges form a face angle, and every two adjacent faces form a dihedral angle. The face angles and dihedral angles are the *parts* of the polyhedral angle.

490. Size of a Polyhedral Angle. The size of a polyhedral angle depends upon the relative position of its faces, and not upon their extent.

491. Convex and Concave Polyhedral Angles. A polyhedral angle is said to be *convex* or *concave* according as a section made by a plane that cuts all its edges at other points than the vertex is a convex or concave polygon.

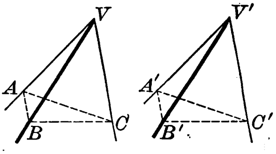
Only convex polyhedral angles are considered in this work.

492. Classes of Polyhedral Angles. A polyhedral angle is called a *trihedral angle* if it has three faces, a *tetrahedral angle* if it has four faces, and so on.

Other names, like pentahedral, hexahedral, heptahedral, etc., for angles with 5, 6, 7, etc., faces, are rarely used.

A polyhedral angle is designated by a letter at the vertex, or by letters representing the vertex and all the faces taken in order. Thus, in the above figure the trihedral angle is designated by V or by $V\text{-}ABC$. A tetrahedral angle would be designated by V or by $V\text{-}ABCD$.

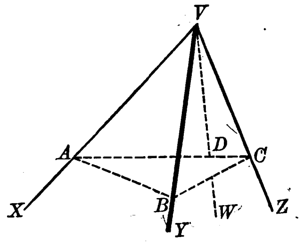
493. Equal Polyhedral Angles. If the corresponding parts of two polyhedral angles are equal and are arranged in the same order, the polyhedral angles are said to be *equal.*



Thus the angles $V\text{-}ABC$ and $V'\text{-}A'B'C'$ are equal. Equal polyhedral angles may evidently be made to coincide by superposition.

PROPOSITION XXV. THEOREM.

494. *The sum of any two face angles of a trihedral angle is greater than the third face angle.*



Given the trihedral angle $V\text{-}XYZ$, with the face angle XVZ greater than either of the face angles XVY or YVZ.

To prove that $\angle XVY + \angle YVZ$ *is greater than* $\angle XVZ$.

Proof. In the $\angle XVZ$ draw VW, making $\angle XVW = \angle XVY$.
Through any point D of VW draw ADC in the plane XVZ.

On VY take VB equal to VD.

Pass a plane through the line AC and the point B.

Then since $AV = AV$, $VD = VB$, and $\angle AVD = \angle AVB$,

$\therefore \triangle AVD$ is congruent to $\triangle AVB$. § 68

$\therefore AD = AB$. § 67

In the $\triangle ABC$, $AB + BC > AC$. § 112

Since $AB = AD$, $\therefore BC > DC$. Ax. 6

In the $\triangle$s BVC and DVC,

$VC = VC$, and $VB = VD$, but $BC > DC$.

$\therefore \angle BVC$ is greater than $\angle DVC$. § 116

$\therefore \angle AVB + \angle BVC$ is greater than $\angle AVD + \angle DVC$. Ax. 6

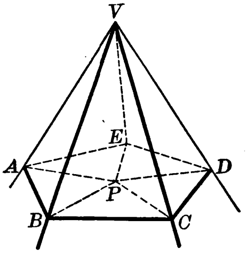
But $\angle AVD + \angle DVC = \angle AVC$. Ax. 11

$\therefore \angle AVB + \angle BVC$ is greater than $\angle AVC$. Ax. 9

That is, $\angle XVY + \angle YVZ$ is greater than $\angle XVZ$. Q. E. D.

PROPOSITION XXVI. THEOREM

495. *The sum of the face angles of any convex polyhedral angle is less than four right angles.*



Given a convex polyhedral angle *V*, all of its edges being cut by a plane making the section *ABCDE*.

To prove that $\angle AVB + \angle BVC$, *etc., is less than four rt.* ∠s.

Proof. From any point P within the polygon draw PA, PB, PC, PD, PE.

The number of the △s having the common vertex P is the same as the number having the common vertex V.

Therefore the sum of the ∠s of all the △s having the common vertex V is equal to the sum of the ∠s of all the △s having the common vertex P.

But in the trihedral ∠s formed at A, B, C, etc.,

$\angle EAV + \angle BAV$ is greater than $\angle BAE$,

$\angle VBA + \angle CBV$ is greater than $\angle CBA$, etc. § 494

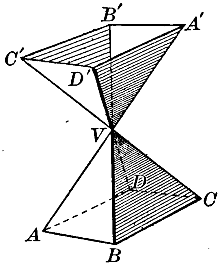
Hence the sum of the ∠s at the bases of the △s whose common vertex is V is greater than the sum of the ∠s at the bases of the △s whose common vertex is P. Ax. 7

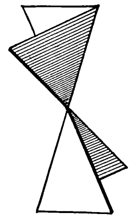
Therefore the sum of the ∠s at the vertex V is less than the sum of the ∠s at the vertex P. Ax. 7

But the sum of the ∠s at P is equal to 4 rt. ∠s. § 41

Therefore the sum of the ∠s at V is less than 4 rt. ∠s. Q.E.D.

496. Symmetric Polyhedral Angles. If the faces of a polyhedral angle $V\text{-}ABCD$ are produced through the vertex V, another polyhedral angle $V\text{-}A'B'C'D'$ is formed, *symmetric* with respect to $\angle V\text{-}ABCD$.





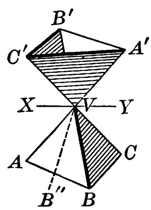
The face angles AVB, BVC, etc., are equal respectively to the face angles $A'VB'$, $B'VC'$, etc. (§ 60).

Also the dihedral angles VA, VB, etc., are equal respectively to the dihedral angles VA', VB', etc. (§ 470). (The second figure shows a pair of these vertical dihedral angles.)

Looked at from the point V, the edges of $\angle V\text{-}ABCD$ are arranged from left to right (counterclockwise) in the order VA, VB, VC, VD, but the edges of $\angle V\text{-}A'B'C'D'$ are arranged from right to left (clockwise) in the order VA', VB', VC', VD'; that is, in an order the reverse of the order of the edges in $\angle V\text{-}ABCD$. Therefore,

Two symmetric polyhedral angles have all their parts equal each to each but arranged in reverse order.

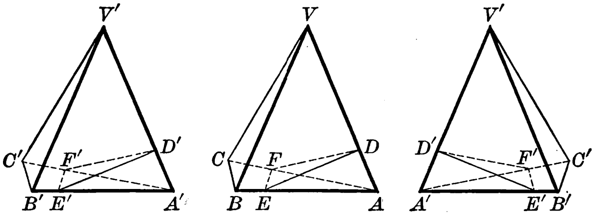
497. Symmetric Polyhedral Angles not Superposable. In general, two symmetric polyhedral angles are not superposable. Thus, if the trihedral angle $V\text{-}A'B'C'$ is made to turn 180° about XY, the bisector of the angle CVA', then VA' will coincide with VC, VC' with VA, and the face $A'VC'$ with AVC; but the dihedral angle VA, and hence the dihedral angle VA', not being equal to VC, the plane $A'VB'$ will not coincide with BVC; and, for a similar reason, the plane $C'VB'$ will not coincide with AVB. Hence the edge VB' takes some position VB'' not coincident with VB; that is, the trihedral angles are not superposable.



An analogous case is seen in a pair of gloves. All the parts of one are equal to the corresponding parts of the other, but the right-hand glove will not fit the left hand.

PROPOSITION XXVII. THEOREM

498. *Two trihedral angles are equal or symmetric when the three face angles of the one are equal respectively to the three face angles of the other.*



Given the trihedral angles V and V', the angles BVA, CVA, CVB being equal respectively to the angles $B'V'A'$, $C'V'A'$, $C'V'B'$.

To prove that the angles V and V' are equal or symmetric.

Proof. On the edges of these angles take the six equal segments VA, VB, VC, $V'A'$, $V'B'$, $V'C'$.

Draw AB, BC, CA, $A'B'$, $B'C'$, $C'A'$.

The isosceles $\triangle$s BAV, CAV, CBV are congruent respectively to the isosceles $\triangle$s $B'A'V'$, $C'A'V'$, $C'B'V'$. § 68

$\therefore$ AB, BC, CA are equal respectively to $A'B'$, $B'C'$, $C'A'$. § 67

$\therefore$ $\triangle BAC$ is congruent to $\triangle B'A'C'$. § 80

From any point D in VA draw DE in the face AVB and DF in the face AVC, each $\perp$ to VA.

These lines meet AB and AC respectively.

(*For the* $\angle$s VAB *and* VAC *are acute, each being one of the equal* $\angle$s *of an isosceles* $\triangle$.)

Draw EF.

On $A'V'$ take $A'D'$ equal to AD.

Draw $D'E'$ in the face $A'V'B'$ and $D'F'$ in the face $A'V'C'$, each $\perp$ to $V'A'$, and draw $E'F'$.

Then since $AD = A'D'$, Const.

and $\angle DAE = \angle D'A'E'$, § 67

$\therefore$ rt. $\triangle ADE$ is congruent to rt. $\triangle A'D'E'$. § 72

$\therefore AE = A'E'$, and $DE = D'E'$. § 67

In like manner $AF = A'F'$, and $DF = D'F'$.

Furthermore, since it has been proved that

$\triangle BAC$ is congruent to $\triangle B'A'C'$,

$\therefore \angle CAB = \angle C'A'B'$. § 67

$\therefore \triangle AFE$ is congruent to $\triangle A'F'E'$. § 68

$\therefore EF = E'F'$. § 67

$\therefore \triangle EDF$ is congruent to $\triangle E'D'F'$. § 80

$\therefore \angle FDE = \angle F'D'E'$. § 67

$\therefore$ dihedral $\angle VA =$ dihedral $\angle V'A'$. § 473

(For $\angle$s FDE and F′D′E′, the measures of these dihedral $\angle$s, are equal.)

In like manner it may be proved that the dihedral angles VB and VC are equal respectively to the dihedral angles $V'B'$ and $V'C'$.

$\therefore$ the trihedral angles V and V' are equal, § 493

or else they are symmetric, by § 496. Q.E.D.

This demonstration applies to either of the two figures denoted by V'-$A'B'C'$, which are symmetric with respect to each other. If the first of these figures is taken, V and V' are equal. If the second is taken, V and V' are symmetric.

499. COROLLARY. *If two trihedral angles have the three face angles of the one equal respectively to the three face angles of the other, then the dihedral angles of the one are equal respectively to the dihedral angles of the other.*

For whether the trihedral angles are equal or symmetric, as stated in the proposition, the dihedral angles are equal (§§ 493, 496).

EXERCISE 81

1. Find the locus of a point in a space of three dimensions equidistant from two given intersecting lines.

2. Find a point at equal distances from four points not all in the same plane.

3. Two dihedral angles which have their edges parallel and their faces perpendicular are equal or supplementary.

4. The projections on a plane of equal and parallel line-segments are equal and parallel.

5. Two trihedral angles are equal when two dihedral angles and the included face angle of the one are equal respectively to two dihedral angles and the included face angle of the other, and are similarly placed.

6. Two trihedral angles are equal when two face angles and the included dihedral angle of the one are equal respectively to two face angles and the included dihedral angle of the other, and are similarly placed.

7. If the face angle AVB of the trihedral angle $V\text{-}ABC$ is bisected by the line VD, the angle CVD is less than, equal to, or greater than half the sum of the angles AVC and BVC, according as $\angle CVD$ is less than, equal to, or greater than 90°.

8. If two face angles of a trihedral angle are equal, the dihedral angles opposite them are equal.

9. A trihedral angle having two of its face angles equal is superposable on its symmetric trihedral angle.

10. Find the locus of a point equidistant from the three edges of a trihedral angle.

11. Find the locus of a point equidistant from the three faces of a trihedral angle.

12. The planes that bisect the dihedral angles of a trihedral angle meet in a straight line.

EXERCISE 82

Problems of Computation

1. From a point P, 4 in. from a plane, a line PX is drawn meeting the plane at X. If PX is 5 in., what is the length of the locus of X in the plane?

2. From a point P, 5 in. from a plane, a line PX is drawn meeting the plane at X. If PX is 12 in., what area is inclosed in the plane by the locus of X? Answer to two decimal places.

3. The base AB of the isosceles triangle ABC in the plane MN is 6 in., and the perimeter of the triangle is 20 in. If the triangle revolves about its base as an axis, what is the greatest distance from the plane that is reached by C? Answer to three decimal places.

4. Two points A and B are 4 in. apart. A point P moves so as to be constantly 5 in. from each of these points. Find the length of the locus of P. Answer to three decimal places.

5. Two parallel planes MN and PQ are cut by a third plane RS so as to make one of the dihedral angles $27° 15' 30''$. Find the other dihedral angles.

6. Two lines are cut by three parallel planes. The segments cut from one line are 3 in. and $5\frac{1}{2}$ in., and those cut from the other line are $7\frac{3}{4}$ in. and x. Find the value of x.

7. Two given planes are at right angles to each other. A point X is 8 in. from each plane. How far is X from the edge of the right dihedral angle?

8. What is the length of the projection on a plane of a line whose length is $10\sqrt{2}$, the inclination of the line to the plane being 45°?

9. From the external point P a perpendicular PP', 9 in. long, is drawn to a plane MN. From P the line PQ is drawn to the plane making the angle $P'PQ$ equal to 30°. Find the length of the projection of PQ on the plane MN.

EXERCISE 83

Review Questions

1. How many and what conditions determine a straight line? How many and what conditions determine a plane?

2. What simple numerical test, following the measurement of certain lengths, determines whether or not one line is perpendicular to another? a line is perpendicular to a plane?

3. How many planes can be passed through a given line perpendicular to a given plane? Is this true for all positions of the given line?

4. Through a given point how many lines can be drawn parallel to a given line? parallel to a given plane? Through a given point how many planes can be passed parallel to a given line? parallel to a given plane?

5. What is the locus, in a line, of a point equidistant from two given points? in a plane? in a space of three dimensions?

6. What is the locus, in a plane, of a point equidistant from two intersecting lines? State a corresponding proposition for solid geometry.

7. What may be said of two lines in one plane perpendicular to the same line? State two corresponding propositions for solid geometry. Does one of these propositions state that two planes perpendicular to the same plane are parallel?

8. What may be said of a line perpendicular to one of two parallel lines? State two corresponding propositions for solid geometry. Is a plane perpendicular to one of two parallel planes perpendicular to the other?

9. If a line is perpendicular to a plane, what may be said of every plane passed through this line? Does a true proposition result from changing the word "perpendicular" to "parallel" in this statement?

BOOK VII

POLYHEDRONS, CYLINDERS, AND CONES

500. Polyhedron. A solid bounded by planes is called a *polyhedron.*

For example, the figures on pages 317 and 318 are polyhedrons.

The bounding planes are called the *faces* of the polyhedron, the intersections of the faces are called the *edges* of the polyhedron, and the intersections of the edges are called the *vertices* of the polyhedron.

A line joining any two vertices not in the same face is called a *diagonal* of the polyhedron.

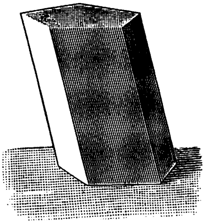
The plural of polyhedron is *polyhedrons* or *polyhedra.*

501. Section of a Polyhedron. If a plane passes through a polyhedron, the intersection of the plane with such faces as it cuts is called a *section* of the polyhedron.

502. Convex Polyhedron. If every section of a polyhedron is a convex polygon, the polyhedron is said to be *convex.*

Only convex polyhedrons are considered in this work.

503. Prism. A polyhedron of which two faces are congruent polygons in parallel planes, the other faces being parallelograms, is called a *prism.*



The parallel polygons are called the *bases* of the prism, the parallelograms are called the *lateral faces*, and the intersections of the lateral faces are called the *lateral edges.*

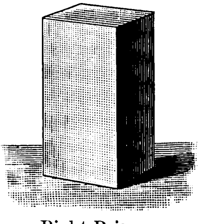
The sum of the areas of the lateral faces is called the *lateral area* of the prism.

The lateral edges of a prism are equal (§ 125).

504. Altitude of a Prism. The perpendicular distance between the planes of the bases of a prism is called its *altitude.*

505. Right Prism. A prism whose lateral edges are perpendicular to its bases is called a *right prism*.

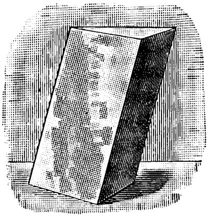
The lateral edges of a right prism are equal to the altitude (§ 455).



Right Prism

506. Oblique Prism. A prism whose lateral edges are oblique to its bases is called an *oblique prism*.

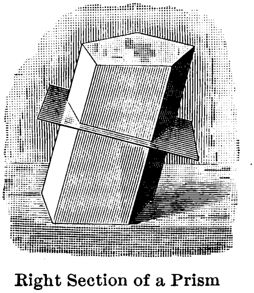
507. Prisms classified as to Bases. Prisms are said to be *triangular*, *quadrangular*, and so on, according as their bases are triangles, quadrilaterals, and so on.



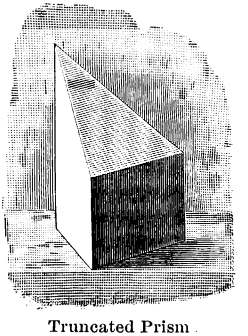
Oblique Triangular Prism

508. Right Section. A section of a prism made by a plane cutting all the lateral edges and perpendicular to them is called a *right section*.

In the case of oblique prisms it is sometimes necessary to produce some of the edges in order that the cutting plane may intersect them.



Right Section of a Prism

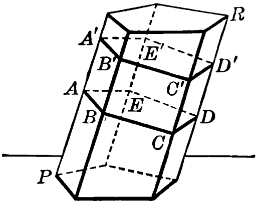


Truncated Prism

509. Truncated Prism. The part of a prism included between the base and a section made by a plane oblique to the base is called a *truncated prism*.

Proposition I. Theorem

510. *The sections of a prism made by parallel planes cutting all the lateral edges are congruent polygons.*



Given the prism *PR* and the parallel sections *AD*, *A'D'* cutting all the lateral edges.

To prove that AD is congruent to A'D'.

Proof. AB is ∥ to $A'B'$, BC is ∥ to $B'C'$, CD is ∥ to $C'D'$,

and so on for all the corresponding sides. § 453

$\therefore AB = A'B'$, $BC = B'C'$, $CD = C'D'$,

and so on for all the corresponding sides, § 127

and $\angle CBA = \angle C'B'A'$, $\angle DCB = \angle D'C'B'$,

and so on for all the corresponding angles. § 461

$\therefore AD$ is congruent to $A'D'$, by § 142. Q.E.D.

Discussion. Is the proof the same whether or not the two parallel planes are parallel to the bases?

If the sections are all parallel to the bases, are they also congruent to the bases?

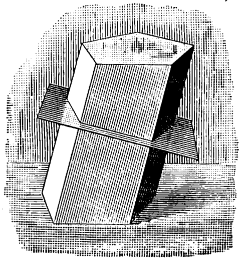
Would the proposition be true if the prism were concave instead of convex?

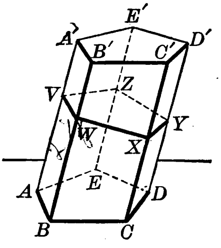
Suppose the bases were squares, what would be known as to the form of the sections?

511. Corollary. *Every section of a prism made by a plane parallel to the base is congruent to the base; and all right sections of a prism are congruent.*

Proposition II. Theorem

512. *The lateral area of a prism is equal to the product of a lateral edge by the perimeter of a right section.*





Given $VWXYZ$ a right section of the prism AD', l the lateral area, e a lateral edge, and p the perimeter of the right section.

To prove that $l = ep$.

Proof. $AA' = BB' = CC' = DD' = EE' = e$. § 503

Furthermore, VW is $\perp$ to BB', WX to CC', XY to DD', YZ to EE', and ZV to AA'. § 508

$\therefore$ the area of $\square\, AB' = BB' \times VW = e \times VW$; § 322

the area of $\square\, BC' = CC' \times WX = e \times WX$,

the area of $\square\, CD' = DD' \times XY = e \times XY$, and so on.

But l is equal to the sum of these parallelograms. § 503

$\therefore l = e(VW + WX + XY + YZ + ZV)$. Ax. 1

But $VW + WX + XY + YZ + ZV = p$. Ax. 11

$\therefore l = ep$, by Ax. 9. Q.E.D.

513. Corollary. *The lateral area of a right prism is equal to the product of the altitude by the perimeter of the base.*

For how would p then compare with $AB + BC + CD + DE + EA$?

The truth of the corollary is easily seen by imagining the right prism laid on one of its lateral faces, and the surface as it were unrolled.

EXERCISE 84

Find the lateral areas of the right prisms whose altitudes and perimeters of bases are as follows:

1. $a = 18$ in., $p = 29$ in.
2. $a = 22$ in., $p = 37$ in.
3. $a = 4.25$ in., $p = 6.75$ in.
4. $a = 1$ ft. 7 in., $p = 2$ ft. 9 in.
5. $a = 3$ ft. 8 in., $p = 5$ ft. 7 in.
6. $a = 12$ ft. 2 in., $p = 27$ ft. 9 in.

Find the lateral areas of the prisms whose lateral edges and perimeters of right sections are as follows:

7. $e = 17$ in., $p = 27$ in.
8. $e = 23$ in., $p = 35$ in.
9. $e = 2\frac{3}{4}$ in., $p = 4\frac{7}{8}$ in.
10. $e = 1$ ft. 3 in., $p = 2$ ft. 3 in.
11. $e = 2$ ft. 7 in., $p = 3$ ft. 9 in.
12. $e = 6$ ft. $1\frac{1}{4}$ in., $p = 8$ ft. $9\frac{1}{8}$ in.

Find the lateral edges of the prisms whose lateral areas and perimeters of right sections are as follows:

13. $l = 187$ sq. in., $p = 11$ in.
14. $l = 357$ sq. in., $p = 21$ in.
15. $l = 169$ sq. in., $p = 1$ ft. 1 in.

16. The lateral surface of an iron bar 5 ft. long is to be gilded. The right section is a square whose area is 2.89 sq. in. How many square inches of gilding are required?

17. A right prism of glass is $2\frac{1}{8}$ in. long. Its right section is an equilateral triangle whose altitude is 0.866 in. ($\frac{1}{2}\sqrt{3}$ in.). Find the lateral surface.

18. Find the total area of a right prism whose base is a square with area 5.29 sq. in., and whose length is twice its thickness.

19. What is the total area of a right prism whose altitude is 32 in., and whose base is a right triangle with hypotenuse 106 in. and with one side 84.8 in.?

20. Every section of a prism made by a plane parallel to the lateral edges is a parallelogram.

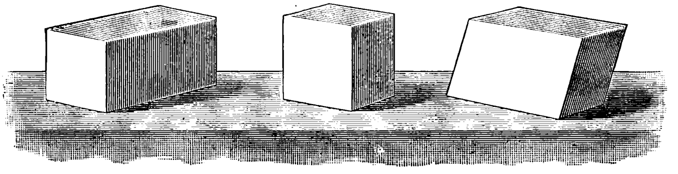
514. Parallelepiped. A prism whose bases are parallelograms is called a *parallelepiped.*

The word is also, with less authority, spelled *parallelopiped.*

515. Right Parallelepiped. A parallelepiped whose edges are perpendicular to the bases is called a *right parallelepiped.*

516. Rectangular Parallelepiped. A right parallelepiped whose bases are rectangles is called a *rectangular parallelepiped.*

By §§ 430 and 453 the four lateral faces are also rectangles.



Rectangular Parallelepiped Cube Oblique Parallelepiped

517. Cube. A parallelepiped whose six faces are all squares is called a *cube.*

We might also say that a hexahedron whose six faces are all squares is a cube, because such a figure would necessarily be a parallelepiped.

518. Unit of Volume. In measuring volumes, a cube whose edges are all equal to the unit of length is taken as the *unit of volume.*

Thus, if we are measuring the contents of a box of which the dimensions are given in feet, we take 1 cubic foot as the unit of volume. If the dimensions are given in inches, we take 1 cubic inch as the unit.

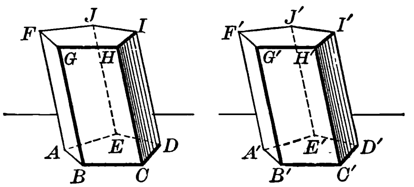
519. Volume. The number of units of volume contained by a solid is called its *volume.*

520. Equivalent Solids. If two solids have equal volumes, they are said to be *equivalent.*

521. Congruent Solids. If two geometric solids are equal in all their parts, and their parts are similarly arranged, the solids are said to be *congruent.*

Proposition III. Theorem

522. *Two prisms are congruent if the three faces which include a trihedral angle of the one are respectively congruent to three faces which include a trihedral angle of the other, and are similarly placed.*



Given the prisms AI and $A'I'$, with the faces AD, AG, AJ respectively congruent to $A'D'$, $A'G'$, $A'J'$, and similarly placed.

To prove that AI *is congruent to* $A'I'$.

Proof. The face $\angle s\ BAE, BAF, EAF$ are equal to the face $\angle s\ B'A'E', B'A'F', E'A'F'$ respectively. § 142

Therefore the trihedral angles A and A' are equal. § 498

Apply the trihedral angle A to its equal A'.

Then the face AD coincides with $A'D'$, AG with $A'G'$, and AJ with $A'J'$; and C falls at C', and D at D'.

The lateral edges of the prisms are parallel. § 446

Therefore CH falls along $C'H'$, and DI along $D'I'$. § 94

Since the points F, G, and J coincide with F', G', and J', each to each, the planes of the upper bases coincide. § 427

Hence H coincides with H', and I with I'.

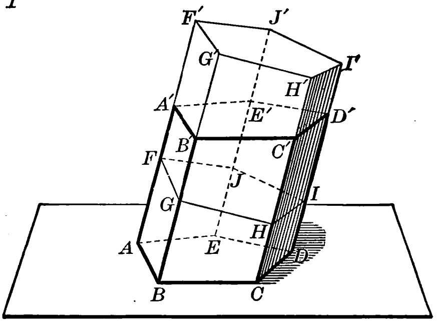
Hence the prisms coincide and are congruent, by § 521. Q.E.D.

523. Corollary 1. *Two truncated prisms are congruent under the conditions given in Proposition III.*

524. Corollary 2. *Two right prisms having congruent bases and equal altitudes are congruent.*

PROPOSITION IV. THEOREM.

525. *An oblique prism is equivalent to a right prism whose base is equal to a right section of the oblique prism, and whose altitude is equal to a lateral edge of the oblique prism.*



Given a right section FI of the oblique prism AD', and FI' a right prism whose lateral edges are equal to the lateral edges of AD'.

To prove that AD' *is equivalent to* FI'.

Proof. If from the equal lateral edges of AD' and FI' we take the lateral edges of FD', which are common to both, the remainders AF and $A'F'$, BG and $B'G'$, etc., are equal. Ax. 2

The bases FI and $F'I'$ are congruent. § 510

Place AI on $A'I'$ so that FI shall coincide with $F'I'$.

Then FA, GB, etc., coincide with $F'A'$, $G'B'$, etc. § 436

Hence the faces GA and $G'A'$, HB and $H'B'$, coincide.

But the faces FI and $F'I'$ coincide.

$\therefore$ the truncated prisms AI and $A'I'$ are congruent. § 523

$\therefore AI + FD' = A'I' + FD'$. Ax. 1

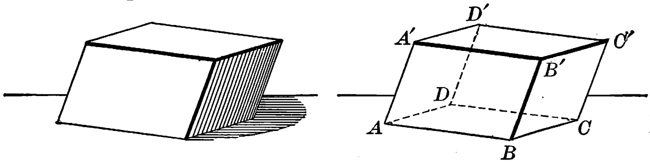
But $AI + FD' = AD'$,

and $A'I' + FD' = FI'$. Ax. 11

Therefore AD' is equivalent to FI', by Ax. 9. Q. E. D.

Proposition V. Theorem

526. *The opposite faces of a parallelepiped are congruent and parallel.*



Given a parallelepiped *ABCD-A′B′C′D′*.

To prove that the opposite faces AB' and DC' are congruent and parallel.

Proof. AB is ∥ to DC, § 118

and $AB = DC$. § 125

Likewise AA' is ∥ and equal to DD'.

$\therefore \angle BAA' = \angle CDD'$. § 461

$\therefore AB'$ is ∥ to DC'. § 461

$\therefore AB'$ is congruent to DC', by § 132. Q.E.D.

EXERCISE 85

1. If in the above figure the three plane angles at A are 80°, 70°, 75°, what are all the other angles in the faces?

2. Given a parallelepiped with the three plane angles at one of the vertices 85°, 75°, 60°, to find all the other angles in the faces.

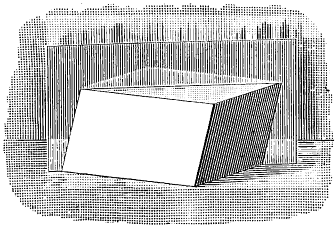
3. Given a rectangular parallelepiped lettered as in the figure above, and with $AB = 4$, $BC = 3$, and $CC' = 3\frac{3}{4}$, to find the length of the diagonal AC'.

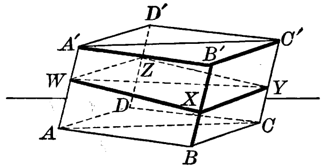
4. The four diagonals of a rectangular parallelepiped are equal.

5. Compute the lengths of the diagonals of a rectangular parallelepiped whose edges from any vertex are a, b, c.

Proposition VI. Theorem

527. *The plane passed through two diagonally opposite edges of a parallelepiped divides the parallelepiped into two equivalent triangular prisms.*





Given the plane $ACC'A'$ passed through the opposite edges AA' and CC' of the parallelepiped AC'.

To prove that the parallelepiped AC' is divided into two equivalent triangular prisms ABC-B' and ACD-D'.

Proof. Let $WXYZ$ be a right section of the parallelepiped.

The opposite faces AB' and DC' are parallel and equal. § 526

Similarly, the faces AD' and BC' are parallel and equal.

$\therefore$ WX is $\parallel$ to ZY, and WZ to XY. § 453

Therefore $WXYZ$ is a parallelogram. § 118

The plane $ACC'A'$ cuts this parallelogram $WXYZ$ in the diagonal WY. § 429

$\therefore$ $\triangle WXY$ is congruent to $\triangle YZW$. § 126

How shall it be proved that prism ABC-B' is equivalent to a right prism with base WXY and altitude AA'?

How shall it be proved that prism CDA-D' is equivalent to a right prism with base YZW and altitude AA'?

How are these two right prisms known to be equivalent?

How does this prove the proposition?

Discussion. What is the corresponding proposition of plane geometry?

EXERCISE 86

1. The lateral faces of a right prism are rectangles.

2. The diagonals of a parallelepiped bisect one another.

3. The three edges of the trihedral angle at one of the vertices of a rectangular parallelepiped are 5 in., 6 in., and 7 in. respectively. Required the total area of the six faces of the parallelepiped.

4. The three face angles at one vertex of a parallelepiped are each 60°, and the three edges of the trihedral angle with that vertex are 3 in., 2 in., 1 in. respectively. Required the total area of the six faces. Answer to two decimal places.

5. In a rectangular parallelepiped the square on any diagonal is equivalent to the sum of the squares on any three edges that meet at one of the vertices.

6. In a box 3 in. deep and 6 in. wide a wire 1 ft. long can be stretched to reach from one corner to the diagonally opposite corner. Required the length of the box. Answer to two decimal places.

7. The diagonal of the base of a rectangular parallelepiped is $31\frac{3}{5}$ in. and the height of the parallelepiped is 23.7 in. Required the length of the diagonal of the parallelepiped.

8. The total area of the six faces of a cube is 18 sq. in. Find the diagonal of the cube.

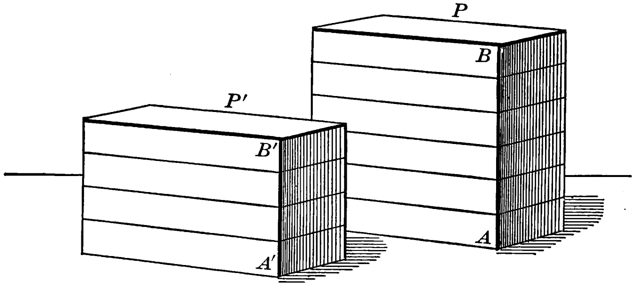
9. The diagonal of the face of a cube equals $\sqrt{14}$. Find the diagonal of the cube.

10. The diagonal of a cube equals $2.75\sqrt{3}$. Find the diagonal of a face of the cube.

11. A water tank is 3 ft. long, 2 ft. 6 in. wide, and 1 ft. 9 in. deep. How many square feet of zinc will be required to line the four sides and the base, allowing $1\frac{7}{8}$ sq. ft. for overlapping and for turning the top edge?

Proposition VII. Theorem

528. *Two rectangular parallelepipeds having congruent bases are to each other as their altitudes.*



Given two rectangular parallelepipeds P and P', with congruent bases and with altitudes AB and $A'B'$.

To prove that $P : P' = AB : A'B'$.

Case 1. *When AB and $A'B'$ are commensurable.*

Proof. Suppose a common measure of AB and $A'B'$ to be contained m times in AB, and n times in $A'B'$.

Then $AB : A'B' = m : n$.

Apply this measure to AB and $A'B'$, and through the several points of division pass planes perpendicular to these lines.

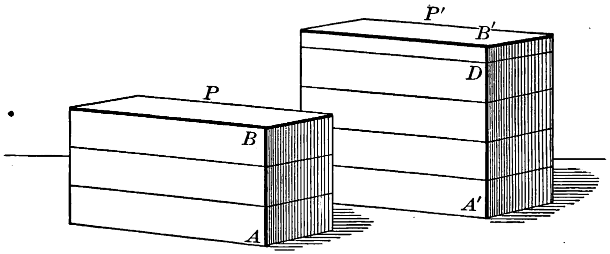
These planes divide the parallelepiped P into m parallelepipeds and the parallelepiped P' into n parallelepipeds, congruent each to each. § 524

$\therefore P : P' = m : n$.

$\therefore P : P' = AB : A'B'$, by Ax. 8. Q.E.D.

The proof for the incommensurable case is similar to that in other propositions of this nature. It may be omitted at the discretion of the teacher without destroying the sequence, if the incommensurable cases are not being considered by the class.

CASE 2. *When AB and A'B' are incommensurable.*



Proof. Divide AB into any number of equal parts, and apply one of these parts to $A'B'$ as a unit of measure as many times as $A'B'$ will contain it.

Since AB and $A'B'$ are incommensurable, a certain number of these parts will extend from A' to a point D, leaving a remainder DB' less than one of the parts.

Through D pass a plane $\perp$ to $A'B'$, and let Q denote the parallelepiped whose base is the same as that of P', and whose altitude is $A'D$.

Then $Q : P = A'D : AB$. Case 1

If the number of parts into which AB is divided is indefinitely increased, the ratio $Q : P$ approaches $P' : P$ as a limit, and the ratio $A'D : AB$ approaches $A'B' : AB$ as a limit. § 204

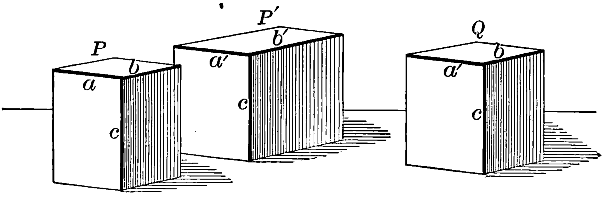
The remainder of the proof of the incommensurable case is substantially as in the proof given on page 297, and it is therefore left for the student.

529. Dimensions. The lengths of the three edges of a rectangular parallelepiped which meet at a common vertex are called its *dimensions*.

530. COROLLARY. *Two rectangular parallelepipeds which have two dimensions in common are to each other as their third dimensions.*

PROPOSITION VIII. THEOREM

531. *Two rectangular parallelepipeds having equal altitudes are to each other as their bases.*



Given two rectangular parallelepipeds, P and P', and a, b, c, and a', b', c, their three dimensions respectively.

To prove that $$\frac{P}{P'} = \frac{ab}{a'b'}.$$

Proof. Let Q be a third rectangular parallelepiped whose dimensions are a', b, and c.

Now Q has the two dimensions b and c in common with P, and the two dimensions a' and c in common with P'.

Therefore $$\frac{P}{Q} = \frac{a}{a'},$$

and $$\frac{Q}{P'} = \frac{b}{b'}. \qquad \S\ 530$$

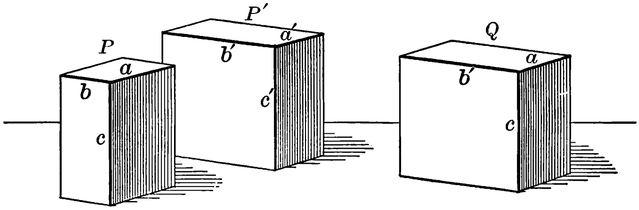
The products of the corresponding members of these two equations give $$\frac{P}{P'} = \frac{ab}{a'b'}, \text{ by Ax. 3.} \qquad \text{Q.E.D.}$$

532. COROLLARY. *Two rectangular parallelepipeds which have one dimension in common are to each other as the products of their other two dimensions.*

For any edge of a rectangular parallelepiped may be taken as the altitude, whence § 531 applies.

Proposition IX. Theorem

533. *Two rectangular parallelepipeds are to each other as the products of their three dimensions.*



Given two rectangular parallelepipeds, P and P', and a, b, c, and a', b', c', their three dimensions respectively.

To prove that $$\frac{P}{P'} = \frac{abc}{a'b'c'}.$$

Proof. Let Q be a third rectangular parallelepiped whose dimensions are a, b', and c.

Then $$\frac{P}{Q} = \frac{b}{b'}, \qquad \text{§ 530}$$

and $$\frac{Q}{P'} = \frac{ac}{a'c'}. \qquad \text{§ 532}$$

$$\therefore \frac{P}{P'} = \frac{abc}{a'b'c'}, \text{ by Ax. 3.} \qquad \text{Q.E.D.}$$

534. Corollary 1. *The volume of a rectangular parallelepiped is equal to the product of its three dimensions.*

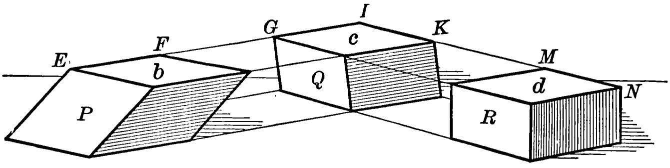
For in the above case, if $a' = b' = c' = 1$, then $P' = 1 \times 1 \times 1 = 1$ (§ 518). But the volume of P (§ 519) is $P : P'$, and $P : P' = abc : 1$ (§ 533). Therefore the volume of P is abc.

535. Corollary 2. *The volume of a rectangular parallelepiped is equal to the product of its base and altitude.*

For the volume of P is abc, and ab equals the base and c the altitude.

Proposition X. Theorem

536. *The volume of any parallelepiped is equal to the product of its base by its altitude.*



Given an oblique parallelepiped P of volume v, with no two of its faces perpendicular, with base b and with altitude a.

To prove that $v = ba$.

Proof. Produce the edge EF and the edges ∥ to EF, and cut them perpendicularly by two parallel planes whose distance apart GI is equal to EF. We then have the oblique parallelepiped Q whose base c is a rectangle.

Produce the edge IK and the edges ∥ to IK, and cut them perpendicularly by two planes whose distance apart MN is equal to IK. We then have the rectangular parallelepiped R.

Now $P = Q$, and $Q = R$. § 525

$\therefore P = R$. Ax. 8

The three parallelepipeds have a common altitude a. § 455

Also $b = c$, § 323

and $c = d$. § 133

$\therefore b = d$. Ax. 8

But the volume of $R = da$. § 535

Putting P for R, and b for d, we have $v = ba$, by Ax. 9. Q.E.D.

537. Corollary. *The volume of any parallelepiped is equal to that of a rectangular parallelepiped of equivalent base and equal altitude.*

EXERCISE 87

1. Find the ratio of two rectangular parallelepipeds, if their dimensions are 3, 4, 5, and 9, 8, 10 respectively.

2. Find the ratio of two rectangular parallelepipeds, if their altitudes are each 6 in., and their bases 5 in. by 4 in., and 10 in. by 8 in. respectively.

3. Find the volume of a rectangular parallelepiped 2 ft. 6 in. long, 1 ft. 8 in. wide, and 1 ft. 6 in. high.

4. Find the volume of a rectangular parallelepiped whose base is 27 sq. in. and whose altitude is $13\frac{1}{2}$ in.

5. The volume of a rectangular parallelepiped is 1152 cu. in. and the area of the base is half a square foot. Find the altitude.

6. The volume of a rectangular parallelepiped with a square base is 273.8 cu. in. and the altitude is 5 in. Find the dimensions.

7. A rectangular tank full of water is 7 ft. 3 in. long by 4 ft. 6 in. wide. How many cubic feet of water must be drawn off in order that the surface may be lowered a foot?

8. Find to two decimal places the length of each side of a cubic reservoir that will contain exactly a gallon (231 cu. in.).

9. A box has as its internal dimensions 18 in., $9\frac{1}{2}$ in., and $4\frac{1}{4}$ in. The box and cover are made of steel $\frac{1}{8}$ in. thick. If steel weighs 490 lb. per cubic foot, what is the weight of the box?

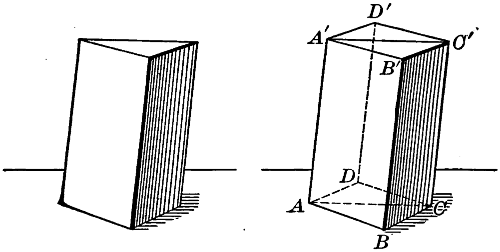
10. A steel rod 4 ft. 8 in. long is 2 in. wide and $1\frac{1}{2}$ in. thick. How much does it weigh, at 490 lb. per cubic foot?

11. If 3 cu. in. of gold beaten into gold leaf will cover 75,000 sq. in. of surface, find the thickness of the leaf.

12. The sum of the squares on the four diagonals of a parallelepiped is equivalent to the sum of the squares on the twelve edges.

Proposition XI. Theorem

538. *The volume of a triangular prism is equal to the product of its base by its altitude.*



Given the triangular prism *ABC-B′*, with volume *v*, base *b*, and altitude *a*.

To prove that $v = ba.$

Proof. Upon the edges AB, BC, BB' construct the parallelepiped $ABCD\text{-}B'$.

Then $ABC\text{-}B' = \frac{1}{2}\,ABCD\text{-}B'.$ § 527

The volume of $ABCD\text{-}B' = ABCD \times a.$ § 536

But $ABCD = 2\,b.$ § 126

$\therefore\ v = \frac{1}{2}(2\,ba) = ba$, by Ax. 9. Q.E.D.

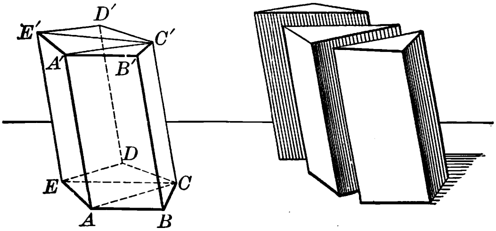
EXERCISE 88

Find the volumes of the triangular prisms whose bases and altitudes are as follows:

1. 17 sq. in., 8 in.
2. 15.75 sq. ft., 3 ft.
3. $3\frac{1}{4}$ sq. ft., 1 ft. 8 in.
4. $5\frac{1}{2}$ sq. ft., 2 ft. 9 in.
5. 15.84 sq. ft., 3 ft. 10 in.
6. $16\frac{2}{3}$ sq. in., $2\frac{2}{5}$ in.
7. $22\frac{1}{2}$ sq. in., $4\frac{1}{2}$ in.
8. $33\frac{1}{3}$ sq. in., $7\frac{1}{5}$ in.
9. $42\frac{7}{8}$ sq. in., $3\frac{3}{8}$ in.
10. $27\frac{3}{5}$ sq. in., $3\frac{3}{4}$ in.
11. 12 sq. ft. 75 sq. in., 2 ft. 7 in.

PROPOSITION XII. THEOREM

539. *The volume of any prism is equal to the product of its base by its altitude.*



Given the prism AC' with volume v, base b, and altitude a.

To prove that $v = ba$.

Proof. It is possible to divide any prism in general into what kind of simpler prisms?

How is this done?

What is the volume of each of these simpler prisms (§ 538)?

What is the sum of the volumes of these simpler prisms?

What is the sum of their bases?

How does the common altitude of these simpler prisms compare with a, the altitude of the given prism?

What conclusion can be drawn from these statements?

Write the proof in full.

540. COROLLARY 1. *Prisms having equivalent bases are to each other as their altitudes; prisms having equal altitudes are to each other as their bases.*

Write the proof in full.

541. COROLLARY 2. *Prisms having equivalent bases and equal altitudes are equivalent.*

Write the proof in full.

EXERCISE 89

1. If the length of a rectangular parallelepiped is 18 in., the width 9 in., and the height 8 in., find the total area of the surface.

2. Find the volume of a triangular prism, if its height is 15 in. and the sides of the base are 6 in., 5 in., and 5 in.

3. Find the volume of a prism whose height is 15 ft., if each side of the triangular base is 10 in.

4. The base of a right prism is a rhombus of which one side is 20 in., and the shorter diagonal 24 in. The height of the prism is 30 in. Find the entire surface and the volume.

5. How many square feet of lead will be required to line an open cistern which is 4 ft. 6 in. long, 2 ft. 8 in. wide, and contains 42 cu. ft.?

6. An open cistern 6 ft. long and $4\frac{1}{2}$ ft. wide holds 108 cu. ft. of water. How many square feet of lead will it take to line the sides and bottom?

7. One edge of a cube is e. Find in terms of e the surface, the volume, and the length of a diagonal of the cube.

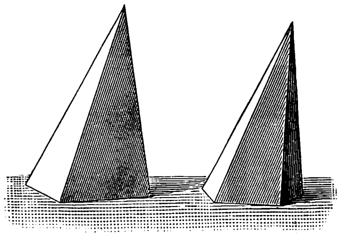
8. The diagonal of one of the faces of a cube is d. Find in terms of d the volume of the cube.

9. The three dimensions of a rectangular parallelepiped are a, b, c. Find in terms of a, b, and c the volume and the area of the surface.

10. Find the volume of a prism with bases regular hexagons, if the height is 10 ft. and each side of the hexagons is 10 in.

11. An open cistern is made of iron $\frac{1}{4}$ in. thick. The inner dimensions are: length, 4 ft. 6 in.; breadth, 3 ft.; depth, 2 ft. 6 in. What will the cistern weigh when empty? when full of water? (A cubic foot of water weighs $62\frac{1}{2}$ lb. Iron is 7.2 times as heavy as water; that is, the specific gravity of iron is 7.2.)

542. Pyramid. A polyhedron of which one face, called the *base,* is a polygon of any number of sides and the other faces are triangles having a common vertex is called a *pyramid.*



The triangular faces having a common vertex are called the *lateral faces*, their intersections are called the *lateral edges*, and their common vertex is called the *vertex* of the pyramid. The base of a pyramid may be any kind of a polygon, but usually a convex polygon is taken.

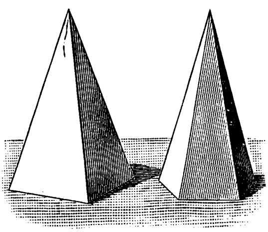
543. Lateral Area. The sum of the areas of the lateral faces of a pyramid is called the *lateral area* of the pyramid.

544. Altitude. The perpendicular distance from the vertex to the plane of the base is called the *altitude* of the pyramid.

545. Pyramids classified as to Bases. Pyramids are said to be *triangular, quadrangular,* and so on, according as their bases are triangles, quadrilaterals, and so on.

A triangular pyramid has four triangular faces and is called a *tetrahedron.* Any one of its faces may be taken as the base.

546. Regular Pyramid. If the base of a pyramid is a regular polygon whose center coincides with the foot of the perpendicular let fall from the vertex to the base, the pyramid is called a *regular pyramid.*

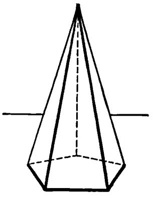


A regular pyramid is also called a *right pyramid.*

547. Slant Height of a Regular Pyramid. The altitude of any one of the lateral faces of a regular pyramid, drawn from the vertex of the pyramid, is called the *slant height.*

The slant height is the same whatever face is taken (§ 439). Only a regular pyramid can have a slant height.

548. Properties of Regular Pyramids. Among the properties of regular pyramids the following are too evident to require further proof than that referred to below:

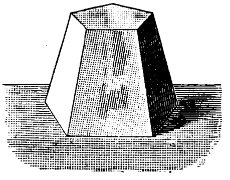


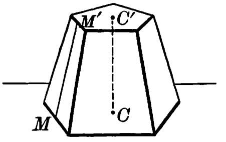
(1) *The lateral edges of a regular pyramid are equal* (§ 439).

(2) *The lateral faces of a regular pyramid are congruent isosceles triangles* (§ 80).

(3) *The slant height of a regular pyramid is the same for all the lateral faces* (§ 439).

549. Frustum of a Pyramid. The portion of a pyramid included between the base and a section parallel to the base is called a *frustum of a pyramid.*





The base of the pyramid and the parallel section are called the *bases* of the frustum.

A more general term, including frustum as a special case, is *truncated pyramid*, the portion of a pyramid included between the base and *any* section made by a plane that cuts all the lateral edges. This term is little used.

550. Altitude of a Frustum. The perpendicular distance between the bases is called the *altitude* of the frustum.

E.g. $C'C$ is the altitude of the frustum in the above figure.

551. Lateral Faces of a Frustum. The portions of the lateral faces of a pyramid that lie between the bases of a frustum are called the *lateral faces* of the frustum.

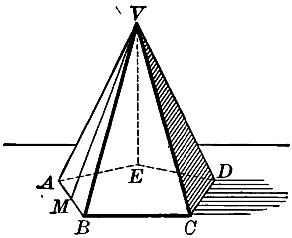
In the case of a frustum of a regular pyramid the lateral faces are congruent isosceles trapezoids. The sum of the areas of the lateral faces is called the *lateral area* of the frustum.

552. Slant Height of a Frustum. The altitude of one of the trapezoid faces of a frustum of a regular pyramid is called the *slant height* of the frustum.

Thus MM' in the above figure is the slant height.

PROPOSITION XIII. THEOREM

553. *The lateral area of a regular pyramid is equal to half the product of its slant height by the perimeter of its base.*



Given the regular pyramid *V-ABCDE*, with *l* the lateral area, *s* the slant height, and *p* the perimeter of the base.

To prove that $l = \frac{1}{2} sp$.

Proof. The △s *VAB*, *VBC*, *VCD*, *VDE*, and *VEA* are congruent. § 548

The area of each $\triangle = \frac{1}{2} s \times$ its base. § 325

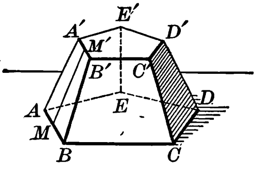
The sum of the bases of the triangles $= p$. Ax. 11

∴ the sum of the areas of these △s $= \frac{1}{2} sp$. Ax. 1

But the sum of the areas of these △s $= l$. § 543

∴ $l = \frac{1}{2} sp$, by Ax. 8. Q. E. D.

554. COROLLARY. *The lateral area of the frustum of a regular pyramid is equal to half the sum of the perimeters of the bases multiplied by the slant height of the frustum.*

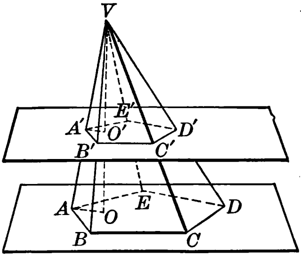


How is the area of a trapezoid found (§ 329)? Are these trapezoids congruent? What is the sum of their lower bases? of their upper bases? What is the sum of their areas? Insert the formula.

Proposition XIV. Theorem

555. *If a pyramid is cut by a plane parallel to the base:*

1. *The edges and altitude are divided proportionally.*
2. *The section is a polygon similar to the base.*



Given the pyramid *V-ABCDE* cut by a plane parallel to its base, intersecting the lateral edges in *A'*, *B'*, *C'*, *D'*, *E'*, and the altitude *VO* in *O'*.

1. *To prove that* $\frac{VA'}{VA} = \frac{VB'}{VB} = \cdots = \frac{VO'}{VO}$.

Proof. Since the plane $A'D'$ is $\parallel$ to the plane AD, Given

$\therefore A'B'$ is $\parallel$ to AB, $B'C'$ is $\parallel$ to BC, $\cdots$, and $A'O'$ is $\parallel$ to AO. § 453

$\therefore \frac{VA'}{VA} = \frac{VB'}{VB} = \cdots = \frac{VO'}{VO}$, by § 274. Q.E.D.

2. *To prove the section $A'B'C'D'E'$ similar to the base $ABCDE$.*

Proof. Since $\triangle VA'B'$ is similar to $\triangle VAB$, $\triangle VB'C'$ similar to $\triangle VBC$, and so on (why?), how can the corresponding sides of the polygons be proved proportional?

Since $A'B'$ is $\parallel$ to AB, $B'C'$ to BC, etc. (why?),

how can the corresponding angles be proved equal?

Then why is $A'B'C'D'E'$ similar to $ABCDE$?

556. Corollary 1. *Any section of a pyramid parallel to the base is to the base as the square of the distance from the vertex is to the square of the altitude of the pyramid.*

For $\frac{VO'}{VO} = \frac{VA'}{VA}$ § 555

$= \frac{A'B'}{AB}.$ § 288

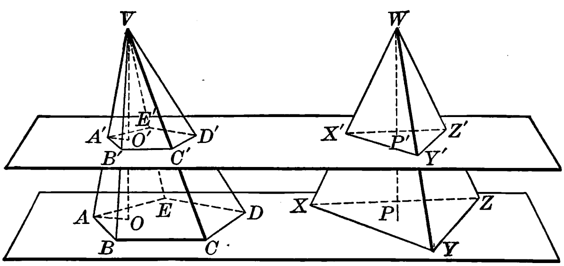
Therefore $\frac{\overline{VO'}^2}{\overline{VO}^2} = \frac{\overline{A'B'}^2}{\overline{AB}^2}.$ § 270

But, from similar polygons,

$\frac{A'B'C'D'E'}{ABCDE} = \frac{\overline{A'B'}^2}{\overline{AB}^2}.$ § 334

Hence, by substituting,

$\frac{A'B'C'D'E'}{ABCDE} = \frac{\overline{VO'}^2}{\overline{VO}^2}.$ Ax. 8

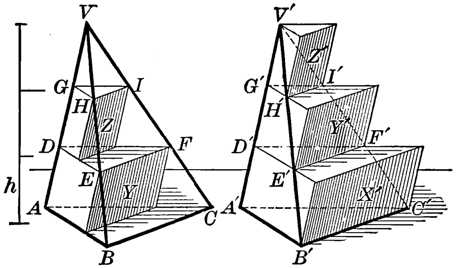


557. Corollary 2. *If two pyramids have equal altitudes and equivalent bases, sections made by planes parallel to the bases, and at equal distances from the vertices, are equivalent.*

What is the ratio of $A'B'C'D'E'$ to $ABCDE$?
How can this be shown to equal $\overline{VO'}^2 : \overline{VO}^2$?
What is the ratio of $X'Y'Z'$ to XYZ?
How can this be shown to equal $\overline{WP'}^2 : \overline{WP}^2$?
Are the ratios $\overline{VO'}^2 : \overline{VO}^2$ and $\overline{WP'}^2 : \overline{WP}^2$ equal?
Since it is given that $ABCDE = XYZ$, what can be said of $A'B'C'D'E'$ and $X'Y'Z'$?

Proposition XV. Theorem

558. *Two triangular pyramids having equivalent bases and equal altitudes are equivalent.*



Given two triangular pyramids, *V-ABC* and *V′-A′B′C′*, having equivalent bases and equal altitudes.

To prove that V-ABC and V′-A′B′C′ are equivalent.

Proof. Suppose the pyramids are not equivalent, and

$$V'\text{-}A'B'C' > V\text{-}ABC.$$

Place the bases in the same plane, and suppose the altitude divided into n equal parts, calling each of these parts h.

Through the points of division pass planes parallel to the base, cutting the pyramids in DEF, GHI, $\cdots$, $D'E'F'$, $G'H'I'$, $\cdots$.

On $A'B'C'$, $D'E'F'$, $G'H'I'$, and other parallel sections, if any, construct prisms with lateral edges parallel to $A'V'$, and with altitude h. In the figure these are represented by X', Y', and Z'.

On DEF, GHI, and other parallel sections, if any, as upper bases, construct the prisms Y, Z, with lateral edges parallel to VA, and with altitude h.

Then since	$DEF = D'E'F'$,	§ 557
and	$h = h$,	Iden.
	$\therefore$ prism $Y =$ prism Y'.	§ 541
Similarly	prism $Z =$ prism Z'.	

But $$X' + Y' + Z' > V'\text{-}A'B'C',$$

and $$Y + Z < V\text{-}ABC. \qquad \text{Ax. 11}$$

$$\therefore V'\text{-}A'B'C' - V\text{-}ABC < X' + Y' + Z' - (Y + Z),$$

or $$V'\text{-}A'B'C' - V\text{-}ABC < X'.$$

That is, the difference between the pyramids must be less than the difference between the sets of prisms.

Now by increasing n indefinitely, and consequently decreasing h indefinitely, X' can be made less than any assigned quantity.

Hence whatever difference we suppose to exist between the pyramids, X' can be made smaller than that supposed difference.

But this is absurd, since we have shown that X' is greater than the difference, if any exists.

Hence it leads to a manifest absurdity to suppose that $V'\text{-}A'B'C' > V\text{-}ABC$.

In the same way it leads to an absurdity to suppose that $V\text{-}ABC > V'\text{-}A'B'C'$.

$$\therefore V\text{-}ABC = V'\text{-}A'B'C'. \qquad \text{Q.E.D.}$$

EXERCISE 90

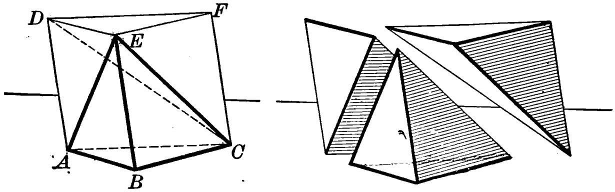
1. The slant height of a regular pyramid is 6 in., and the base is an equilateral triangle of altitude $2\sqrt{3}$ in. Find the lateral area of the pyramid.

2. The slant height of a regular triangular pyramid equals the altitude of the base. The area of the base is $\sqrt{3}$ sq. ft. Find the total area of the pyramid.

3. A pyramid has for its base a right triangle with hypotenuse 5 and shortest side 3. Another one of equal altitude has for its base an equilateral triangle with side $2\sqrt{2\sqrt{3}}$. Prove the pyramids equivalent.

Proposition XVI. Theorem

559. *The volume of a triangular pyramid is equal to one third the product of its base by its altitude.*



Given the triangular pyramid *E-ABC*, with volume *v*, base *b*, and altitude *a*.

To prove that $v = \frac{1}{3} ba.$

Proof. On the base ABC construct a prism ABC-DEF.

Through DE and EC pass a plane CDE.

Then the prism is composed of three triangular pyramids E-ABC, E-CFD, and E-ACD.

Now the pyramids E-CFD and E-ACD have the same altitude

and equal bases CFD and ACD. § 126

$\therefore E\text{-}CFD = E\text{-}ACD.$ § 558

But pyramid E-CFD is the same as pyramid C-DEF,

which has the same altitude as pyramid E-ABC,

and has base DEF equal to base ABC. § 511

$\therefore E\text{-}CFD = E\text{-}ABC.$ § 558

$\therefore E\text{-}ABC = E\text{-}CFD = E\text{-}ACD.$ Ax. 8

$\therefore$ pyramid $E\text{-}ABC = \frac{1}{3}$ prism ABC-DEF.

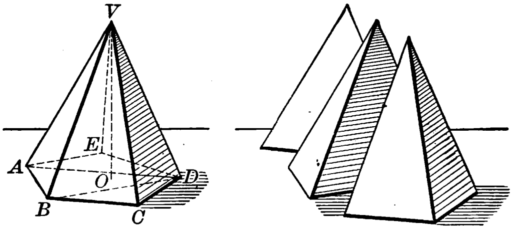
But the volume of $ABC\text{-}DEF = ba.$ § 539

$\therefore v = \frac{1}{3} ba$, by Ax. 4. Q.E.D.

560. Corollary. *The volume of a triangular pyramid is equal to one third the volume of a triangular prism of the same base and altitude.*

Proposition XVII. Theorem

561. *The volume of any pyramid is equal to one third the product of its base by its altitude.*



Given the pyramid *V-ABCDE*, with volume *v*, base *b*, and altitude *a*.

To prove that $v = \frac{1}{3} ba.$

Proof. Through the edge *VD* and the diagonals of the base, *DA*, *DB*, pass planes.

These planes divide the pyramid *V-ABCDE* into three triangular pyramids.

What can be said as to the altitudes of the original pyramid and of the triangular pyramids?

What can be said as to the base of the original pyramid in relation to the bases of the triangular pyramids?

What is the volume of each triangular pyramid?

What is the sum of the volumes of the triangular pyramids?

Complete the proof.

562. Corollary. *The volumes of two pyramids are to each other as the products of their bases and altitudes; pyramids having equivalent bases are to each other as their altitudes; pyramids having equal altitudes are to each other as their bases; pyramids having equivalent bases and equal altitudes are equivalent.*

EXERCISE 91

Find the lateral areas of regular pyramids, given the slant heights and the perimeters of the bases, as follows:

1. $s = 34$ in., $p = 57$ in.
2. $s = 8\frac{3}{4}$ in., $p = 17\frac{1}{8}$ in.
3. $s = 2$ ft. 7 in., $p = 4$ ft. 6 in.
4. $s = 127$ ft. 5 in., $p = 63$ ft. 2 in.

Find the lateral areas of frustums of regular pyramids, given the slant heights of the frustums and the perimeters of the bases, as follows:

5. $s = 4$ in., $p = 8$ in., $p' = 6$ in.
6. $s = 5\frac{1}{8}$ in., $p = 9\frac{3}{4}$ in., $p' = 7\frac{3}{8}$ in.
7. $s = 2$ ft. 3 in., $p = 4$ ft. 8 in., $p' = 3$ ft. 9 in.

Find the volumes of pyramids, given the altitudes and the areas of the bases, as follows:

8. $a = 7$ in., $b = 9$ sq. in.
9. $a = 6$ in., $b = 23$ sq. in.
10. $a = 17$ in., $b = 51$ sq. in.
11. $a = 3\frac{1}{8}$ in., $b = 5\frac{1}{8}$ sq. in.
12. $a = 4\frac{3}{8}$ in., $b = 19$ sq. in.
13. $a = 27.5$ ft., $b = 325$ sq. ft.

Find the lateral areas of regular pyramids, given the slant heights, the number of sides of the bases, and the length of each side, as follows:

14. $s = 2.3$ in., $n = 4$, $l = 2.1$ in.
15. $s = 3.7$ in., $n = 6$, $l = 2.9$ in.
16. $s = 5.33$ in., $n = 8$, $l = 3$ in.

Find the volumes of pyramids, given the altitudes and a description of the bases, as follows:

17. $a = 7$ in., the base a square with side 2 in.
18. $a = 6\frac{3}{4}$ in., the base a square with diagonal $3\sqrt{2}$ in.
19. $a = 8.9$ in., the base a triangle with each side 3.7 in.

20. Find the lateral area of a regular pyramid, if the slant height is 16 ft. and the base is a hexagon with side 12 ft.

21. Find the lateral area of a regular pyramid, if the slant height is 8 ft. and the base is a pentagon with side 5 ft.

22. Find the total surface of a regular pyramid, if the slant height is 6 ft. and the base is a square with side 4 ft.

23. Find the total surface of a regular pyramid, if the slant height is 18 ft. and the base is a square with side 8 ft.

24. Find the total surface of a regular pyramid, if the slant height is 16 ft. and the base is a triangle with side 8 ft.

25. The volume of a pyramid is 26 cu. ft. 936 cu. in. and each side of its square base is 3 ft. 6 in. Find the height.

26. The volume of a pyramid is 20 cu. ft. and the sides of its triangular base are 5 ft., 4 ft., and 3 ft. respectively. Find the height.

27. Find the volume of a regular pyramid with a square base whose side is 40 ft., the lateral edge being 101 ft.

28. Find the volume of a regular pyramid whose slant height is 12 ft. and whose base is an equilateral triangle inscribed in a circle of radius 10 ft.

29. Having given the base edge a and the total surface t of a regular pyramid with a square base, find the height h.

30. Having given the base edge a and the total surface t of a regular pyramid with a square base, find the volume v.

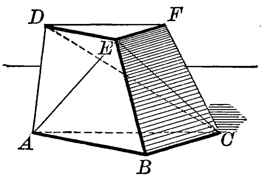
31. The eight edges of a regular pyramid with a square base are equal and the total surface is t. Find the edge.

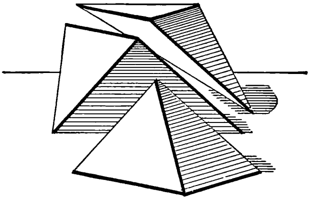
32. Find the base edge a of a regular pyramid with a square base, having given the height h and the total surface t.

33. Show how to find the volume of any polyhedron by dividing the polyhedron into pyramids.

Proposition XVIII. Theorem

563. *The frustum of a triangular pyramid is equivalent to the sum of three pyramids whose common altitude is the altitude of the frustum and whose bases are the lower base, the upper base, and the mean proportional between the two bases of the frustum.*





Given the frustum of a triangular pyramid, *ABC-DEF*, having *ABC*, or *b*, for its lower base; *DEF*, or *b'*, for its upper base; and the altitude *a*.

To prove that $ABC\text{-}DEF = \frac{1}{3}ab + \frac{1}{3}ab' + \frac{1}{3}a\sqrt{bb'}$.

Proof. Through A, E, and C, and also through C, D, and E, pass planes dividing the frustum into three pyramids.

Then $E\text{-}ABC = \frac{1}{3}ab$,

and $C\text{-}DEF = \frac{1}{3}ab'$. § 559

It therefore remains only to prove that $E\text{-}ACD = \frac{1}{3}a\sqrt{bb'}$.

We see by the figure that we may speak of $E\text{-}ABC$ as $C\text{-}ABE$, and of $E\text{-}ACD$ as $C\text{-}AED$.

But $C\text{-}ABE : C\text{-}AED = \triangle ABE : \triangle AED$. § 562

Since $\triangle$s ABE and AED have for a common altitude the altitude of the trapezoid $ABED$,

$\therefore \triangle ABE : \triangle AED = AB : DE$. § 327

$\therefore C\text{-}ABE : C\text{-}AED = AB : DE$, Ax. 8

or $E\text{-}ABC : E\text{-}ACD = AB : DE$. Ax. 9

In like manner $E\text{-}ACD$ and $E\text{-}CFD$ have a common vertex E and have their bases in the same plane, $ACFD$, so that

$$E\text{-}ACD : E\text{-}CFD = \triangle ACD : \triangle CFD. \qquad \S\ 562$$

Since $\triangle$s ACD and CFD have for a common altitude the altitude of the trapezoid $ACFD$,

$$\therefore \triangle ACD : \triangle CFD = AC : DF. \qquad \S\ 327$$

$$\therefore E\text{-}ACD : E\text{-}CFD = AC : DF. \qquad \text{Ax. } 8$$

But $\triangle DEF$ is similar to $\triangle ABC$. § 555

$$\therefore AB : DE = AC : DF. \qquad \S\ 282$$

$$\therefore E\text{-}ABC : E\text{-}ACD = AC : DF. \qquad \text{Ax. } 8$$

$$\therefore E\text{-}ABC : E\text{-}ACD = E\text{-}ACD : E\text{-}CFD. \qquad \text{Ax. } 8$$

But $E\text{-}CFD$ is the same as $C\text{-}DEF$, which has been shown to equal $\frac{1}{3}\,ab'$.

$$\therefore \tfrac{1}{3}\,ab : E\text{-}ACD = E\text{-}ACD : \tfrac{1}{3}\,ab'. \qquad \text{Ax. } 9$$

$$\therefore E\text{-}ACD = \sqrt{\tfrac{1}{3}\,ab \times \tfrac{1}{3}\,ab'} \qquad \S\ 262$$

$$= \tfrac{1}{3}\,a\sqrt{bb'}.$$

$$\therefore E\text{-}ABC + C\text{-}DEF + E\text{-}ACD = \tfrac{1}{3}\,ab + \tfrac{1}{3}\,ab' + \tfrac{1}{3}\,a\sqrt{bb'}. \qquad \text{Ax. } 1$$

That is, $ABC\text{-}DEF = \frac{1}{3}\,ab + \frac{1}{3}\,ab' + \frac{1}{3}\,a\sqrt{bb'}$, by Ax. 9. Q.E.D.

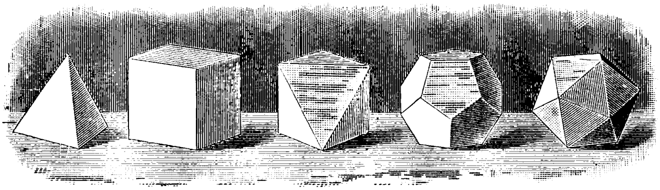
564. Corollary 1. *The volume of a frustum of a triangular pyramid may be expressed as* $\frac{1}{3}\,a\left(b + b' + \sqrt{bb'}\right)$.

For we may factor by $\frac{1}{3}\,a$.

565. Corollary 2. *The volume of a frustum of any pyramid is equal to the sum of the volumes of three pyramids whose common altitude is the altitude of the frustum, and whose bases are the lower base, the upper base, and the mean proportional between the bases of the frustum.*

Extend the faces of the frustum F, forming a pyramid P. From a triangular pyramid P' of equivalent base b and equal altitude, cut off a frustum F' of the same altitude a as F. Then $P = P'$ and $F = F'$. But F and F' have equivalent bases, and $F' = \frac{1}{3}\,a\left(b + b' + \sqrt{bb'}\right)$. Hence $F = \frac{1}{3}\,a\left(b + b' + \sqrt{bb'}\right)$.

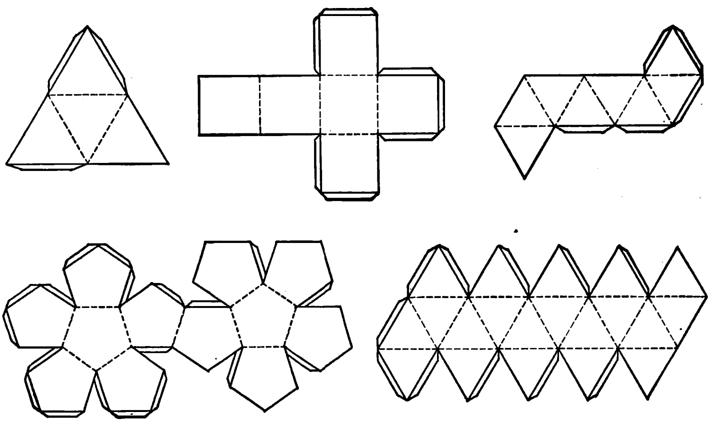
566. Polyhedrons classified as to Faces. A polyhedron of four faces is called a *tetrahedron;* one of six faces, a *hexahedron;* one of eight faces, an *octahedron;* one of twelve faces, a *dodecahedron;* one of twenty faces, an *icosahedron.*



Tetrahedron Hexahedron Octahedron Dodecahedron Icosahedron

567. Regular Polyhedron. A polyhedron whose faces are congruent regular polygons, and whose polyhedral angles are equal, is called a *regular polyhedron.*

It is proved on page 351 that it is possible to have only five regular polyhedrons. They may be constructed from paper as follows:



Draw on stiff paper the diagrams given above. Cut through the full lines and paste strips of paper on the edges as shown. Fold on the dotted lines, and keep the edges in contact by the pasted strips of paper.

PROPOSITION XIX. PROBLEM

568. *To determine the number of regular convex polyhedrons possible.*

A convex polyhedral angle must have at least three faces, and the sum of its face angles must be less than 360° (§ 495).

1. Since each angle of an equilateral triangle is 60°, convex polyhedral angles may be formed by combining three, four, or five equilateral triangles. The sum of six such angles is 360°, and therefore is greater than the sum of the face angles of a convex polyhedral angle. Hence three regular convex polyhedrons are possible with equilateral triangles for faces.

2. Since each angle of a square is 90°, a convex polyhedral angle may be formed by combining three squares. The sum of four such angles is 360°, and therefore is greater than the sum of the face angles of a convex polyhedral angle. Hence one regular convex polyhedron is possible with squares.

3. Since each angle of a regular pentagon is 108° (§ 145), a convex polyhedral angle may be formed by combining three regular pentagons. The sum of four such angles is 432°, and therefore is greater than the sum of the face angles of a convex polyhedral angle. Hence one regular convex polyhedron is possible with regular pentagons.

4. The sum of three angles of a regular hexagon is 360°, of a regular heptagon is greater than 360°, and so on.

Hence only five regular convex polyhedrons are possible.

The regular polyhedrons are the regular *tetrahedron*, the regular *hexahedron*, or cube, the regular *octahedron*, the regular *dodecahedron*, and the regular *icosahedron*. Q.E.F.

It adds greatly to a clear understanding of the five regular polyhedrons if they are constructed from paper as suggested in § 567.

Since these solids were extensively studied by the pupils of Plato, the great Greek philosopher, they are often called the Platonic Bodies.

EXERCISE 92

Find the volumes of frustums of pyramids, the altitudes and the bases of the frustums being given, as follows:

1. $a = 3$ in., $b = 8$ sq. in., $b' = 2$ sq. in.

2. $a = 4\frac{1}{2}$ in., $b = 8\frac{1}{3}$ sq. in., $b' = 3$ sq. in.

3. $a = 3.2$ in., $b = 2$ sq. in., $b' = 0.18$ sq. in.

4. $a = 2$ ft. 6 in., $b = 10$ sq. ft., $b' = 2$ sq. ft. 72 sq. in.

5. $a = 3$ ft. 7 in., $b = 24$ sq. ft. 72 sq. in., $b' = 2$ sq. ft.

6. A pyramid 2 in. high, with a base whose area is 8 sq. in., is cut by a plane parallel to the base 1 in. from the vertex. Find the volume of the frustum.

7. A pyramid 3 in. high, with a base whose area is 81 sq. in., is cut by a plane parallel to the base 2 in. from the base. Find the volume of the frustum.

8. The lower base of a frustum of a pyramid is a square 4 in. on a side. The side of the upper base is half that of the lower base, and the altitude of the frustum is the same as the side of the upper base. Find the volume of the frustum.

9. The lower base of a frustum of a pyramid is a square 3 in. on a side. The area of the upper base is half that of the lower base, and the altitude of the frustum is 2 in. Find to two decimal places the volume of the frustum.

10. A pyramid has six edges, each 1 in. long. Find to two decimal places the volume of the pyramid.

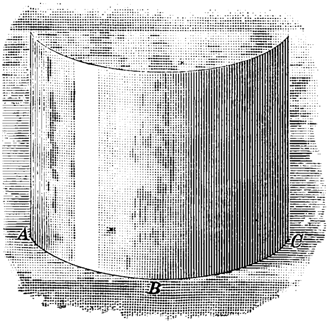
11. A regular tetrahedron has a volume $2\sqrt{2}$ cu. in. Find to two decimal places the length of an edge.

12. The base of a regular pyramid is a square l ft. on a side. The slant height is s ft. Find the area of the entire surface.

13. Consider the formula $v = \frac{1}{3}a\left(b + b' + \sqrt{bb'}\right)$, of § 564, when $b' = 0$. Discuss the meaning of the result. Also discuss the case in which $b = b'$.

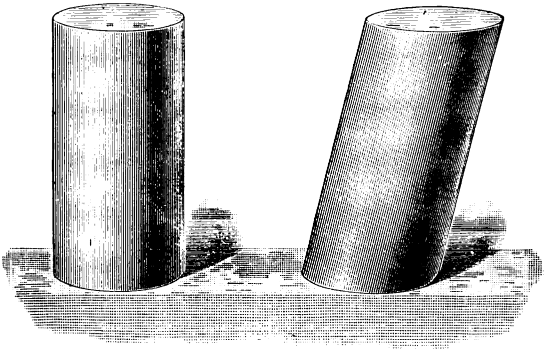
569. Cylindric Surface. A surface generated by a straight line which is constantly parallel to a fixed straight line, and touches a fixed curve not in the plane of the straight line, is called a *cylindric surface,* or a cylindrical surface.

The moving line is called the *generatrix* and the fixed curve the *directrix.* In the figure *ABC* is the directrix.



570. Element. The generatrix in any position is called an *element* of the cylindric surface.

571. Cylinder. A solid bounded by a cylindric surface and two parallel plane surfaces is called a *cylinder.*



It follows, therefore, *that all the elements of a cylinder are equal.*

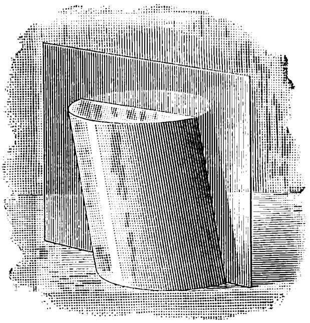
The terms *bases*, *lateral surface*, and *altitude* are used as with prisms.

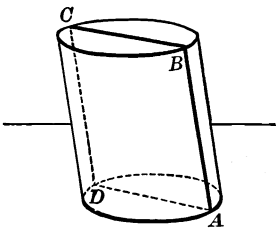
572. Right and Oblique Cylinders. A cylinder whose elements are perpendicular to its bases is called a *right cylinder;* otherwise a cylinder is called an *oblique cylinder.*

573. Section of a Cylinder. A figure formed by the intersection of a plane and a cylinder is called a *section of the cylinder.*

Proposition XX. Theorem

574. *Every section of a cylinder made by a plane passing through an element is a parallelogram.*





Given a cylinder *AC*, and a section *ABCD* made by a plane passing through the element *AB*.

To prove that *ABCD is a parallelogram.*

Proof. Through D draw a line in the plane $ABCD \parallel$ to AB.

This line is an element of the cylindric surface. § 570

Since this line is in both the plane and the cylindric surface, it must be their intersection and must coincide with DC.

Hence DC coincides with a straight line parallel to AB.

Therefore DC is a straight line $\parallel$ to AB.

Also AD is a straight line $\parallel$ to BC. § 453

$\therefore$ $ABCD$ is a parallelogram, by § 118. Q.E.D.

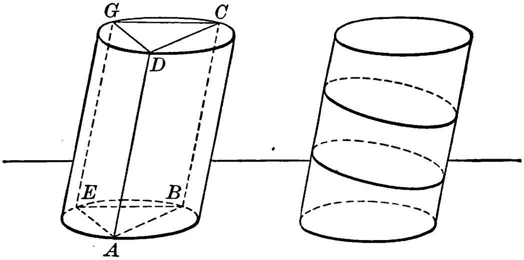
575. Corollary. *Every section of a right cylinder made by a plane passing through an element is a rectangle.*

576. Circular Cylinder. A cylinder whose bases are circles is called a *circular cylinder*.

A right circular cylinder, being generated by the revolution of a rectangle about one side as an axis, is also called a *cylinder of revolution*.

Proposition XXI. Theorem

577. *The bases of a cylinder are congruent.*



Given the cylinder *AC*, with bases *ABE* and *DCG*.

To prove that ABE *is congruent to* DCG.

Proof. Let A, B, E be any three points in the perimeter of the lower base, and AD, BC, EG be elements of the surface.

Draw AB, AE, EB, DC, DG, GC.

Then AD, BC, EG are equal, § 571

and parallel. § 569

$\therefore AB = DC,\ AE = DG,\ EB = GC.$ § 130

$\therefore \triangle ABE$ is congruent to $\triangle DCG$. § 80

Place the lower base on the upper base so that the $\triangle ABE$ shall fall on the $\triangle DCG$. Then A, B, E will fall on D, C, G.

Therefore all points in either perimeter will coincide with points in the other, and the bases are congruent, by § 66. Q.E.D.

578. Corollary 1. *Any two parallel sections of a cylinder, cutting all the elements, are congruent.*

579. Corollary 2. *Any section of a cylinder parallel to the base is congruent to the base.*

580. Corollary 3. *The straight line joining the centers of the bases of a circular cylinder passes through the centers of all sections of the cylinder parallel to the bases.*

581. Tangent Plane. A plane which contains an element of a cylinder, but does not cut the surface, is called a *tangent plane* to the cylinder.

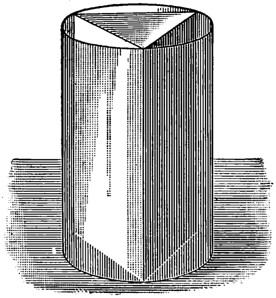
582. Construction of Tangent Planes. From a consideration of the nature of a tangent plane and of the construction of a cylindric surface it is evident that:

A plane passing through a tangent to the base of a circular cylinder and the element drawn through the point of contact is tangent to the cylinder.

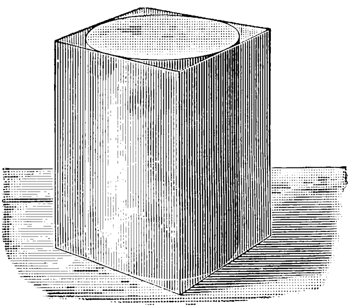
If a plane is tangent to a circular cylinder, its intersection with the plane of the base is tangent to the base.

583. Inscribed Prism. A prism whose lateral edges are elements of a cylinder and whose bases are inscribed in the bases of the cylinder is called an *inscribed prism.*

In this case the cylinder is said to be *circumscribed* about the prism.



Inscribed Prism



Circumscribed Prism

584. Circumscribed Prism. A prism whose lateral faces are tangent to the lateral surface of a cylinder and whose bases are circumscribed about the bases of the cylinder is called a *circumscribed prism.*

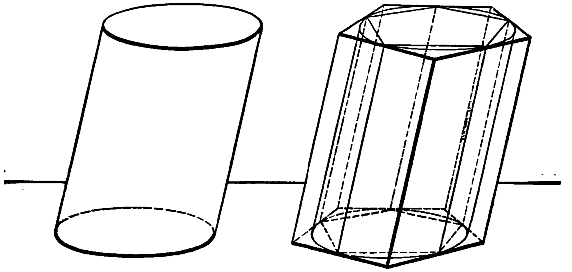
In this case the cylinder is said to be *inscribed* in the prism.

585. Right Section. A section of a cylinder made by a plane that cuts all the elements and is perpendicular to them is called a *right section* of the cylinder.

586. Cylinder as a Limit. From the work already done in connection with limits, and from the nature of the inscribed and circumscribed prisms, the following properties of the cylinder may now be assumed without further proof than that given below:

If a prism whose base is a regular polygon is inscribed in or circumscribed about a circular cylinder, and if the number of sides of the prism is indefinitely increased,

1. *The volume of the cylinder is the limit of the volume of the prism.*
2. *The lateral area of the cylinder is the limit of the lateral area of the prism.*
3. *The perimeter of a right section of the cylinder is the limit of the perimeter of a right section of the prism.*

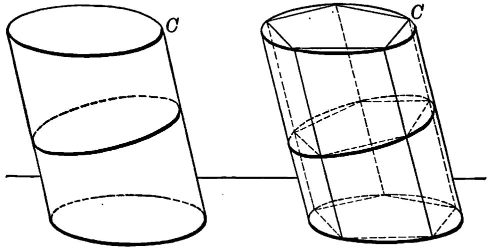


For as we increase the number of sides of the base of the inscribed or circumscribed prism whose base is a regular polygon, the perimeter of the base approaches the circle as its limit (§ 381).

This brings the lateral surface of each prism nearer and nearer the lateral surface of the cylinder. It also brings the volume of each prism nearer and nearer the volume of the cylinder. In the same way it brings the right section of each prism nearer and nearer the right section of the cylinder.

PROPOSITION XXII. THEOREM

587. *The lateral area of a circular cylinder is equal to the product of an element by the perimeter of a right section of the cylinder.*



Given a circular cylinder C, l being the lateral area, p the perimeter of a right section, and e an element.

To prove that $l = ep$.

Proof. Suppose a prism with base a regular polygon to be inscribed in C, l' being its lateral area and p' being the perimeter of its right section.

Then $l' = ep'$. § 512

If the number of lateral faces of the prism is indefinitely increased,

l' approaches l as a limit,

and p' approaches p as a limit, § 586

and consequently ep' approaches ep as a limit.

$\therefore l = ep$, by § 207. Q.E.D.

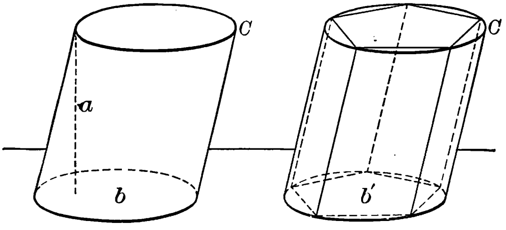
588. COROLLARY. *The lateral area of a cylinder of revolution is equal to the product of the altitude by the circumference of the base.*

In the case of a right circular cylinder of altitude a, lateral area l, total area t, and radius of base r, we have

$$l = 2\pi ra, \text{ and } t = 2\pi ra + 2\pi r^2 = 2\pi r(a + r).$$

PROPOSITION XXIII. THEOREM

589. *The volume of a circular cylinder is equal to the product of its base by its altitude.*



Given a circular cylinder *C*, *b* being the base, *v* the volume, and *a* the altitude.

To prove that $v = ba$.

Proof. Suppose a prism with base a regular polygon to be inscribed in C, b' being its base and v' being its volume.

Then $v' = b'a$. § 539

If the number of lateral faces of the prism is indefinitely increased,

v' approaches v as a limit, § 586

b' approaches b as a limit, § 381

and consequently $b'a$ approaches ba as a limit.

But $v' = b'a$, whatever the number of sides. § 539

$\therefore v = ba$, by § 207. Q.E.D.

590. COROLLARY. *The volume of a cylinder of revolution with radius r and altitude a is* $\pi r^2 a$.

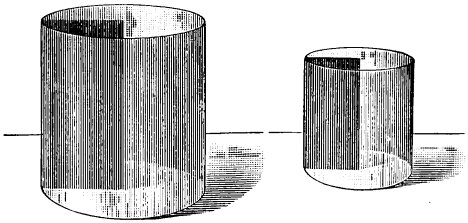
What is the area of the base? By what is this to be multiplied?

591. Similar Cylinders. Cylinders generated by the revolution of similar rectangles about corresponding sides are called *similar cylinders of revolution.*

§§ 591 and 592 may be omitted without destroying the sequence.

PROPOSITION XXIV. THEOREM

592. *The lateral areas, or the total areas, of similar cylinders of revolution are to each other as the squares of their altitudes or as the squares of their radii; and their volumes are to each other as the cubes of their altitudes or as the cubes of their radii.*



Given two similar cylinders of revolution, l and l' denoting their lateral areas, t and t' their total areas, v and v' their volumes, a and a' their altitudes, and r and r' their radii.

To prove that $l : l' = t : t' = a^2 : a'^2 = r^2 : r'^2$,

and that $v : v' = a^3 : a'^3 = r^3 : r'^3$.

Proof. Since the generating rectangles are similar, § 591

$$\therefore \frac{a}{a'} = \frac{r}{r'} = \frac{a+r}{a'+r'}. \qquad \text{§ 269}$$

Also we have by this proportion and § 588,

$$\frac{l}{l'} = \frac{2\pi ra}{2\pi r'a'} = \frac{ra}{r'a'} = \frac{r^2}{r'^2} = \frac{a^2}{a'^2}.$$

But $t = 2\pi ra + 2\pi r^2$ (§ 588), and $v = \pi r^2 a$. § 590

$$\therefore \frac{t}{t'} = \frac{2\pi r(a+r)}{2\pi r'(a'+r')} = \frac{r(a+r)}{r'(a'+r')} = \frac{r^2}{r'^2} = \frac{a^2}{a'^2},$$

and $$\frac{v}{v'} = \frac{\pi r^2 a}{\pi r'^2 a'} = \frac{r^2}{r'^2} \times \frac{a}{a'} = \frac{r^3}{r'^3} = \frac{a^3}{a'^3}.$$ Q.E.D.

EXERCISE 93

1. The diameter of a well is 6 ft. and the water is 7 ft. deep. How many gallons of water are there in the well, reckoning $7\frac{1}{2}$ gal. to the cubic foot?

2. When a body is placed under water in a right circular cylinder 60 centimeters in diameter, the level of the water rises 40 centimeters. Find the volume of the body.

3. How many cubic yards of earth must be removed in constructing a tunnel 100 yd. long, the section being a semicircle with a radius of 18 ft.?

4. How many square feet of sheet iron are required to make a pipe 18 in. in diameter and 40 ft. long?

5. Find the radius of a cylindric pail 14 in. high that will hold exactly 2 cu. ft.

6. The height of a cylindric vessel that will hold 20 liters is equal to the diameter. Find the altitude and the radius.

7. If the total surface of a right circular cylinder is t and the radius of the base is r, find the altitude a.

8. If the lateral surface of a right circular cylinder is l and the volume is v, find the radius r and the altitude a.

9. If the circumference of the base of a right circular cylinder is c and the altitude is a, find the volume v.

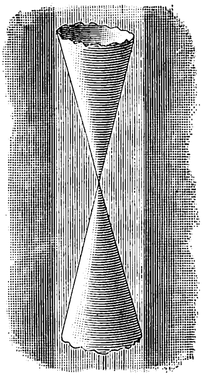
10. If the circumference of the base of a right circular cylinder is c and the total surface is t, find the volume v.

11. If the volume of a right circular cylinder is v and the altitude is a, find the total surface t.

12. If v is the volume of a right circular cylinder in which the altitude equals the diameter, find the altitude a and the total surface t.

13. From the formula $t = 2\pi r(a + r)$ (§ 588) find the value of r. (Omit unless quadratics have been studied.)

593. Conic Surface. A surface generated by a straight line which constantly touches a fixed plane curve and passes through a fixed point not in the plane of the curve is called a *conic surface* or a conical surface.



The moving line is called the *generatrix*, the fixed curve the *directrix*, and the fixed point the *vertex*.

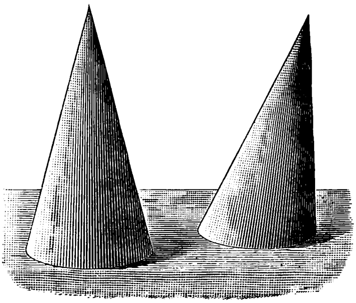
Hold a pencil by the point and let the other end swing around a circle, and the pencil will generate a conic surface.

We may also swing a blackboard pointer about any point near the middle, so that either end shall touch any fixed plane curve, and thus generate a conic surface. Such a surface is represented in the annexed figure.

594. Element. The generatrix in any position is called an *element* of the conic surface.

If the generatrix is of indefinite length, the surface consists of two portions, one above and the other below the vertex, which are called the *upper nappe* and *lower nappe* respectively. The two nappes are shown in the above figure.

595. Cone. A solid bounded by a conic surface and a plane cutting all the elements is called a *cone*.



The conic surface is called the *lateral surface* of the cone, and the plane surface is called the *base* of the cone.

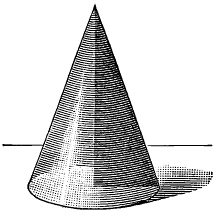
The vertex of the conic surface is called the *vertex* of the cone, and the elements of the conic surface are called the *elements* of the cone.

The perpendicular distance from the vertex to the plane of the base is called the *altitude* of the cone.

596. Circular Cone. A cone whose base is a circle is called a *circular cone.*

The straight line joining the vertex of a circular cone and the center of the base is called the *axis* of the cone.

597. Right and Oblique Cones. A circular cone whose axis is perpendicular to the base is called a *right cone;* otherwise a circular cone is called an *oblique cone.*



598. Cone of Revolution. Since a right circular cone may be generated by the revolution of a right triangle about one of the sides of the right angle, it is called a *cone of revolution.*

In this case the hypotenuse corresponds to an element of the surface and is called the *slant height.*

599. Conic Section. A section formed by the intersection of a plane and the conic surface of a cone of revolution is called a *conic section.*

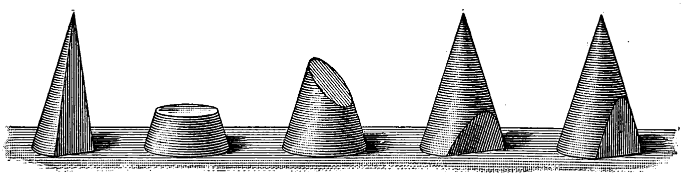


FIG. 1 FIG. 2 FIG. 3 FIG. 4 FIG. 5

In Fig. 1 the conic section is two intersecting straight lines, and this is discussed in § 600. This is true for all kinds of cones.

In Fig. 2 the conic section is a circle, and this is discussed in § 601.

In Fig. 3 the conic section is called an *ellipse*, the form a circle seems to take when looked at obliquely. The orbit of a planet is an ellipse.

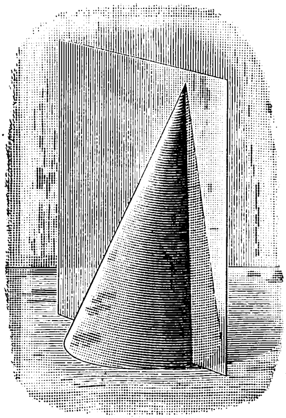
In Fig. 4 the conic section is a *parabola*, the path of a projectile (in a vacuum). Here the cutting plane is parallel to an element.

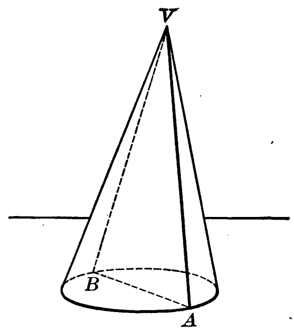
In Fig. 5 the conic section is an *hyperbola.*

The general study of conic sections is not a part of elementary geometry, but the names of the sections may profitably be known.

Proposition XXV. Theorem

600. *Every section of a cone made by a plane passing through its vertex is a triangle.*





Given a cone, with *AVB* a section made by a plane passing through the vertex *V*.

To prove that *AVB is a triangle.*

Proof. AB is a straight line. § 429

Draw the straight lines VA and VB.

The lines VA and VB are both elements of the surface of the given cone. § 594

These lines lie in the cutting plane, since their extremities are in the plane. § 422

Hence VA and VB are the intersections of the conic surface with the cutting plane.

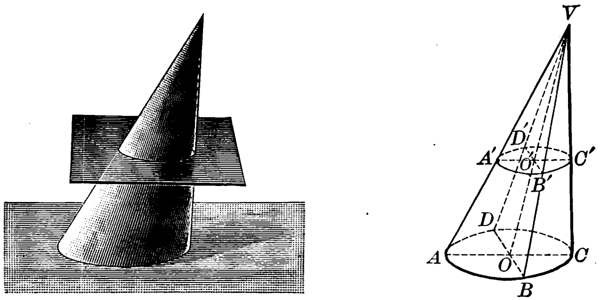
But VA and VB are straight lines. Const.

Therefore the intersections of the conic surface and the plane are straight lines.

Therefore the section AVB is a triangle, by § 28. Q.E.D.

Proposition XXVI. Theorem

601. *In a circular cone a section made by a plane parallel to the base is a circle.*



Given the circular cone *V-ABCD*, with the section *A′B′C′D′* parallel to the base.

To prove that $A'B'C'D'$ *is a circle.*

Proof. Let O be the center of the base, and let O' be the point in which the axis VO pierces the plane of the conic section.

Through VO and any elements VA, VB, pass planes cutting the base in the radii OA, OB, and cutting the section $A'B'C'D'$ in the straight lines $O'A'$, $O'B'$.

Then $O'A'$ and $O'B'$ are ∥ respectively to OA and OB. § 453

Therefore the ▵ AOV and OBV are similar respectively to the ▵ $A'O'V$ and $O'B'V$. § 285

$$\therefore \frac{OA}{O'A'} = \frac{VO}{VO'} = \frac{OB}{O'B'}. \qquad \S\ 282$$

But $OA = OB$. § 162

$\therefore$ $O'A' = O'B'$ (§ 263), and $A'B'C'D'$ is a circle, by § 159. Q.E.D.

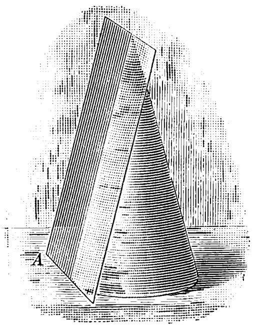
602. Corollary. *The axis of a circular cone passes through the center of every section which is parallel to the base.*

603. Tangent Plane. A plane which contains an element of a cone, but does not cut the surface, is called a *tangent plane* to the cone.

604. Construction of Tangent Planes. It is evident that:

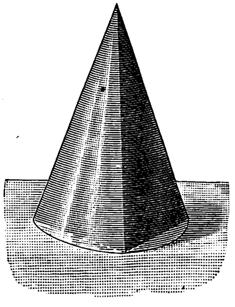
A plane passing through a tangent to the base of a circular cone and the element drawn through the point of contact is tangent to the cone.

If a plane is tangent to a circular cone its intersection with the plane of the base is tangent to the base.

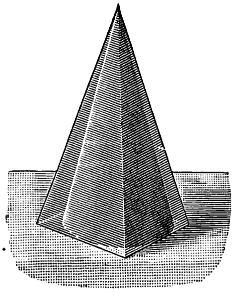


605. Inscribed Pyramid. A pyramid whose lateral edges are elements of a cone and whose base is inscribed in the base of the cone is called an *inscribed pyramid.*

In this case the cone is said to be *circumscribed* about the pyramid.



Inscribed Pyramid

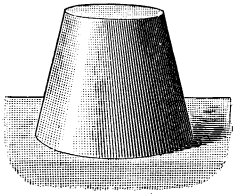


Circumscribed Pyramid

606. Circumscribed Pyramid. A pyramid whose lateral faces are tangent to the lateral surface of a cone and whose base is circumscribed about the base of the cone is called a *circumscribed pyramid.*

In this case the cone is said to be *inscribed* in the pyramid.

607. Frustum of a Cone. The portion of a cone included between the base and a section parallel to the base is called a *frustum of a cone.*

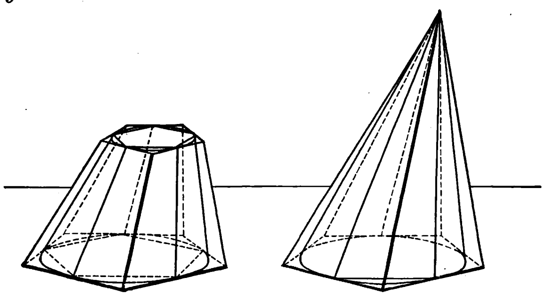


The base of the cone and the parallel section are together called the *bases* of the frustum.

The terms *altitude* and *lateral area* of a frustum of a cone, and *slant height* of a frustum of a right circular cone, are used in substantially the same manner as with the frustum of a pyramid (§§ 550, 551, 552).

608. Cones and Frustums as Limits. The following properties, similar to those of § 586, are assumed without proof:

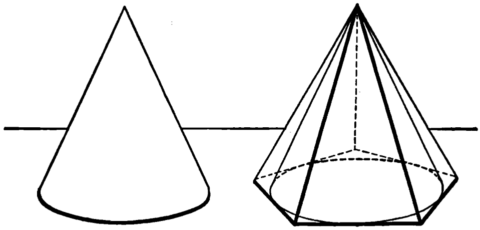
If a pyramid whose base is a regular polygon is inscribed in or circumscribed about a circular cone, and if the number of sides of the base of the pyramid is indefinitely increased, the volume of the cone is the limit of the volume of the pyramid, and the lateral area of the cone is the limit of the lateral area of the pyramid.



The volume of a frustum of a cone is the limit of the volumes of the frustums of the inscribed and circumscribed pyramids, if the number of lateral faces is indefinitely increased, and the lateral area of the frustum of a cone is the limit of the lateral areas of the frustums of the inscribed and circumscribed pyramids, the bases being regular polygons.

PROPOSITION XXVII. THEOREM

609. *The lateral area of a cone of revolution is equal to half the product of the slant height by the circumference of the base.*



Given a cone of lateral area l, circumference of base c, and slant height s.

To prove that $l = \frac{1}{2} sc$.

Proof. Suppose a regular pyramid to be circumscribed about the cone, the perimeter of its base being p and its lateral area l'.

Then $l' = \frac{1}{2} sp$. § 553

If the number of the lateral faces of the circumscribed pyramid is indefinitely increased,

l' approaches l as a limit, § 608

p approaches c as a limit, § 381

and consequently $\frac{1}{2} sp$ approaches $\frac{1}{2} sc$ as a limit.

But $l' = \frac{1}{2} sp$, whatever the number of sides. § 553

$\therefore l = \frac{1}{2} sc$, by § 207. Q.E.D.

610. COROLLARY. *If l denotes the lateral area, t the total area, s the slant height, and r the radius of the base of a cone of revolution, then*

$$l = \frac{1}{2}(2\pi r \times s) = \pi rs;$$

$$t = \pi rs + \pi r^2 = \pi r(s + r).$$

EXERCISE 94

Find the lateral areas of cones of revolution, given the slant heights and the circumferences of the bases respectively as follows:

1. $2\frac{7}{8}$ in., $5\frac{3}{8}$ in.	**4.** 3.7 in., 5.8 in.	**7.** 2 ft. 6 in., 4 ft. 8 in.
2. $4\frac{3}{8}$ in., $8\frac{1}{4}$ in.	**5.** 5.3 in., 9.7 in.	**8.** 3 ft. 7 in., 8 ft. 6 in.
3. $6\frac{5}{16}$ in., $10\frac{1}{2}$ in.	**6.** 6.5 in., 11.6 in.	**9.** 5 ft. 8 in., 12 ft. 4 in.

Find the lateral areas of cones of revolution, given the slant heights and the radii of the bases respectively as follows:

10. $3\frac{3}{4}$ in., $2\frac{1}{2}$ in.	**13.** 6.4 in., 4.8 in.	**16.** 2 ft. 3 in., 8 in.
11. $2\frac{1}{2}$ in., $1\frac{3}{4}$ in.	**14.** 7.2 in., 5.3 in.	**17.** 4 ft. 6 in., 2 ft.
12. $4\frac{7}{8}$ in., $3\frac{1}{4}$ in.	**15.** 8.9 in., 5.6 in.	**18.** 6 ft. 9 in., 3 ft. 2 in.

Find the total areas of cones of revolution, given the slant heights and the radii of the bases respectively as follows:

19. 3 in., 2 in.	**21.** 7 in., 4 in.	**23.** 6 ft., 4 ft.
20. 5 in., 3 in.	**22.** 9 in., 5 in.	**24.** 12 ft., 5 ft.

25. Deduce a formula for finding the lateral area of a cone of revolution in terms of the radius of the base and the altitude.

26. Deduce a formula for finding the slant height in terms of the lateral area and the circumference of the base.

27. Deduce a formula for finding the slant height in terms of the lateral area and the radius of the base.

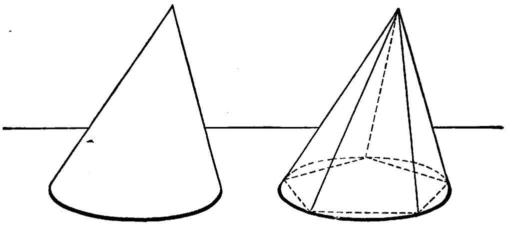
28. Deduce a formula for finding the radius of the base in terms of the lateral area and the slant height.

29. Deduce a formula for finding the slant height in terms of the total area and the radius of the base.

30. Deduce a formula for finding the circumference of the base in terms of the lateral area and the slant height.

Proposition XXVIII. Theorem

611. *The volume of a circular cone is equal to one third the product of its base by its altitude.*



Given a circular cone of volume v, base b, and altitude a.

To prove that $v = \frac{1}{3} ba$.

Proof. Suppose a pyramid with base a regular polygon to be inscribed in the cone, b' being its base and v' its volume.

Then $v' = \frac{1}{3} b'a$. § 561

If the number of lateral faces of the pyramid is indefinitely increased,

v' approaches v as a limit, § 608

b' approaches b as a limit, § 381

and consequently $b'a$ approaches ba as a limit.

$\therefore v = \frac{1}{3} ba$, by § 207. Q.E.D.

612. Corollary. *In a circular cone of radius r and altitude a, $v = \frac{1}{3} \pi r^2 a$.*

For the area of the base is πr^2 (§ 389).

613. Similar Cones. Cones generated by the revolution of similar right triangles about corresponding sides are called *similar cones* of revolution.

In case § 614 is omitted this definition may also be omitted.

EXERCISE 95

Find the volumes of circular cones, given the altitudes and the areas of the bases respectively as follows:

1. 4 in., 8 sq. in.
2. $3\frac{1}{4}$ in., $9\frac{3}{8}$ sq. in.
3. $5\frac{3}{8}$ in., $10\frac{1}{2}$ sq. in.
4. 6.3 in., 3.8 sq. in.
5. 7.8 in., 6.9 sq. in.
6. 9.3 in., 16.8 sq. in.

Find the volumes of circular cones, given the altitudes and the radii of the bases respectively as follows:

7. 4 in., 3 in.
8. 6 in., 4 in.
9. 8 in., 5 in.
10. 9.8 in., 4.3 in.
11. 10.5 in., 6.2 in.
12. 14.9 in., 9.6 in.

13. How many cubic feet in a conical tent 10 ft. in diameter and 7 ft. high?

14. How many cubic feet in a conical pile of earth 15 ft. in diameter and 8 ft. high?

15. Deduce a formula for finding the altitude of a circular cone in terms of the volume and the area of the base.

16. Deduce a formula for finding the area of the base of a circular cone in terms of the volume and the altitude.

17. Deduce a formula for finding the altitude of a circular cone in terms of the volume and the radius of the base.

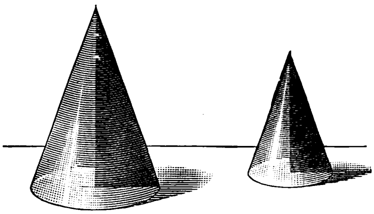
18. Deduce a formula for finding the radius of the base of a circular cone in terms of the volume and the altitude.

19. Deduce a formula for finding the volume of a cone of revolution in terms of the slant height and the radius of the base.

20. Deduce formulas for finding the slant height and the altitude of a cone of revolution in terms of the volume and the radius of the base.

Proposition XXIX. Theorem

614. *The lateral areas, or the total areas, of two similar cones of revolution are to each other as the squares of their altitudes, as the squares of their radii, or as the squares of their slant heights; and their volumes are to each other as the cubes of their altitudes, as the cubes of their radii, or as the cubes of their slant heights.*



Given two similar cones of revolution, with lateral areas l and l', total areas t and t', volumes v and v', altitudes a and a', radii r and r', and slant heights s and s' respectively.

To prove that $l : l' = t : t' = a^2 : a'^2 = r^2 : r'^2 = s^2 : s'^2$,

and that $v : v' = a^3 : a'^3 = r^3 : r'^3 = s^3 : s'^3$.

Proof. $$\frac{a}{a'} = \frac{r}{r'} = \frac{s}{s'} = \frac{s+r}{s'+r'}.$$ §§ 282, 269

$$\frac{l}{l'} = \frac{\pi rs}{\pi r's'} = \frac{r}{r'} \times \frac{s}{s'} = \frac{r^2}{r'^2} = \frac{s^2}{s'^2} = \frac{a^2}{a'^2}.$$ § 610

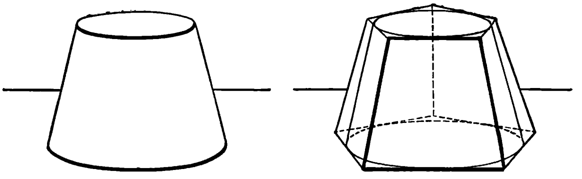
$$\frac{t}{t'} = \frac{\pi r(s+r)}{\pi r'(s'+r')} = \frac{r}{r'} \times \frac{s+r}{s'+r'} = \frac{r^2}{r'^2} = \frac{s^2}{s'^2} = \frac{a^2}{a'^2}.$$ § 610

$$\frac{v}{v'} = \frac{\frac{1}{3}\pi r^2 a}{\frac{1}{3}\pi r'^2 a'} = \frac{r^2}{r'^2} \times \frac{a}{a'} = \frac{r^3}{r'^3} = \frac{a^3}{a'^3} = \frac{s^3}{s'^3},$$ by § 612. Q.E.D.

§§ 613 and 614, like §§ 591 and 592, are occasionally demanded in college entrance examinations. They are not needed for any exercises and they may be omitted without destroying the sequence.

Proposition XXX. Theorem

615. *The lateral area of a frustum of a cone of revolution is equal to half the sum of the circumferences of its bases multiplied by the slant height.*



Given a frustum of a cone of revolution, with lateral area l, circumferences of bases c and c', and slant height s.

To prove that $l = \frac{1}{2}(c + c')s.$

Proof. Suppose a frustum of a regular pyramid circumscribed about the frustum of the cone, as a pyramid is circumscribed about a cone.

Let the lateral area of the circumscribed frustum be l', and let p and p' be the perimeters of the bases corresponding to c and c' respectively. The slant height is s, the same as that of the frustum of the cone.

Then $l' = \frac{1}{2}(p + p')s.$ § 554

If the number of lateral faces of the circumscribed frustum is indefinitely increased, what limits do l' and $p + p'$ approach?

Therefore what limit does $\frac{1}{2}(p + p')s$ approach?

What conclusion may be drawn, as in § 587?

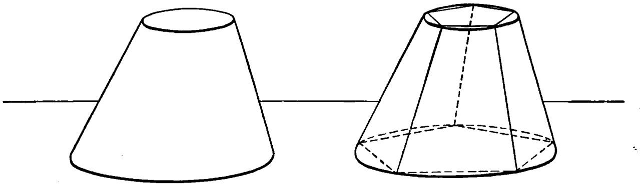
Complete the proof.

616. Corollary. *The lateral area of a frustum of a cone of revolution is equal to the circumference of a section equidistant from its bases multiplied by its slant height.*

How can it be proved that $\frac{1}{2}(c + c')$ equals the circumference of this section? How are the radii related?

PROPOSITION XXXI. THEOREM

617. *A frustum of a circular cone is equivalent to the sum of three cones whose common altitude is the altitude of the frustum and whose bases are the lower base, the upper base, and the mean proportional between the bases of the frustum.*



Given a frustum of a circular cone, with volume v, bases b and b', and altitude a.

To prove that $v = \frac{1}{3} a(b + b' + \sqrt{bb'})$.

Proof. Suppose a frustum of a pyramid with base a regular polygon to be inscribed in the frustum of the cone, as a pyramid is inscribed in a cone.

Let v' be the volume, and let x and x' be the bases corresponding to b and b' respectively. The altitude is a, the same as that of the frustum of the cone.

Then $v' = \frac{1}{3} a(x + x' + \sqrt{xx'})$. § 565

If the number of lateral faces of the inscribed frustum is indefinitely increased, what limits do v', x, x', and xx' approach?

Therefore what limit does $\frac{1}{3} a(x + x' + \sqrt{xx'})$ approach?

What conclusion may be drawn?

Complete the proof.

618. COROLLARY. *In a frustum of a cone of revolution, r and r' being the radii of the bases,* $v = \frac{1}{3} \pi a (r^2 + r'^2 + rr')$.

For $b = \pi r^2$, $b' = \pi r'^2$. $\therefore \sqrt{bb'} = \sqrt{\pi r^2 \times \pi r'^2} = \pi r r'$.

EXERCISE 96

Find the lateral areas of frustums of cones, given the circumferences of the bases and the slant heights respectively as follows:

1. $c = 4$ in., $c' = 3$ in., $s = 0.5$ in.

2. $c = 6$ in., $c' = 5$ in., $s = 1.4$ in.

3. $c = 7\frac{1}{2}$ in., $c' = 5\frac{3}{4}$ in., $s = 2\frac{1}{8}$ in.

4. $c = 23$ in., $c' = 18$ in., $s = 16$ in.

Find to two decimal places the volumes of frustums of cones, given the altitudes and the areas of the bases respectively as follows:

5. $a = 3$ in., $b = 4\frac{1}{2}$ sq. in., $b' = 2$ sq. in.

6. $a = 4$ in., $b = 8\frac{1}{3}$ sq. in., $b' = 3$ sq. in.

7. $a = 5\frac{1}{2}$ in., $b = 16$ sq. in., $b' = 9$ sq. in.

8. $a = 6$ in., $b = 17$ sq. in., $b' = 11$ sq. in.

Find to two decimal places the volumes of frustums of cones of revolution, given the altitudes and the radii of the bases respectively as follows:

9. $a = 4$ in., $r = 3$ in., $r' = 2$ in.

10. $a = 5$ in., $r = 3\frac{1}{2}$ in., $r' = 2\frac{1}{4}$ in.

11. $a = 6$ in., $r = 3.7$ in., $r' = 3.1$ in.

12. $a = 7\frac{1}{2}$ in., $r = 4\frac{3}{4}$ in., $r' = 3\frac{1}{8}$ in.

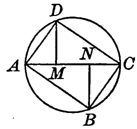
13. Deduce a formula for finding the altitude of a frustum of a circular cone in terms of the volume and the areas of the bases.

14. Deduce a formula for finding the altitude of a frustum of a cone of revolution in terms of the volume and the radii of the bases.

EXERCISE 97

Industrial Problems

1. There is a rule for calculating the strongest beam that can be cut from a cylindric log, as follows:



Erect perpendiculars MD and NB on opposite sides of a diameter AC, at the trisection points M and N, meeting the circle in D and B. Then $ABCD$ is a section of the beam.

Calculate the dimensions, the log being 16 in. in diameter.

2. A cylindric funnel for a steamboat is 4 ft. 3 in. in diameter. It is built up of four plates in girth, and the lap of each joint is $1\frac{3}{8}$ in. Find one dimension of each plate.

3. A tubular boiler has 124 tubes each $3\frac{7}{8}$ in. in diameter and 18 ft. long. Required the total tube surface. Answer to the nearest square foot.

4. A room in a factory is heated by steam pipes. There are 235 ft. of 2-inch pipe and 26 ft. 3 in. of 3-inch pipe, besides 2 ft. 8 in. of $4\frac{1}{4}$-inch feed pipe. Required the total heating surface. Answer to the nearest square foot.

5. A triangular plate of wrought iron $\frac{5}{8}$ in. thick is 2 ft. 7 in. on each side. If the weight of a plate 1 ft. square and $\frac{1}{8}$ in. thick is 5 lb., find to the nearest pound the weight of the given triangular plate.

6. The water surface of an upright cylindric boiler is 2 ft. 8 in. below the top of the boiler, and is 12.57 sq. ft. in area. What is the volume of the steam space?

7. A cylinder 16 in. in diameter is required to hold 50 gal. of water. What must be its height, to the nearest tenth of an inch, allowing 231 cu. in. to the gallon?

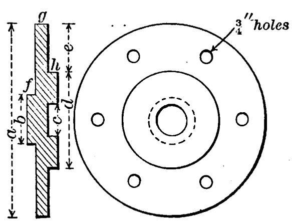
8. How many square feet of tin are required to make a funnel, if the diameters of the top and bottom are 30 in. and 15 in. respectively, and the height is 25 in.?

9. Find to two decimal places the weight of a steel plate 4 ft. by 3 ft. 2 in. by $1\frac{3}{8}$ in., allowing 490 lb. per cubic foot.

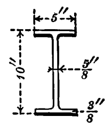
10. A steel plate for a steamship is 5 ft. long, 3 ft. 6 in. wide, and $\frac{1}{2}$ in. thick. A porthole 10 in. in diameter is cut through the plate. Required the weight of the finished plate, allowing 0.29 lb. per cubic inch. Answer to two decimal places.

11. A cast-iron base for a column is in the form of a frustum of a pyramid, the lower base being a square 2 ft. on a side, and the upper base having a fourth of the area of the lower base. The altitude of the frustum is 9 in. Required the weight to the nearest pound, allowing 460 lb. per cubic foot.

12. A cylinder head for a steam engine has the shape shown in the figure, where the dimensions in inches are: $a = 12$, $b = 3$, $c = 2$, $d = 6$, $e = 3$, $f = \frac{1}{2}$, $g = \frac{7}{8}$, and $h = \frac{3}{4}$. There are six $\frac{3}{4}$-inch holes for bolts. Compute the weight of the plate, allowing 41 lb. for the weight of a steel plate 1 ft. square and 1 in. thick. Answer to the nearest tenth of a pound.



13. A steel beam 10 in. by 5 in., in the form here shown, is 18 ft. long. The thickness of the beam is $\frac{5}{8}$ in. and the average thickness of the flanges is $\frac{3}{8}$ in. Find the weight of the beam to the nearest pound, allowing 0.29 lb. per cubic inch.



14. A hollow steel shaft 12 ft. long is 18 in. in exterior diameter and 8 in. in interior diameter. Find the weight to the nearest pound, allowing 0.29 lb. per cubic inch.

15. Find the expense, at 70 cents a square foot, of polishing the curved surface of a marble column in the shape of the frustum of a right circular cone whose slant height is 12 ft. and the radii of whose bases are 3 ft. 6 in. and 2 ft. 4 in. respectively.

EXERCISE 98

Miscellaneous Problems

1. The slant height of the frustum of a regular pyramid is 25 ft., and the sides of its square bases are 54 ft. and 24 ft. respectively. Find the volume.

2. If the bases of the frustum of a pyramid are regular hexagons whose sides are 1 ft. and 2 ft. respectively, and the volume of the frustum is 12 cu. ft., find the altitude.

3. From a right circular cone whose slant height is 30 ft., and the circumference of whose base is 10 ft., there is cut off by a plane parallel to the base a cone whose slant height is 6 ft. Find the lateral area and the volume of the frustum.

4. Find the difference between the volume of the frustum of a pyramid whose altitude is 9 ft. and whose bases are squares, 8 ft. and 6 ft. respectively on a side, and the volume of a prism of the same altitude whose base is a section of the frustum parallel to its bases and equidistant from them.

5. A Dutch stone windmill in the shape of the frustum of a right cone is 12 meters high. The outer diameters at the bottom and the top are 16 meters and 12 meters, the inner diameters 12 meters and 10 meters. How many cubic meters of stone were required to build it?

6. The chimney of a factory has the shape of a frustum of a regular pyramid. Its height is 180 ft., and its upper and lower bases are squares whose sides are 10 ft. and 16 ft. respectively. The flue throughout is a square whose side is 7 ft. How many cubic feet of material does the chimney contain?

7. Two right triangles with bases 15 in. and 21 in., and with hypotenuses 25 in. and 35 in. respectively, revolve about their third sides. Find the ratio of the total areas of the solids generated and find their volumes.

EXERCISE 99

Equivalent Solids

1. A cube each edge of which is 12 in. is transformed into a right prism whose base is a rectangle 16 in. long and 12 in. wide. Find the height of the prism and the difference between its total area and the total area of the cube.

2. The dimensions of a rectangular parallelepiped are a, b, c. Find the height of an equivalent right circular cylinder, having a for the radius of its base; the height of an equivalent right circular cone having a for the radius of its base.

3. A regular pyramid 12 ft. high is transformed into a regular prism with an equivalent base. Find the height of the prism.

4. The diameter of a cylinder is 14 ft. and its height 8 ft. Find the height of an equivalent right prism, the base of which is a square with a side 4 ft. long.

5. If one edge of a cube is e, what is the height h of an equivalent right circular cylinder whose radius is r?

6. The heights of two equivalent right circular cylinders are in the ratio $4:9$. If the diameter of the first is 6 ft., what is the diameter of the second?

7. A right circular cylinder 6 ft. in diameter is equivalent to a right circular cone 7 ft. in diameter. If the height of the cone is 8 ft., what is the height of the cylinder?

8. The frustum of a regular pyramid 6 ft. high has for bases squares 5 ft. and 8 ft. on a side. Find the height of an equivalent regular pyramid whose base is a square 12 ft. on a side.

9. The frustum of a cone of revolution is 5 ft. high and the diameters of its bases are 2 ft. and 3 ft. respectively. Find the height of an equivalent right circular cylinder whose base is equal in area to the section of the frustum made by a plane parallel to the bases and equidistant from them.

EXERCISE 100

Review Questions

1. Define polyhedron. Is a cylinder a polyhedron?

2. Define prism, and classify prisms according to their bases.

3. How is the lateral area of a prism computed? Is the method the same for right as for oblique prisms?

4. Define parallelepiped; rectangular parallelepiped; cube. Is a rectangular parallelepiped always a cube? Is a cube always a rectangular parallelepiped?

5. Distinguish between equivalent and congruent solids. Are two cubes with the same altitudes always equivalent? always congruent? Is this true for parallelepipeds?

6. What are the conditions of congruence of two prisms? of two right prisms? of two cubes?

7. The opposite angles of a parallelogram are equal. What is a corresponding proposition concerning parallelepipeds?

8. How do you find the volume of a parallelepiped? What is the corresponding proposition in plane geometry?

9. How do you find the volume of a prism? of a cylinder? of a pyramid? of a cone?

10. Define pyramid. How many bases has a pyramid? Is there any kind of a pyramid in which more than one face may be taken as the base?

11. How do you find the lateral area of a pyramid? of a right cone? of a frustum of a pyramid? of a frustum of a right cone?

12. How many regular convex polyhedrons are possible? What are their names?

13. Given the radius of the base and the altitude of a cone of revolution, how do you find the volume? the lateral area? the total area?

BOOK VIII

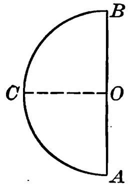
THE SPHERE

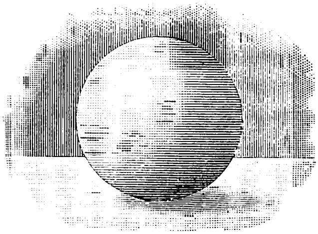
619. Sphere. A solid bounded by a surface all points of which are equidistant from a point within is called a *sphere.*

The point within, from which all points on the surface are equally distant, is called the *center*. The surface is called the *spherical surface*, and sometimes the *sphere*. Half of a sphere is called a *hemisphere*. The terms *radius* and *diameter* are used as in the case of a circle.

620. Generation of a Spherical Surface. By the definition of sphere it appears that a spherical surface may be generated by the revolution of a semicircle about its diameter as an axis.

Thus, if the semicircle ACB revolves about AB, a spherical surface is generated. It is therefore assumed that *a sphere may be described with any given point as a center and any given line as a radius.*





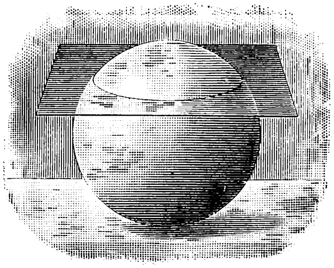
621. Equality of Radii and Diameters. It follows that:

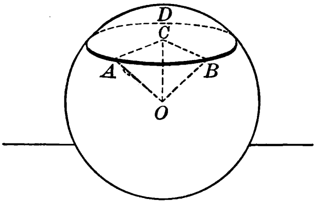
All radii of the same sphere are equal, and all diameters of the same sphere are equal.

Equal spheres have equal radii, and spheres having equal radii are equal.

Proposition I. Theorem

622. *Every intersection of a spherical surface by a plane is a circle.*





Given a sphere with center *O*, and *ABD* any section of its surface made by a plane.

To prove that the section ABD is a circle.

Proof. Draw the radii OA, OB, to any two points A, B, in the section, and draw $OC \perp$ to the plane of the section.

Then in $\triangle$s OCA and OCB, $\angle$s OCA and OCB are rt. $\angle$s, § 430

OC is common, and $OA = OB$. § 621

$\therefore \triangle OCA$ is congruent to $\triangle OCB$. § 89

$\therefore CA = CB$. § 67

$\therefore$ any points A and B, and hence all points, in the section are equidistant from C, and ABD is a $\odot$, by § 159. Q. E. D.

623. Corollary 1. *The line joining the center of a sphere and the center of a circle of the sphere is perpendicular to the plane of the circle.*

624. Corollary 2. *Circles of a sphere made by planes equidistant from the center are equal; and of two circles made by planes not equidistant from the center the one made by the plane nearer the center is the greater.*

625. Great Circle. The intersection of a spherical surface by a plane passing through the center is called a *great circle* of the sphere.

626. Small Circle. The intersection of a spherical surface by a plane which does not pass through the center is called a *small circle* of the sphere.

627. Poles of a Circle. If a diameter of a sphere is perpendicular to the plane of a circle of the sphere, the extremities are called the *poles* of the circle.

628. COROLLARY 1. *Parallel circles have the same poles.*

629. COROLLARY 2. *All great circles of a sphere are equal.*

630. COROLLARY 3. *Every great circle bisects the spherical surface.*

631. COROLLARY 4. *Two great circles bisect each other.*

The intersection of the planes passes through what point?

632. COROLLARY 5. *If the planes of two great circles are perpendicular, each circle passes through the poles of the other.*

Draw the figure and state the reason.

633. COROLLARY 6. *Through two given points on the surface of a sphere an arc of a great circle may always be drawn.*

Do these two points, together with the center of the sphere, generally determine a plane? Consider the special case in which the two points are ends of a diameter.

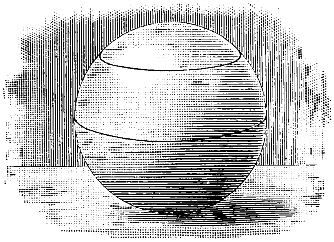
634. COROLLARY 7. *Through three given points on the surface of a sphere one circle and only one can be drawn.*

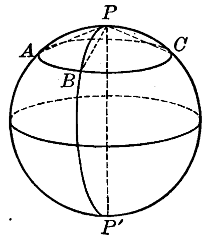
How many points determine a plane?

635. Spherical Distance. The length of the smaller arc of the great circle joining two points on the surface of a sphere is called the *spherical distance* between the points, or, where no confusion is likely to arise, simply the *distance*.

PROPOSITION II. THEOREM

636. *The spherical distances of all points on a circle of a sphere from either pole of the circle are equal.*





Given P, P', the poles of the circle ABC, and A, B, C, any points on the circle.

To prove that the great-circle arcs PA, PB, PC are equal.

Proof. The straight lines PA, PB, PC are equal. § 439

Therefore the arcs PA, PB, PC are equal, by § 172. Q. E. D.

In like manner, the great-circle arcs $P'A$, $P'B$, $P'C$ may be proved equal.

637. Polar Distance. The spherical distance from the nearer pole of a circle to any point on the circle is called the *polar distance of the circle.*

The spherical distance of a great circle from either of its poles may be taken as the polar distance of the circle.

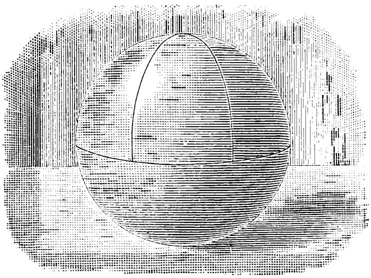
638. Quadrant. One fourth of a great circle is called a *quadrant.*

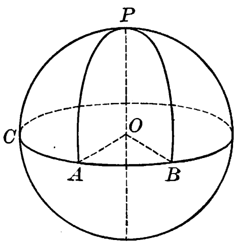
639. COROLLARY 1. *The polar distance of a great circle is a quadrant.*

640. COROLLARY 2. *The straight lines joining points on a circle to either pole of the circle are equal.*

Proposition III. Theorem

641. *A point on a sphere, which is at the distance of a quadrant from each of two other points, not the extremities of a diameter, is a pole of the great circle passing through these points.*





Given a point P on a sphere, PA and PB quadrants, and ABC the great circle passing through A and B.

To prove that P *is the pole of* $\odot ABC$.

Proof. What kind of angles are the $\angle$s AOP and BOP?

How is PO related to the plane of $\odot ABC$?

Does this prove that P is the pole of $\odot ABC$?

642. Describing Circles on a Sphere. This proposition proves that we may describe a great circle on a sphere of a given radius so that it shall pass through two given points.

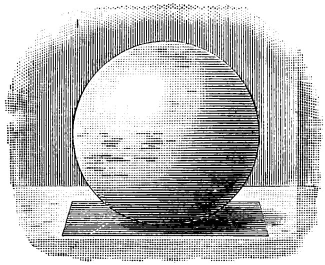
Open the compasses the length of chord $PA = \sqrt{r^2 + r^2} = r\sqrt{2}$.

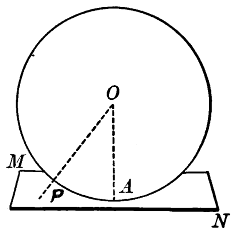
643. Tangent Lines and Planes. A line or plane that has one point and only one point in common with a sphere, however far produced, is said to be *tangent* to the sphere, and the sphere to be *tangent* to the line or plane.

644. Tangent Spheres. Two spheres whose surfaces have one point and only one point in common are said to be *tangent*.

Proposition IV. Theorem

645. *A plane perpendicular to a radius at its extremity is tangent to the sphere.*





Given the plane *MN* perpendicular to the radius *OA* at *A*.

To prove that MN is tangent to the sphere.

Proof. Let *P* be any point except *A* in *MN*.

Then which is longer, *OP* or *OA*, and why?

Therefore, is *P* inside, on, or outside the sphere, and why?

What does this tell us concerning all points, except *A*, on *MN*?

How, then, do we know that *MN* is tangent to the sphere?

646. Corollary. *A plane tangent to a sphere is perpendicular to the radius drawn to the point of contact.*

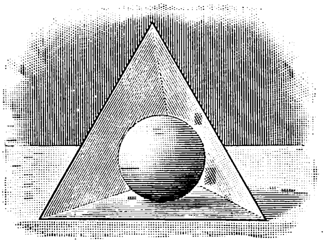
What are the proposition and corollary of plane geometry corresponding to §§ 645 and 646? Do they suggest the proof of this corollary?

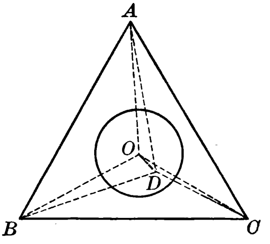
647. Inscribed Sphere. If a sphere is tangent to all the faces of a polyhedron, it is said to be *inscribed* in the polyhedron, and the polyhedron to be circumscribed about the sphere.

648. Circumscribed Sphere. If all the vertices of a polyhedron lie on a spherical surface, the sphere is said to be *circumscribed* about the polyhedron, and the polyhedron to be inscribed in the sphere.

Proposition V. Theorem

649. *A sphere may be inscribed in any given tetrahedron.*





Given the tetrahedron *ABCD*.

To prove that a sphere may be inscribed in ABCD.

Proof. Bisect the dihedral ∠s at the edges *AB*, *BC*, and *CA* by the planes *OAB*, *OBC*, and *OCA* respectively.

Every point in the plane *OAB* is equidistant from the faces *ABC* and *ABD*. § 479

For a like reason every point in the plane *OBC* is equidistant from the faces *ABC* and *DBC*; and every point in the plane *OCA* is equidistant from the faces *ABC* and *ADC*.

Therefore the point *O*, the common intersection of these three planes, is equidistant from the four faces of the tetrahedron and is the center of the sphere inscribed in the tetrahedron, by § 647. Q.E.D.

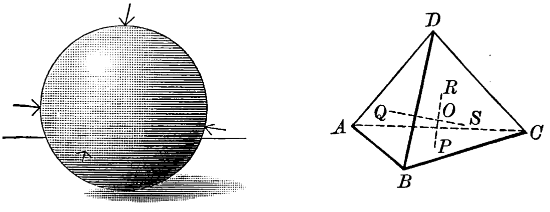
Discussion. What is the corresponding proposition in plane geometry? Is the line of proof similar?

It is shown in plane geometry that the three lines which bisect the three angles of a triangle meet in a point. What is the corresponding proposition with reference to planes in a tetrahedron? Is it substantially proved in this proposition?

It is proved in plane geometry that a circle may be inscribed in what kind of a polygon? What corresponding proposition may be inferred in solid geometry?

Proposition VI. Theorem

650. *A sphere may be circumscribed about any given tetrahedron.*



Given the tetrahedron *ABCD*.

To prove that a sphere may be circumscribed about ABCD.

Proof. Let P, Q respectively be the centers of the circles circumscribed about the faces ABC, ABD.

Let PR be $\perp$ to the face ABC, and $QS \perp$ to the face ABD.

Then PR is the locus of a point equidistant from A, B, C, and QS is the locus of a point equidistant from A, B, D. § 442

Therefore PR and QS lie in the same plane, the plane $\perp$ to AB at its mid-point. § 443

If QS were $\parallel$ to PR, it would be $\perp$ to the face ABC. § 445

But this is impossible, for QS is $\perp$ to the face ABD which intersects the face ABC. Given

Since PR and QS cannot be $\parallel$, and since they lie in the same plane, they must therefore meet at some point O.

$\therefore$ O is equidistant from A, B, C, and D,

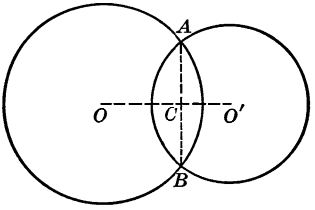
and is the center of the required sphere, by § 648. Q.E.D.

651. Corollary. *Through four points not in the same plane one spherical surface and only one can be passed.*

The center of any sphere whose surface passes through the four points must be in the perpendiculars mentioned in the proof, and since there is only one point of intersection, there can be only one sphere.

PROPOSITION VII. THEOREM

652. *The intersection of two spherical surfaces is a circle whose plane is perpendicular to the line which joins the centers of the spheres and whose center is in that line.*



Given two intersecting spherical surfaces, with centers *O* and *O′*.

To prove that the spherical surfaces intersect in a circle whose plane is perpendicular to OO', and whose center is in OO'.

Proof. Let the two great circles formed by any plane through O and O' intersect in A and B.

Then OO' is a $\perp$ bisector of AB. § 195

If this plane revolves about OO', the circles generate the spherical surfaces, and A describes their line of intersection.

But during the revolution AC remains constant in length and perpendicular to OO'.

Therefore A generates a circle with center C, whose plane is perpendicular to OO', by § 432. Q.E.D.

653. Spherical Angle. The opening between two great-circle arcs that intersect is called a *spherical angle.* A spherical angle is considered equal to the plane angle formed by the tangents to the arcs at their point of intersection.

Draw a figure illustrating this definition.

In elementary geometry we do not consider angles formed by arcs of small circles.

EXERCISE 101

1. The four perpendiculars erected at the centers of the circles circumscribed about the faces of a tetrahedron meet in the same point.

2. The six planes perpendicular to the edges of a tetrahedron at their mid-points intersect in the same point.

3. The six planes which bisect the six dihedral angles of a tetrahedron intersect in the same point.

4. Circles on the same sphere having equal polar distances are equal.

5. Equal circles on the same sphere have equal polar distances.

6. Find the locus of a point in a plane at a given distance from a given point. Also of a point in a three-dimensional space.

7. A line tangent to a great circle of a sphere lies in the plane tangent to the sphere at the point of contact.

8. Any line in a tangent plane drawn through the point of contact is tangent to the sphere at that point.

9. One plane and only one plane can be passed through a given point on a given sphere tangent to the sphere.

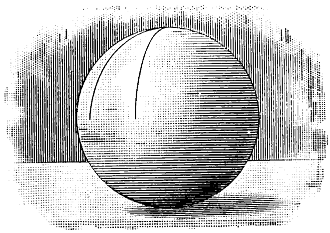
10. Find a point in a plane equidistant from two intersecting lines in the plane, and at a given distance from a given point not in the plane. Discuss the solution.

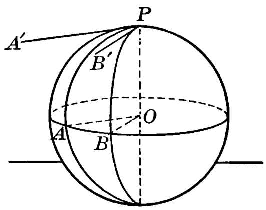
11. How many points determine a straight line? a circle? a spherical surface? Prove that two spherical surfaces coincide if they have this number of points in common.

12. If two planes which intersect in the line AB touch a sphere at the points C and D respectively, the line CD is perpendicular to AB in the sense mentioned in the discussion under § 450, — that a plane can be passed through CD perpendicular to AB.

PROPOSITION VIII. THEOREM

654. *A spherical angle is measured by the arc of the great circle described from its vertex as a pole and included between its sides, produced if necessary.*





Given PA and PB, arcs of great circles intersecting at P; PA' and PB', the tangents to these arcs at P; AB, the arc of the great circle described from P as a pole and included between PA and PB.

To prove that the spherical $\angle APB$ is measured by arc AB.

Proof. In the plane POB, PB' is $\perp$ to PO, § 185

and OB is $\perp$ to PO. § 213

$\therefore PB'$ is $\parallel$ to OB. § 95

Similarly PA' is $\parallel$ to OA.

$\therefore \angle A'PB' = \angle AOB$. § 461

But $\angle AOB$ is measured by arc AB. § 213

$\therefore \angle A'PB'$ is measured by arc AB.

$\therefore \angle APB$ is measured by arc AB, by § 653. Q.E.D.

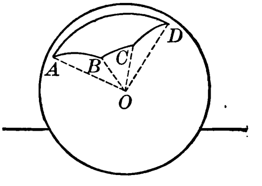
655. COROLLARY 1. *A spherical angle has the same measure as the dihedral angle formed by the planes of the two circles.*

656. COROLLARY 2. *All arcs of great circles drawn through the pole of a given great circle are perpendicular to the given circle.*

657. Spherical Polygon. A portion of a spherical surface bounded by three or more arcs of great circles is called a *spherical polygon.*

The bounding arcs are called the *sides* of the polygon, the angles between the sides are called the *angles* of the polygon, and the points of intersection of the sides are called the *vertices* of the polygon.

658. Relation of Polygons to Polyhedral Angles. The planes of the sides of a spherical polygon form a polyhedral angle whose vertex is the center of the sphere, whose face angles are measured by the sides of the polygon, and whose dihedral angles have the same numerical measure as the angles of the polygon.



Thus the planes of the sides of the polygon *ABCD* form the polyhedral angle *O-ABCD*. The face angles *BOA*, *COB*, and so on, are measured by the sides *AB*, *BC*, and so on, of the polygon. The dihedral angle whose edge is *OA* has the same measure as the spherical angle *BAD*, and so on.

Hence from any property of polyhedral angles we may infer an analogous property of spherical polygons; and conversely.

659. Convex Spherical Polygon. If a polyhedral angle at the center of a sphere is convex (§ 491), the corresponding spherical polygon is said to be *convex.*

Every spherical polygon is assumed to be convex unless the contrary is stated.

660. Diagonal. An arc of a great circle joining two non-consecutive vertices of a spherical polygon is called a *diagonal.*

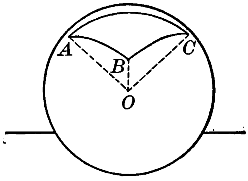
661. Spherical Triangle. A spherical polygon of three sides is called a *spherical triangle.*

A spherical triangle may be *right*, *obtuse*, or *acute*. It may also be *equilateral*, *isosceles*, or *scalene*.

662. Congruent Spherical Polygons. If two spherical polygons can be applied, one to the other, so as to coincide, they are said to be *congruent.*

PROPOSITION IX. THEOREM

663. *Each side of a spherical triangle is less than the sum of the other two sides.*



Given a spherical triangle *ABC*, *CA* being the longest side.

To prove that $CA < AB + BC$.

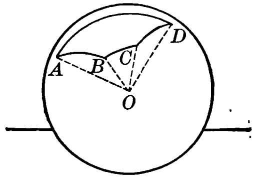
Proof. In the corresponding trihedral angle $O\text{-}ABC$,

$\angle COA$ is less than $\angle BOA + \angle COB$. § 494

$\therefore CA < AB + BC$, by § 658. Q.E.D.

PROPOSITION X. THEOREM

664. *The sum of the sides of a spherical polygon is less than 360°.*



Given a spherical polygon *ABCD*.

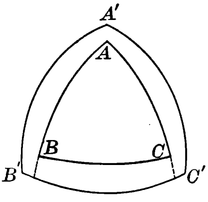
To prove that $AB + BC + CD + DA < 360°$.

Proof. In the corresponding polyhedral angle $O\text{-}ABCD$,

$\angle BOA + \angle COB + \angle DOC + \angle DOA < 360°$. § 495

$\therefore AB + BC + CD + DA < 360°$, by § 658. Q.E.D.

665. Polar Triangle. If from the vertices of a spherical triangle as poles arcs of great circles are described, another spherical triangle is formed which is called the *polar triangle* of the first.



Thus, if A is the pole of the arc of the great circle $B'C'$, B of $C'A'$, C of $A'B'$, $A'B'C'$ is the polar triangle of ABC.

If, with A, B, C as poles, entire great circles are described, these circles divide the surface of the sphere into *eight* spherical triangles.

Of these eight triangles, that one is the polar of ABC whose vertex A', corresponding to A, lies on the same side of BC as the vertex A; and similarly for the other vertices.

EXERCISE 102

1. To bisect a given great-circle arc.

What must be done to the angle at the center?

2. If two great-circle arcs intersect, the vertical angles are equal.

3. To describe an arc of a great circle through a given point and perpendicular to a given arc of a great circle.

4. Every point lying on a great circle which bisects a given arc of another great circle at right angles is equidistant (§ 635) from the extremities of the given arc.

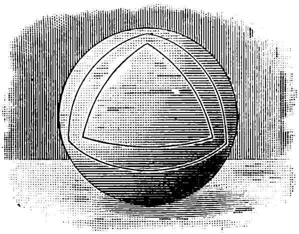
5. Two sides of a spherical triangle are respectively 82° 47′ and 67° 39′. What is known concerning the number of degrees in the third side?

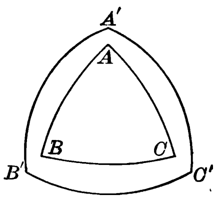
6. Three sides of a spherical quadrilateral are respectively 86° 29′, 73° 47′, and 69° 54′. What is known concerning the number of degrees in the fourth side?

7. Draw a picture of a sphere, and of an equilateral spherical triangle on the sphere, each side being 90°. Then draw a picture of the polar triangle.

Proposition XI. Theorem

666. *If one spherical triangle is the polar triangle of another, then reciprocally the second is the polar triangle of the first.*





Given the triangle ABC and its polar triangle $A'B'C'$.

To prove that ABC is the polar triangle of $A'B'C'$.

Proof. Since A is the pole of $B'C'$,

and C is the pole of $A'B'$, § 665

$\therefore B'$ is at a quadrant's distance from A and C. § 639

$\therefore B'$ is the pole of arc AC. § 641

Similarly A' is the pole of BC,

and C' is the pole of AB.

$\therefore ABC$ is the polar triangle of $A'B'C'$, by § 665. Q.E.D.

Discussion. Is it necessary that one of the triangles should be wholly within the other? Draw the figures approximately, without using instruments, starting with $\triangle ABC$ having $AB = 100°$, $AC = 100°$, $BC = 30°$.

Also draw the figures having $AB = 120°$, $AC = 80°$, $BC = 40°$.

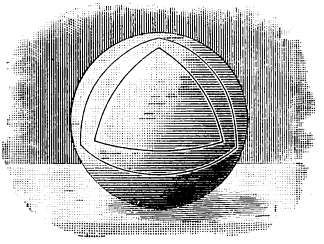
Also draw the figures suggested in Ex. 7, on page 394, where $AB = BC = CA = 90°$. Consider the proposition with these figures.

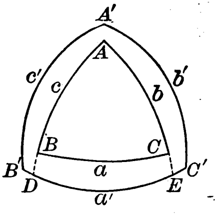
The proposition may also be considered by starting with $\triangle A'B'C'$ as the polar triangle of $\triangle A'B'C'$, and proving that $\triangle A'B'C'$ is the polar triangle of $\triangle ABC$.

It is desirable in the study of spherical triangles to have a spherical blackboard. Where this is not available, any wooden ball will serve the purpose.

Proposition XII. Theorem

667. *In two polar triangles each angle of the one is the supplement of the opposite side in the other.*





Given two polar triangles *ABC* and *A'B'C'*, the letter at the vertex of each angle denoting its value in degrees, and the small letter denoting the value of the opposite side in degrees.

To prove that $A + a' = 180°$, $B + b' = 180°$, $C + c' = 180°$;
$A' + a = 180°$, $B' + b = 180°$, $C' + c = 180°$.

Proof. Produce the arcs AB, AC until they meet $B'C'$ at the points D, E respectively.

Since B' is the pole of AE, $\therefore B'E = 90°$. § 639

And since C' is the pole of AD, $\therefore DC' = 90°$.

$\therefore B'E + DC' = 180°$. Ax. 1

That is, $B'D + DE + DC' = 180°$, Ax. 9

or $DE + B'C' = 180°$. Ax. 9

But DE is the measure of the $\angle A$, § 654

and $B'C' = a'$.

$\therefore A + a' = 180°$.

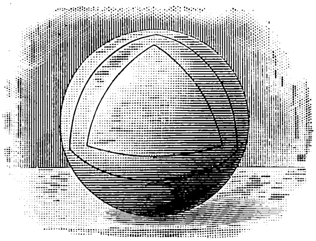
Similarly $B + b' = 180°$,

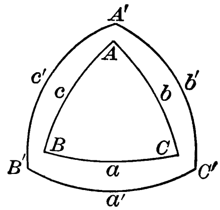
and $C + c' = 180°$.

In a similar way, starting with $\triangle A'B'C'$ and producing the sides of $\triangle ABC$, all the other relations are proved. Q.E.D.

Proposition XIII. Theorem

668. *The sum of the angles of a spherical triangle is greater than 180° and less than 540°.*





Given a spherical triangle *ABC*, the letter at the vertex of each angle denoting its value in degrees, and the small letter denoting the value of the opposite side in degrees.

To prove that $A + B + C > 180°$ *and* $< 540°$.

Proof. Let $\triangle A'B'C'$ be the polar triangle of $\triangle ABC$.

Then $A + a' = 180°$, $B + b' = 180°$, $C + c' = 180°$. § 667

$\therefore A + B + C + a' + b' + c' = 540°$. Ax. 1

$\therefore A + B + C = 540° - (a' + b' + c')$. Ax. 2

Now $a' + b' + c' < 360°$. § 664

$\therefore A + B + C = 540° -$ some value less than 360°.

$\therefore A + B + C > 180°$.

Again $a' + b' + c'$ is greater than 0°.

$\therefore A + B + C < 540°$. Q.E.D.

669. Corollary. *A spherical triangle may have two, or even three, right angles; and a spherical triangle may have two, or even three, obtuse angles.*

670. Triangles classified as to Right Angles. A spherical triangle having two right angles is said to be *birectangular*; one having three right angles is said to be *trirectangular*.

The same terms may be applied to the corresponding trihedral angles.

EXERCISE 103

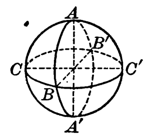
1. If two sides of a spherical triangle are quadrants, the third side measures the opposite angle.

2. In a birectangular spherical triangle the sides opposite the right angles are quadrants, and the side opposite the third angle measures that angle.

Since the ∠s are rt. ∠s, what two planes are ⊥ to a third plane? What two arcs must therefore (§ 632) pass through the pole of a third arc? Then what two arcs are quadrants? Then how is the third angle (§ 654) measured?

3. Each side of a trirectangular spherical triangle is a quadrant.

4. Three planes passed through the center of a sphere, each perpendicular to the other two, divide the spherical surface into eight congruent trirectangular triangles.



Find the number of degrees in the sides of a spherical triangle given the angles of its polar triangle as follows:

5. $82°, 77°, 69°$.

6. $84\frac{1}{2}°, 81\frac{3}{4}°, 72\frac{1}{6}°$.

7. $78° 30', 89°, 102°$.

8. $83° 40', 48° 57', 103° 43'$.

9. $96° 37' 40'', 82° 29' 30'', 68° 47'$.

10. $43° 29' 37'', 98° 22' 53'', 87° 36' 39''$.

Find the number of degrees in the angles of a spherical triangle, given the sides of its polar triangle as follows:

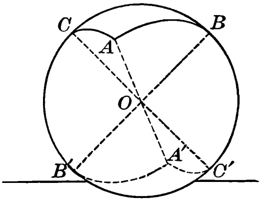
11. $68° 42' 39'', 93° 48' 7'', 89° 38' 14''$.

12. $78° 47' 29'', 106° 36' 42''$, a quadrant.

13. A quadrant, half a quadrant, three fourths of a quadrant.

14. From the center of a sphere are drawn three radii, each perpendicular to the other two. Find the number of degrees in the sides and angles of the spherical triangle determined by their extremities.

671. Symmetric Spherical Triangles. If through the center O of a sphere three diameters AA', BB', CC' are drawn, and the points A, B, C are joined by arcs of great circles, and also the points A', B', C', the two spherical triangles ABC and $A'B'C'$ are called *symmetric spherical triangles.*

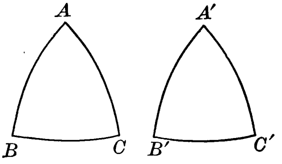


In the same way we may form two symmetric polygons of any number of sides. Having thus formed the symmetric polygons, we may place them in any position we choose upon the surface of the sphere.

672. Relation of Symmetric Triangles. Two symmetric triangles are mutually equilateral and mutually equiangular; yet in general they are not congruent, for they cannot be made to coincide by superposition. If in the above figure the triangle ABC is made to slide on the surface of the sphere until the vertex A falls on A', it is evident that the two triangles cannot be made to coincide for the reason that the corresponding parts of the triangles occur in *reverse order.*

To try to make two symmetric spherical polygons coincide is very much like trying to put the right-hand glove on the left hand. The relation of two symmetric spherical triangles may be illustrated by cutting them out of the peel of an orange or an apple.

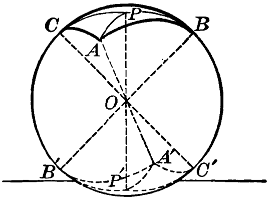
673. Symmetric Isosceles Triangles. If, however, we have two symmetric triangles ABC and $A'B'C'$ such that $AB = AC$, and $A'B' = A'C'$, that is, if the two symmetric triangles are *isosceles,* then because AB, AC, $A'B'$, $A'C'$ are all equal and the angles A and A' are equal, being originally formed by vertical dihedral angles (§ 671), the two triangles can be made to coincide. Therefore,



If two symmetric spherical triangles are isosceles, they are superposable and therefore are congruent.

Proposition XIV. Theorem

674. *Two symmetric spherical triangles are equivalent.*



Given two symmetric spherical triangles *ABC*, *A'B'C'*, having their corresponding vertices opposite each to each with respect to the center of the sphere.

To prove that the triangles ABC, A'B'C' are equivalent.

Proof. Let P be the pole of a small circle passing through the points A, B, C, and let POP' be a diameter.

Draw the great-circle arcs PA, PB, PC, $P'A'$, $P'B'$, $P'C'$.

Then $PA = PB = PC$. § 636

Now $P'A' = PA$, $P'B' = PB$, $P'C' = PC$. § 672

$\therefore P'A' = P'B' = P'C'$. Ax. 8

$\therefore$ the two symmetric $\triangle$s PCA and $P'C'A'$ are isosceles.

$\therefore \triangle PCA$ is congruent to $\triangle P'C'A'$. § 673

Similarly $\triangle PAB$ is congruent to $\triangle P'A'B'$,

and $\triangle PBC$ is congruent to $\triangle P'B'C'$.

Now $\triangle ABC = \triangle PCA + \triangle PAB + \triangle PBC$,

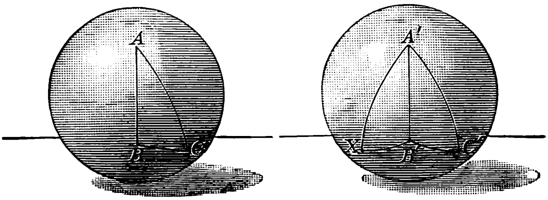
and $\triangle A'B'C' = \triangle P'C'A' + \triangle P'A'B' + \triangle P'B'C'$. Ax. 11

$\therefore \triangle ABC$ is equivalent to $\triangle A'B'C'$, by Ax. 9. Q.E.D.

Discussion. If the pole P should fall without the $\triangle ABC$, then P' would fall without $\triangle A'B'C'$, and each triangle would be equivalent to the sum of two symmetric isosceles triangles diminished by the third; so that the result would be the same as before.

PROPOSITION XV. THEOREM

675. *Two triangles on the same sphere or on equal spheres are either congruent or symmetric if two sides and the included angle of the one are respectively equal to the corresponding parts of the other.*



Given two spherical triangles ABC and $A'B'C'$, with $AB = A'B'$, $AC = A'C'$, and angle A = angle A', and similarly arranged; and given the triangle $A'B'X$ symmetric with respect to the triangle $A'B'C'$.

To prove that $\triangle ABC$ is congruent to $\triangle A'B'C'$, and that $\triangle ABC$ is symmetric with respect to $\triangle A'B'X$.

Proof. Superpose $\triangle ABC$ on $\triangle A'B'C'$, the proof being similar to that of the corresponding case in plane geometry. § 68

$\therefore \triangle ABC$ is congruent to $\triangle A'B'C'$. § 662

Since $\triangle A'B'X$ is symmetric with respect to $\triangle A'B'C'$,

and $\triangle ABC$ is congruent to $\triangle A'B'C'$,

$\therefore AC = A'X,\ AB = A'B',\ \angle A = \angle XA'B'$.

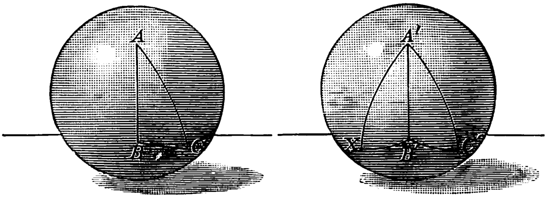
But $\triangle ABC$ is congruent to $\triangle A'B'C'$ and may be made to coincide with it.

$\therefore \triangle ABC$ is symmetric with respect to $\triangle A'B'X$. Q.E.D.

Discussion. In the case of plane triangles, if the corresponding parts are arranged in reverse order, we can still prove the triangles congruent. Why can we not do so in the case of spherical triangles?

Proposition XVI. Theorem

676. *Two triangles on the same sphere or on equal spheres are either congruent or symmetric if two angles and the included side of the one are respectively equal to the corresponding parts of the other.*



Given two spherical triangles ABC and $A'B'C'$, with angle $A =$ angle A', angle $C =$ angle C', and $AC = A'C'$, and similarly arranged; and given the triangle $A'B'X$ symmetric with respect to the triangle $A'B'C'$.

To prove that $\triangle ABC$ is congruent to $\triangle A'B'C'$, and that $\triangle ABC$ is symmetric with respect to $\triangle A'B'X$.

Proof. Superpose $\triangle ABC$ on $\triangle A'B'C'$, the proof being similar to that of the corresponding case in plane geometry. § 72

$\therefore \triangle ABC$ is congruent to $\triangle A'B'C'$. § 662

Since $\triangle A'B'X$ is symmetric with respect to $\triangle A'B'C'$, and $\triangle ABC$ is congruent to $\triangle A'B'C'$,

$$\therefore \angle A = \angle XA'B',\ \angle C = \angle X, \text{ and } AC = A'X.$$

But $\triangle ABC$ is congruent to $\triangle A'B'C'$ and may be made to coincide with it.

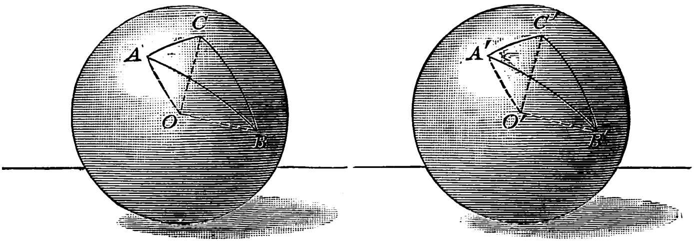
$\therefore \triangle ABC$ is symmetric with respect to $\triangle A'B'X$. Q.E.D.

Discussion. Under what circumstances are the two triangles both congruent and symmetric?

In plane geometry what is the case that corresponds to the one in which the spherical triangles are both congruent and symmetric?

Proposition XVII. Theorem

677. *Two mutually equilateral triangles on the same sphere or on equal spheres are mutually equiangular, and are either congruent or symmetric.*



Given two spherical triangles, ABC, $A'B'C'$, on equal spheres, such that $AB = A'B'$, $BC = B'C'$, $CA = C'A'$.

To prove that $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$, and that △ ABC and $A'B'C'$ are either congruent or symmetric.

Proof. Let O and O' be the centers of the spheres.

Pass a plane through each pair of vertices of each triangle and the center of its sphere.

Then in the trihedral angles at O and O' the face angles are equal each to its corresponding face angle. § 167

∴ the corresponding dihedral ∠s are respectively equal. § 499

∴ the ∠s of the spherical △ are respectively equal. § 655

∴ the △ are either congruent or symmetric, by § 676. Q.E.D.

Discussion. In the figures the parts are arranged in the same order, so that the triangles are congruent. They might be arranged as in the figures of § 676.

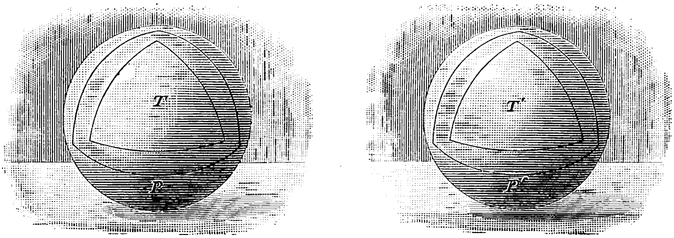
Discuss the proposition when the triangles are equilateral and each side is a quadrant.

Discuss the proposition when two sides of each triangle are quadrants.

What is the corresponding proposition in plane geometry, and why does not the form of proof there given hold here?

Proposition XVIII. Theorem

678. *Two mutually equiangular triangles on the same sphere or on equal spheres are mutually equilateral, and are either congruent or symmetric.*



Given two mutually equiangular spherical triangles T and T' on equal spheres.

To prove that T and T' are mutually equilateral, and are either congruent or symmetric.

Proof. Let the $\triangle P$ be the polar triangle of $\triangle T$, and the $\triangle P'$ be the polar triangle of $\triangle T'$.

Since the $\triangle$s T and T' are mutually equiangular, Given

$\therefore$ the polar $\triangle$s P and P' are mutually equilateral. § 667

$\therefore$ the polar $\triangle$s P and P' are mutually equiangular. § 677

But the $\triangle$s T and T' are the polar $\triangle$s of $\triangle$s P and P'. § 666

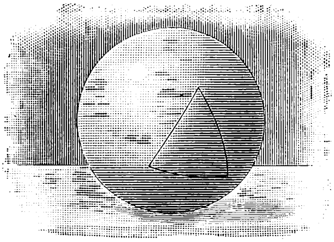
$\therefore$ the $\triangle$s T and T' are mutually equilateral. § 667

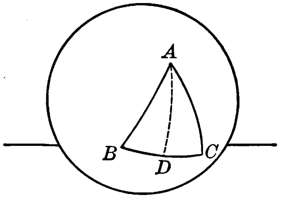
Therefore the $\triangle$s T and T' are either congruent or symmetric, by § 677. Q.E.D.

Discussion. The statement that mutually equiangular spherical triangles are mutually equilateral, and are either congruent or symmetric, is true only when they are on the same sphere or on equal spheres. When the spheres are unequal, the spherical triangles are unequal. In this case, however, their sides have the same arc measure, and therefore have the same ratio as the circumferences or as the radii of the spheres (§ 382).

PROPOSITION XIX. THEOREM

679. *In an isosceles spherical triangle the angles opposite the equal sides are equal.*





Given the spherical triangle *ABC*, with *AB* equal to *AC*.

To prove that $\angle B = \angle C$.

Proof. Draw the arc AD of a great circle, from the vertex A to the mid-point of the base BC.

Then △s ABD and ACD are mutually equilateral.

∴ △s ABD and ACD are mutually equiangular. § 677

$\therefore \angle B = \angle C$. Q.E.D.

EXERCISE 104

1. The radius of a sphere is 4 in. From any point on the surface as a pole a circle is described upon the sphere with an opening of the compasses equal to 3 in. Find the area of this circle.

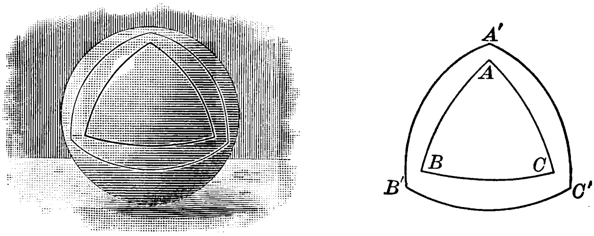
2. The edge of a regular tetrahedron is a. Find the radii r, r' of the inscribed and circumscribed spheres.

3. Find the diameter of the section of a sphere of diameter 10 in. made by a plane 3 in. from the center.

4. The arc of a great circle drawn from the vertex of an isosceles spherical triangle to the mid-point of the base bisects the vertical angle, is perpendicular to the base, and divides the triangle into two symmetric triangles.

PROPOSITION XX. THEOREM

680. *If two angles of a spherical triangle are equal, the sides opposite these angles are equal and the triangle is isosceles.*



Given the spherical triangle *ABC*, with angle *B* equal to angle *C*.

To prove that $AC = AB$.

Proof. Let $\triangle A'B'C'$ be the polar triangle of $\triangle ABC$.

Since $\angle B = \angle C, \quad \therefore A'C' = A'B'$. § 667

$\therefore \angle B' = \angle C'$. § 679

$\therefore AC = AB$, by § 667. Q.E.D.

EXERCISE 105

1. To bisect a given spherical angle.

2. To construct a spherical triangle, given two sides and the included angle.

3. To construct a spherical triangle, given two angles and the included side.

4. To construct a spherical triangle, given the three sides.

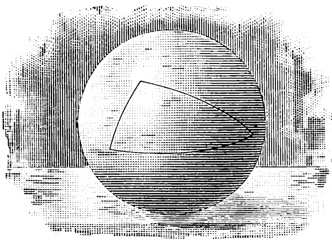
5. To construct a spherical triangle, given the three angles.

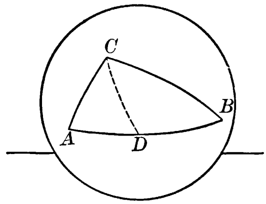
6. To pass a plane tangent to a given sphere at a given point on the surface of the sphere.

7. To pass a plane tangent to a given sphere through a given straight line without the sphere.

Proposition XXI. Theorem

681. *If two angles of a spherical triangle are unequal, the sides opposite these angles are unequal, and the side opposite the greater angle is the greater; and if two sides are unequal, the angles opposite these sides are unequal, and the angle opposite the greater side is the greater.*





Given the triangle *ABC*, with angle *C* greater than angle *B*.

To prove that $AB > AC$.

Proof. Draw the arc CD of a great circle, making $\angle DCB$ equal to $\angle B$. Then $DB = DC$. § 680

Now $AD + DC > AC$. § 663

$\therefore AD + DB > AC$, or $AB > AC$, by Ax. 9. Q.E.D.

Given the triangle *ABC*, with *AB* greater than *AC*.

To prove that $\angle C$ *is greater than* $\angle B$.

Proof. The $\angle C$ must be equal to, less than, or greater than the $\angle B$.

If $\angle C = \angle B$, then $AB = AC$; § 680

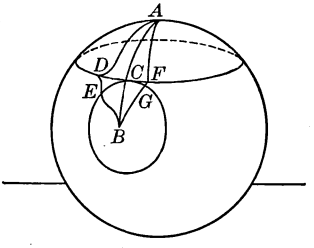
and if $\angle C$ is less than $\angle B$, then $AB < AC$, as above.

But both of these conclusions are contrary to what is given.

$\therefore \angle C$ is greater than $\angle B$. Q.E.D.

Proposition XXII. Theorem

682. *The shortest line that can be drawn on the surface of a sphere between two points is the arc of a great circle joining the two points, not greater than a semicircle.*



Given AB, the arc of a great circle, not greater than a semicircle, joining the points A and B.

To prove that AB is the shortest line that can be drawn on the surface joining A and B.

Proof. Let C be any point in AB.

With A and B as poles and AC and BC as polar distances, describe two arcs DCF and GCE.

The arcs DCF and GCE have only the point C in common. For if F is any other point in DCF, and if arcs of great circles AF and BF are drawn, then

$$AF = AC. \qquad \S\ 636$$

But $$AF + BF > AC + BC. \qquad \S\ 663$$

Take away AF from the left member of the inequality, and its equal AC from the right member.

Then $$BF > BC. \qquad \text{Ax. 6}$$

Therefore $$BF > BG, \text{ the equal of } BC. \qquad \text{Ax. 9}$$

Hence F lies outside the circle whose pole is B, and the arcs DCF and GCE have only the point C in common.

Now let $ADEB$ be any line from A to B on the surface of the sphere, which does not pass through C.

This line will cut the arcs DCF and GCE in separate points D and E; and if we revolve the line AD about A as a fixed point until D coincides with C, we shall have a line from A to C equal to the line AD.

In like manner, we can draw a line from B to C equal to the line BE.

Therefore a line can be drawn from A to B through C that is equal to the sum of the lines AD and BE, and hence is less than the line $ADEB$ by the line DE.

Therefore no line which does not pass through C can be the shortest line from A to B.

Therefore the shortest line from A to B passes through C.

But C is any point in the arc AB.

Therefore the shortest line from A to B passes through every point of the arc AB, and consequently coincides with the arc AB.

Therefore the shortest line from A to B is the great-circle arc AB. Q.E.D.

EXERCISE 106

1. The three medians of a spherical triangle are concurrent.

2. To construct with a given radius a spherical surface that passes through three given points.

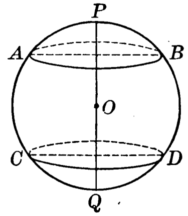
3. To construct with a given radius a spherical surface that passes through two given points and is tangent to a given plane.

4. To construct with a given radius a spherical surface that passes through two given points and is tangent to a given sphere.

5. The smallest circle on a given sphere whose plane passes through a given point within the sphere is the circle whose plane is perpendicular to the radius through the given point.

683. Zone. A portion of a spherical surface included between two parallel planes is called a *zone.*

Thus on the earth we have the torrid zone included between the planes of the tropics of Cancer and Capricorn.



The circles made by the planes are called the *bases* of the zone, and the distance between the planes is called the *altitude* of the zone.

If one of the planes is tangent to the sphere and the other plane cuts the sphere, the zone is called a *zone of one base.*

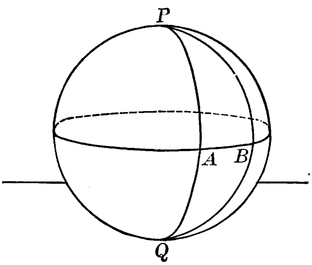
If both planes are tangent to the sphere, the zone is a complete spherical surface.

684. Generation of a Zone. If a great circle revolves about its diameter as an axis, any arc of the circle generates a zone.

Thus, in the figure of § 683, if the great circle $PACQ$ revolves about its diameter PQ as an axis, the arc AC generates the zone AD, of which the altitude is the distance between the parallel planes. Similarly, the arc AP generates the zone ABP, and the arc CQ generates the zone CDQ, these both being zones of one base.

685. Lune. A portion of a spherical surface bounded by the halves of two great circles is called a *lune.*

Thus $PAQB$ is a lune. A lune is evidently generated by the partial or complete revolution of half of a great circle about its diameter as an axis.



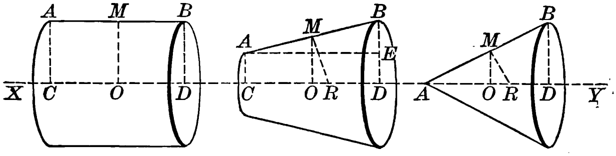
686. Angle of a Lune. The angle between the semicircles bounding a lune is called the *angle of the lune.*

Thus $\angle APB$ is the angle of the lune $PAQB$.

A lune is usually taken as having an angle less than a straight angle. This is not necessary, for we may consider a hemispherical surface as a lune with an angle of 180°. We may also conceive of lunes with angles greater than a straight angle, and we may even think of an entire spherical surface as a lune whose angle is 360°.

Proposition XXIII. Theorem

687. *The area of the surface generated by a straight line revolving about an axis in its plane is equal to the product of the projection of the line on the axis by the circle whose radius is a perpendicular erected at the mid-point of the line and terminated by the axis.*



Given an axis *XY* about which a line *AB* in the same plane with *XY* revolves, *M* being the mid-point of *AB*, *CD* being the projection of *AB* on *XY*, *MO* being perpendicular to *XY*, *MR* being perpendicular to *AB*, and *a* being the area generated by *AB*.

To prove that $a = CD \times 2\pi MR$.

Proof. 1. If AB is ∥ to XY, $CD = AB$, MR coincides with MO, and a is the lateral area of a right cylinder. § 588

2. If AB is not ∥ to XY, and does not cut XY, a is the lateral area of the frustum of a cone of revolution.

$\therefore a = AB \times 2\pi MO.$ § 616

Draw AE ∥ to XY.

The △ AEB and MOR are similar. § 290

$\therefore MO : AE = MR : AB.$ § 282

$\therefore AB \times MO = AE \times MR,$ § 261

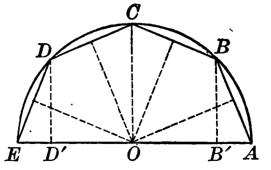
or $AB \times MO = CD \times MR.$ Ax. 9

Substituting, $a = CD \times 2\pi MR.$

3. If A lies in the axis XY, then AE and CD coincide, and $a = CD \times 2\pi MR$, by § 609. Q.E.D.

Proposition XXIV. Theorem

688. *The area of the surface of a sphere is equal to the product of the diameter by the circumference of a great circle.*



Given a sphere generated by the semicircle *ABCDE* revolving about the diameter *AE* as an axis, *s* being the area of the surface, *r* being the radius, and *d* being the diameter.

To prove that $s = 2\,\pi rd.$

Proof. Inscribe in the semicircle half of a regular polygon having an even number of sides, as $ABCDE$.

From the center O draw ⊥s to the chords AB, BC, CD, DE.

These ⊥s bisect the chords (§ 174) and are equal. § 178

Let l denote the length of each of these ⊥s.

From B, C, and D drop perpendiculars to AE.

Then area of surface generated by $AB = AB' \times 2\,\pi l$, § 687

area of surface generated by $BC = B'O \times 2\,\pi l$, etc.

∴ area of surface generated by $ABCDE = AE \times 2\,\pi l$ Ax. 1

$= 2\,\pi ld.$ Ax. 9

Denote the area of the surface generated by $ABCDE$ by s', and let the number of sides of $ABCDE$ be indefinitely increased.

Then s' approaches s as a limit,

l approaches r as a limit, § 377

and consequently $2\,\pi ld$ approaches $2\,\pi rd$ as a limit.

But $s' = 2\,\pi ld$, always. § 687

∴ $s = 2\,\pi rd$, by § 207. Q.E.D.

689. COROLLARY 1. *The area of the surface of a sphere is equivalent to the area of four great circles, or to* $4\pi r^2$.

In $s = 2\pi rd$, what is the value of d in terms of r? Then what is the value of s in terms of r?

For example, if the radius is 10 in., the area of the surface of the sphere is $4\pi \cdot 100$ sq. in., or 1256.64 sq. in.

690. COROLLARY 2. *The areas of the surfaces of two spheres are to each other as the squares on their radii, or as the squares on their diameters.*

If the radii are r and r', the diameters d and d', and the surfaces s and s', then what is the ratio of s to s', according to § 689? Show that this also equals $r^2 : r'^2$, and $d^2 : d'^2$.

691. COROLLARY 3. *The area of a zone is equal to the product of the altitude by the circumference of a great circle.*

If we apply the reasoning of § 688 to the zone generated by the revolution of the arc BCD, we obtain

$$\text{the area of zone } BCD = B'D' \times 2\pi r,$$

where $B'D'$ is the altitude of the zone and $2\pi r$ the circumference of a great circle.

For example, if the radius is 10 in., and the altitude is 5 in., the area of the zone is $5 \cdot 2\pi \cdot 10$ sq. in., or 314.16 sq. in.

692. COROLLARY 4. *The area of a zone of one base is equivalent to the area of a circle whose radius is the chord of the generating arc.*

The arc AB generates a zone of one base.

$$\therefore \text{the area of the zone } AB = AB' \times 2\pi r = \pi AB' \times AE.$$

But $$AB' \times AE = \overline{AB}^2. \qquad \S\ 298$$

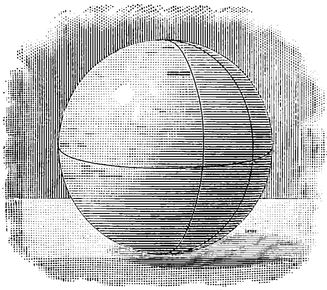
$$\therefore \text{the area of the zone } AB = \pi\overline{AB}^2.$$

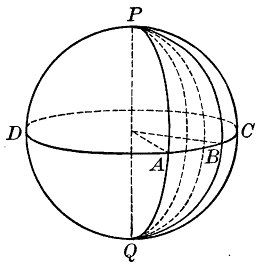
693. Spherical Excess of a Triangle. The excess of the sum of the angles of a spherical triangle over 180° is called the *spherical excess* of the triangle.

For example, if the angles of a spherical triangle are 80°, 90°, and 100°, the spherical excess of the triangle is 90°.

Proposition XXV. Theorem

694. *The area of a lune is to the area of the surface of the sphere as the angle of the lune is to four right angles.*





Given a lune *PAQB*, the great circle *ABCD* whose pole is *P*, *a* the value in degrees of the angle of the lune, *l* the area of the lune, and *s* the area of the surface of the sphere.

To prove that $l : s = a : 4$ *rt.* ∠s.

Proof. The arc AB measures the $\angle a$ of the lune. § 654

Hence arc AB : circle $ABCD = a : 4$ rt. ∠s. § 212

If AB and $ABCD$ are commensurable, let their common measure be contained m times in AB, and n times in $ABCD$.

Then arc AB : circle $ABCD = m : n$.

$\therefore a : 4$ rt. ∠s $= m : n$.

Pass an arc of a great circle through the poles P and Q and each point of division of $ABCD$.

These arcs will divide the entire surface into n equal lunes, of which the lune $PAQB$ will contain m.

$\therefore l : s = m : n$.

$\therefore l : s = a : 4$ rt. ∠s. Ax. 8

If AB and $ABCD$ are incommensurable, the theorem can be proved by the method of limits as in § 472. Q.E.D.

EXERCISE 107

Using $\pi = 3.1416$ for all examples in this exercise, find the areas of spheres whose radii are as follows:

1. 2 in.	**3.** $3\frac{1}{2}$ in.	**5.** 2 ft. 1 in.	**7.** 48.8 in.
2. 7 in.	**4.** $5\frac{7}{8}$ in.	**6.** 3 ft. 6 in.	**8.** 4000 mi.

Find the radii of spheres whose areas are as follows:

9. 12.5664 sq. in.	**11.** 1 sq. ft.	**13.** s.
10. 50.2656 sq. in.	**12.** $100\,\pi$ sq. in.	**14.** $4\,\pi^3$.

On a sphere whose radius is 20 in., find the areas of zones whose altitudes are as follows:

15. 2 in.	**17.** 7 in.	**19.** 1 ft.	**21.** 3.45 in.
16. 3 in.	**18.** 10 in.	**20.** $2\frac{1}{2}$ in.	**22.** 6.83 in.

On a sphere whose radius is 10 in., find the areas of lunes whose angles are as follows:

23. 30°.	**25.** 90°.	**27.** 22° 30′.	**29.** 52° 20′ 20″.
24. 45°.	**26.** 180°.	**28.** 7° 30′.	**30.** 48° 35′ 10″.

31. Two lunes on the same sphere or on equal spheres have the same ratio as their angles.

32. The area of a lune is equal to one ninetieth of the area of a great circle multiplied by the number of degrees in the angle of the lune.

33. Zones on the same sphere or on equal spheres are to each other as their altitudes.

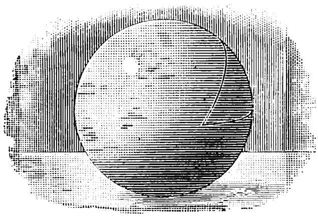
34. Given the radius of a sphere 15 in., find the area of a lune whose angle is 30°.

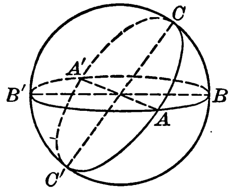
35. Given the diameter of a sphere 16 in., find the area of a lune whose angle is 75°.

36. What is the spherical excess of a trirectangular triangle?

PROPOSITION XXVI. THEOREM

695. *A spherical triangle is equivalent to a lune whose angle is half the spherical excess of the triangle.*





Given the spherical triangle *ABC* on a sphere of surface *s*.

To prove that $\triangle ABC$ *is equivalent to a lune whose angle is* $\frac{1}{2}(\angle A + \angle B + \angle C - 180°)$.

Proof. Produce the sides of the $\triangle ABC$ to complete circles.

Now $\triangle$s $AB'C'$ and $A'BC$ are symmetric. Const.

$\therefore \triangle AB'C'$ is equivalent to $\triangle A'BC$. § 674

$\therefore$ lune $ABA'C = \triangle ABC + \triangle AB'C'$. Ax. 9

But $\triangle CB'A + \triangle AC'B + \triangle AB'C' + \triangle ABC = \frac{1}{2}s$. Ax. 11

$\therefore$ (lune $BCB'A - \triangle ABC$) + (lune $CAC'B - \triangle ABC$) + lune $ABA'C = \frac{1}{2}s$. Ax. 9

$\therefore 2\triangle ABC =$ lune $BCB'A$ + lune $CAC'B$ + lune $ABA'C - \frac{1}{2}s$. Axs. 1, 2

$\therefore \triangle ABC = \frac{1}{2}$ (lune $BCB'A$ + lune $CAC'B$ + lune $ABA'C - \frac{1}{2}s$). Ax. 4

But $\frac{1}{2}s =$ a lune whose angle is 180°. § 694

$\therefore \triangle ABC =$ a lune whose angle is $\frac{1}{2}(\angle A + \angle B + \angle C - 180°)$. Q.E.D.

Discussion. Since we have found (§ 694) how to compute the area of a lune, we can now compute the area of a spherical triangle when the angles are known.

696. Corollary. *If two great-circle arcs intersect within a great circle, the sum of the two opposite spherical triangles which they form with the great circle is equivalent to a lune whose angle is the angle between the arcs.*

697. Computation of Area. To illustrate the computation involved in § 695, find the area of a triangle whose angles are 110°, 100°, and 95°, on the surface of a sphere whose radius is 6 in.

$$\text{Spherical excess} = 110° + 100° + 95° - 180° = 125°.$$

$$\therefore \text{ angle of lune} = 62\tfrac{1}{2}°.$$

$$\therefore \text{ area of lune} = \frac{62\frac{1}{2}}{360} \text{ of the spherical surface.}$$

$$\therefore \text{ area of lune} = \frac{62\frac{1}{2}}{360} \times 4 \times 3.1416 \times 36 \text{ sq. in.}$$

$$\therefore \text{ area of triangle} = 78.54 \text{ sq. in.}$$

698. Spherical Excess of a Polygon. The excess of the sum of the angles of a spherical polygon of n sides over $(n-2) \times 180°$ is called the *spherical excess* of the polygon.

EXERCISE 108

Compute the areas of triangles on spheres of the given diameters, the angles being as follows:

1. 100°, 120°, 140°, $d = 16$ in.

2. 105°, 130°, 125°, $d = 10$ in.

3. 127°, 132°, 90°, $d = 20$ in.

4. 115°, 124°, 85°, $d = 30$ in.

5. 135°, 110°, 92°, $d = 40$ in.

6. 148°, 93°, 68°, $d = 25.8$ in.

7. 115° 27′ 30″, 102° 32′ 48″, 68° 27′ 39″, $d = 8000$ mi.

Compute the areas of triangles on spheres of the given radii, the angles being as follows:

8. 120°, 100°, 90°, $r = 9$ in.

9. 130°, 90°, 80°, $r = 10$ in.

10. 105°, 75°, 65°, $r = 18$ in.

11. 115°, 102°, 30°, $r = 36$ in.

12. 140°, 120°, 85°, $r = 90$ in.

13. 136°, 117°, 93°, $r = 1.8$ in.

Compute the areas of triangles on spheres of the given circumferences, the angles being as follows:

14. 93°, 94°, 120°, $c = 31.416$ in.

15. 82°, 105°, 98°, $c = 62.832$ in.

16. 148°, 27°, 125°, $c = 15.708$ in.

17. 162°, 39°, 120°, $c = 78.54$ in.

18. 149°, 41°, 116°, $c = 39.27$ in.

19. 126° 30′ 42″, 105° 26′ 15″, 63° 15′ 3″, $c = 314.16$ in.

20. What is the area of a triangle on the earth's surface the vertices of which are the north pole and two points on the equator, one at 37° W. and the other at 16° E., the earth being considered a sphere with a radius of 4000 mi.?

21. If the radii of two spheres are 6 in. and 4 in. respectively, and the distance between the centers is 5 in., what is the area of the circle of intersection of the spheres?

22. Find the radius of the circle determined on a sphere of 5 in. diameter by a plane 1 in. from the center.

23. If the radii of two concentric spheres are r and r', and if a plane is passed tangent to the interior sphere, what is the area of the section made in the other sphere?

24. Two points A and B are 8 in. apart. Find the locus in space of a point 5 in. from A and 7 in. from B.

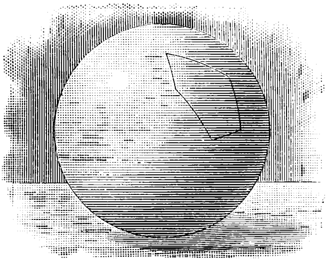
25. Two points A and B are 10 in. apart. Find the locus in space of a point 7 in. from A and 3 in. from B.

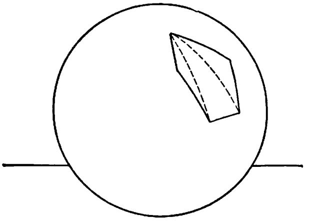
26. The radii of two parallel sections of the same sphere are a and b respectively, and the distance between the sections is d. Find the radius of the sphere.

27. The diameter of a certain sphere is $\sqrt{2}$. The chords of the arcs that form the sides of a triangle on the surface of the sphere are respectively 1, 1, and $\frac{1}{2}\sqrt{2}$. Find the area of the spherical triangle.

Proposition XXVII. Theorem

699. *A spherical polygon is equivalent to a lune whose angle is half the spherical excess of the polygon.*





Given a spherical polygon P of n sides, the sum of the angles being s.

To prove that P is equivalent to a lune whose angle is $\frac{1}{2}(s-\overline{n-2}\times 180°)$.

Proof. Draw all the diagonals from any vertex.

Since there is a distinct triangle for each side except those meeting at the vertex chosen, there are $(n-2)$ triangles.

Since each triangle is equivalent to a lune whose angle is half the excess of the sum of its angles over 180°, § 695

therefore the $(n-2)$ triangles are equivalent to a lune whose angle is half the excess of the sum of all the angles of the polygon over $(n-2)\times 180°$.

$\therefore P=$ a lune whose angle is $\frac{1}{2}(s-\overline{n-2}\times 180°)$. Q.E.D.

700. Computation of Area. Find the area of a spherical polygon whose angles are 100°, 110°, 120°, and 170°, r being 6 in.

Spherical excess $= 100° + 110° + 120° + 170° - 2 \times 180° = 140°$.

$\therefore$ angle of lune $= 70°$.

$\therefore$ area of lune $= \frac{70}{360}$ of $4\pi r^2$

$= \frac{7}{36}$ of $4 \times 3.1416 \times 36$ sq. in.

$= 87.9648$ sq. in.

EXERCISE 109

Find the areas of spherical polygons on spheres of the given areas, the angles being as follows:

1. 30°, 90°, 120°, 130°, $a = 2$ sq. ft.
2. 45°, 60°, 100°, 165°, $a = 288$ sq. in.
3. 70°, 168°, 92°, 120°, $a = 500$ sq. in.
4. 68° 30′, 149° 50′, 96° 54′, 136° 52′, $a = 750$ sq. in.
5. 122° 27′ 40″, 130° 32′ 50″, 98° 31′ 30″, 96° 48′, $a = 600$ sq. in.
6. 132°, 96°, 154°, 120°, 150°, $a = 3$ sq. ft. 120 sq. in.
7. 130°, 156°, 172°, 95°, 120°, 100°, $a = 157.2$ sq. in.

Find the areas of spherical polygons on spheres of the given radii, the angles being as follows:

8. 130°, 150°, 80°, 90°, $r = 10$ in.
9. 148°, 157°, 90°, 100°, 120°, $r = 20$ in.
10. 172°, 169°, 86°, 141°, 100°, 90°, $r = 24$ in.
11. 135° 30′, 148° 42′, 96° 37′, 102° 11′, $r = 10$ in.

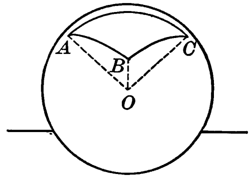
Find the areas of spherical polygons on spheres of the given diameters, the angles being as follows:

12. 148°, 92°, 60°, 120°, $d = 10$ in.
13. 172°, 168°, 93°, 37°, 100°, $d = 22$ in.
14. 102°, 162°, 139°, 141°, 138°, 126°, $d = 20$ in.
15. 82° 50′ 42″, 120° 29′ 18″, 98° 37′ 15″, 141° 22′ 45″, $d = 20$ in.

Find the areas of spherical polygons on spheres of the given circumferences, the angles being as follows:

16. 39°, 148°, 172°, 168°, $c = 3.1416$ in.
17. 128°, 92°, 168°, 109°, $c = 31.416$ in.
18. 146°, 129°, 102°, 137°, 100°, $c = 6.2832$ in.
19. 128°, 145°, 139°, 82°, 161°, 137°, $c = 18.8496$ in.

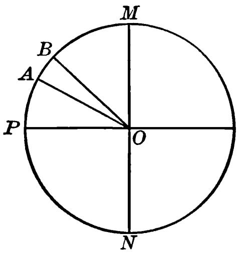
701. Spherical Pyramid. A portion of a sphere bounded by a spherical polygon and the planes of its sides is called a *spherical pyramid.*

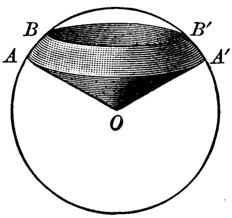


The center of the sphere is called the *vertex* of the spherical pyramid, and the spherical polygon is called the *base.*

Thus $O\text{-}ABC$ is a spherical pyramid.

702. Spherical Sector. A portion of a sphere generated by the revolution of a circular sector about any diameter of the circle of which the sector is a part is called a *spherical sector.*





Thus if the sector AOB revolves about the diameter MN as an axis, it generates the spherical sector $AB\text{-}O\text{-}A'B'$.

The zone generated by the arc of the generating sector is called the *base* of the spherical sector.

703. Spherical Segment. A portion of a sphere contained between two parallel planes is called a *spherical segment.*

The sections of the sphere made by the parallel planes are called the *bases* of the spherical segment, and the distance between these bases is called the *altitude* of the spherical segment.

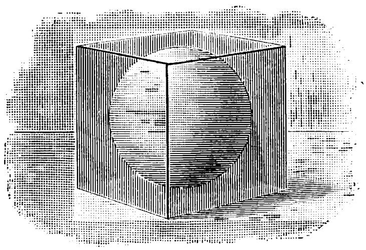
If one of the parallel planes is tangent to the sphere, the segment is called a *spherical segment of one base.*

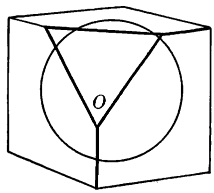
A spherical segment of one base may be generated by the revolution of a circular segment about the diameter perpendicular to its base.

704. Spherical Wedge. A portion of a sphere bounded by a lune and the planes of two great circles is called a *spherical wedge.*

PROPOSITION XXVIII. THEOREM

705. *The volume of a sphere is equal to the product of the area of its surface by one third of its radius.*





Given a sphere of radius *r*, area of surface *s*, volume *v*, and center *O*.

To prove that $v = s \times \frac{1}{3} r.$

Proof. We may imagine a cube of edge $2r$ circumscribed about the sphere.

Connect O with each of the vertices of this cube.

These connecting lines are the edges of six pyramids whose bases are the faces of the cube and whose altitudes all equal r.

The volume of each pyramid is a face of the cube multiplied by $\frac{1}{3} r$, and the volume of the six pyramids, or of the whole cube, is the area of the surface of the cube multiplied by $\frac{1}{3} r$.

Now imagine planes drawn tangent to the sphere, at the points where the edges of the pyramids cut its surface. We then have a circumscribed solid whose volume is nearer that of the sphere than is the volume of the circumscribed cube, but is still greater than the sphere. Ax. 11

Proceeding as before, connect O with the vertices of the new polyhedron. These connecting lines are the edges of pyramids whose bases are together equal to the bases of the polyhedron and whose common altitude is r. § 646

Then the sum of the volumes of these pyramids is again the area of the surface of the polyhedron multiplied by $\frac{1}{3}r$. Denoting this volume by v' and the area of the surface by s', we have

$$v' = s' \times \tfrac{1}{3}r.$$

If we continue to draw tangent planes to the sphere, we continue to diminish the circumscribed solid.

By continuing this process indefinitely we can make the difference between the volume of the sphere and the volume of the circumscribed solid less than any assigned positive quantity, however small, the difference between the surface of the sphere and the surface of the circumscribed solid becoming and remaining less than any assigned value, however small.

$\therefore v$ is the limit of v', and s is the limit of s'. § 204

And since it has been shown that

$$v' = s' \times \tfrac{1}{3}r, \text{ always,}$$

$$\therefore v = s \times \tfrac{1}{3}r, \text{ by § 207.}$$ Q.E.D.

706. Corollary 1. *The volume of a sphere of radius r and diameter d is equal to* $\frac{4}{3}\pi r^3$ *or* $\frac{1}{6}\pi d^3$.

For in $v = s \times \frac{1}{3}r$ what is the value of s in terms of r? What is the value of d in terms of r? Then what is the value of v in terms of d?

707. Corollary 2. *The volumes of two spheres are to each other as the cubes of their radii.*

What is the ratio of $\frac{4}{3}\pi r^3$ to $\frac{4}{3}\pi r'^3$?

By the same reasoning, the volumes are to each other as the cubes of the diameters.

708. Corollary 3. *The volume of a spherical sector is equal to one third the product of the area of the zone which forms its base multiplied by the radius of the sphere.*

Suppose the base divided into spherical triangles. The planes determined by their vertices are the bases of triangular pyramids with vertices at O. What is the limit of the sum of the volumes of these pyramids as the bases decrease in size?

EXERCISE 110

Problems of Computation

Find the volumes of spheres whose radii are:

1. 3 in. **4.** $2\frac{1}{2}$ in. **7.** 20.7 ft.

2. 5 in. **5.** $4\frac{3}{8}$ in. **8.** 2 ft. 3 in.

3. 7 in. **6.** $9\frac{7}{8}$ in. **9.** 4000 mi.

Find the volumes of spheres whose diameters are:

10. 24 in. **13.** 2.8 in. **16.** 2 ft. 1 in.

11. 36 in. **14.** 3.4 in. **17.** 3 ft. 4 in.

12. 48 in. **15.** 4.5 in. **18.** 8 ft. 6 in.

Find the volumes of spheres whose circumferences are:

19. 6.2832 in. **20.** 12.5664 in. **21.** 18.8496 in.

Find the volumes of spheres whose surface areas are:

22. 12.5664 sq. in. **23.** 50.2656 sq. in. **24.** 113.0976 sq. in.

Find the radii of spheres whose volumes are:

25. 4.1888 cu. in. **26.** 33.5104 cu. in. **27.** 113.0976 cu. in.

28. The circumference of a hemispherical dome is 66 ft. How many square feet of lead are required to cover it?

29. If the ball on the top of St. Paul's Cathedral in London is 6 ft. in diameter, how much would it cost to gild it at 9 cents per square inch?

30. The dihedral angles made by the faces of a spherical pyramid are 80°, 100°, 120°, and 150°, and the length of a lateral edge is 42 ft. Find the area of the base.

31. The dihedral angles made by the faces of a spherical pyramid are 60°, 80°, and 100°, and the area of the base is 4π sq. ft. Find the radius.

32. What is the area of the surface of the earth?

Assume that the earth is a sphere with a radius of 4000 mi., and make the same assumption in subsequent examples relating to the earth.

33. The altitude of the torrid zone is 3200 mi. Find its area.

34. What is the area of the north temperate zone if its altitude is 1800 mi.?

35. Find the number of square miles of the earth's surface that can be seen from an aëroplane 1500 ft. above the surface.

36. How far in one direction can a man see from the deck of an ocean steamer if his eye is 40 ft. above the water?

37. To what height must a man be raised above the earth in order to see one sixth of its surface?

38. How much of the earth's surface would a man see if he were raised to the height of the radius above it?

39. If the atmosphere extends 50 mi. above the surface of the earth, find the volume of the atmosphere.

40. If an iron ball 4 in. in diameter weighs 9 lb., find the weight of a spherical iron shell 2 in. thick, the external diameter being 20 in.

41. What is the angle of a spherical wedge if its volume is $1\frac{1}{4}$ cu. ft. and the volume of the entire sphere is $8\frac{3}{4}$ cu. ft.?

42. The inside of a washbasin is in the shape of the segment of a sphere. The distance across the top is 16 in. and its greatest depth is 8 in. How many pints of water will it hold, allowing 7 gal. to the cubic foot?

43. Prove that the volume of a spherical pyramid is equal to the product of the base by one third of the radius, and find the volume if the base is one eighth of the surface of a sphere of radius 10 in.

44. Find the volume of a spherical sector whose base is a zone of area z, the radius of the sphere being r, following a process of reasoning similar to that in § **705**.

EXERCISE 111

Formulas

1. Find the area z of the zone of a sphere of radius r, illuminated by a lamp placed at the height h above the surface.

2. Find the volume v of a sphere in terms of c, the circumference.

3. Find the radius r of a sphere in terms of v, the volume.

4. Find the diameter d of a sphere in terms of s, the area of the surface.

5. Find the circumference c of a sphere in terms of s, the area of the surface.

6. What is the altitude a of a zone, if its area is z and the volume of the sphere is v?

7. Show that in a spherical pyramid $v = \frac{1}{3} br$. Find r in terms of v and b; also b in terms of v and r.

8. Find a formula for the volume of the metal in a spherical iron shell, the inside radius being r and the thickness of the metal being t.

9. Find a formula for the weight of a spherical shell, the inside radius being r, the thickness of the metal being t, and the weight of a cubic unit of metal being w.

10. If the area of a zone z equals $2\pi ra$ (§ 691), find a formula for a in terms of z and r.

11. If the area of a zone is expressed by the formula $z = 2\pi ra$, what is the diameter of the sphere upon which a zone z has an altitude a?

12. Find the area z of a zone of altitude a on a sphere whose area of surface is s.

13. Find a formula for the area a of that part of the surface of a sphere of radius r seen from a point at a distance d above the surface.

EXERCISE 112

Problems of Loci

Find the locus of a point:

1. At a given distance from a given point.

2. At a given distance from a given straight line.

3. At a given distance from a given plane.

4. At a given distance from a given cylindric surface.

5. At a given distance from a given spherical surface.

6. Equidistant from two given points.

7. Equidistant from two given planes.

8. At a given distance from a given point and at another given distance from a given straight line.

9. At a given distance from a given point and at another given distance from a given plane.

10. At a given distance from a given point and equidistant from two other given points.

11. At a given distance from a given point and equidistant from two given planes.

Find one or more points:

12. At a distance d_1 from a given point, at a distance d_2 from a given straight line, and at a distance d_3 from a given plane.

13. At a distance d_1 from a given point, at a distance d_2 from a given plane, and equidistant from two other given planes.

14. Equidistant from two given points, equidistant from two given planes, and at a distance r from a given point.

15. Find the locus of the center of a sphere whose surface touches two given planes and passes through two given points that lie between the planes.

EXERCISE 113

MISCELLANEOUS EXERCISES

1. The volume of a sphere is to the volume of the inscribed cube as π is to $\frac{2}{3}\sqrt{3}$.

2. The volume of a sphere is to the volume of the circumscribed cube as π is to 6.

3. Find the ratio of the volume of a cube inscribed in a sphere to that of a circumscribed cube.

4. Find the difference between the volumes of two cubes, one inscribed in a sphere of radius 10 in. and the other circumscribed about it.

5. The planes perpendicular to the three faces of a trihedral angle, and bisecting the face angles, meet in a straight line.

6. The planes that pass through the edges of a trihedral angle, and are perpendicular to the opposite faces, meet in a straight line.

7. The altitude of a regular tetrahedron is equal to the sum of four perpendiculars let fall from any point within the tetrahedron upon the four faces.

8. To cut a given tetrahedral angle by a plane so that the section shall be a parallelogram.

9. Compare the volumes of the solids generated by the revolution of a rectangle successively about two adjacent sides, the sides being a and b respectively.

10. Find the difference between the volume of a frustum of a pyramid and the volume of a prism each 24 ft. high, if the bases of the frustum are squares with sides 20 ft. and 16 ft. respectively, and the base of the prism is the section of the frustum parallel to the bases and midway between them.

11. To draw a line through the vertex of any trihedral angle, making equal angles with its edges.

12. The lines drawn from each vertex of a tetrahedron to the point of intersection of the medians of the opposite face all meet in a point called the *center of gravity* of the tetrahedron, which divides each line so that the ratio of the shorter segment to the whole line is 1 : 4.

13. The lines joining the mid-points of the opposite edges of a tetrahedron all pass through the center of gravity and are bisected by it.

14. The plane which bisects a dihedral angle of a tetrahedron divides the opposite edge into segments proportional to the areas of the faces that include the dihedral angle.

15. To cut a given cube by a plane so that the section shall be a regular hexagon.

16. The volume of a right circular cylinder is equal to the product of the lateral area by half the radius.

17. The volume of a right circular cylinder is equal to the product of the area of the rectangle which generates it, by the length of the circumference generated by the point of intersection of the diagonals of the rectangle.

18. If the altitude of a right circular cylinder is equal to the diameter of the base, the volume is equal to the total area multiplied by a third of the radius.

19. The surface of a sphere is two thirds the total surface of the circumscribed cylinder.

20. The volume of a sphere is two thirds the volume of the circumscribed cylinder.

21. Given a sphere, a cylinder circumscribed about the sphere, and a cone of two nappes inscribed in the cylinder. If any two planes are drawn perpendicular to the axis of the three figures, the spherical segment between the planes is equivalent to the difference between the corresponding cylindric and conic segments.

EXERCISE 114

Review Questions

1. How is a sphere generated?

2. What are two tests of equality of spheres?

3. If a plane cuts a sphere, what figure is formed? Is the same true of a plane cutting a cone?

4. What is the test of equal circles on a given sphere?

5. What is a great circle of a sphere? Name four properties of great circles.

6. What is meant by a plane being tangent to a sphere? State any proposition concerning a tangent plane, and the corresponding proposition in plane geometry.

7. Complete this statement: A sphere may be inscribed in ···. State the corresponding proposition in plane geometry.

8. Complete this statement: A sphere may be circumscribed about ···. State the corresponding proposition in plane geometry.

9. Complete this statement: A spherical surface is determined by ··· points not in the same plane. State the corresponding proposition in plane geometry.

10. What is the limit of the sum of the sides of a spherical polygon? What are the limits of the sum of the angles of a spherical triangle?

11. What is a polar triangle? State two propositions relating to polar triangles.

12. What is meant by symmetric spherical triangles? State two propositions relating to such triangles.

13. State two propositions relating to congruent spherical triangles.

14. How is the area of a spherical triangle found? How is the area of a spherical polygon found?

APPENDIX

709. Subjects Treated. As with plane geometry, so with solid geometry, there are many topics that might be taken in addition to those given in any textbook. The theorems and problems already given in this work are standard propositions that are looked upon as basal, and are usually required as preliminary to more advanced work, and these, with a reasonable selection from the exercises, will be all that most schools have time to consider. It occasionally happens, however, that a school is able to do more than this, and then more exercises may be selected from the large number contained in this work, and a few additional topics may be studied. For this latter purpose the appendix is added, but its study should not be undertaken at the expense of good work on the fundamental propositions and the exercises depending upon them.

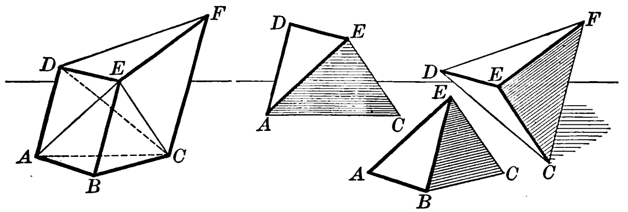
The subjects treated are certain additional propositions in the mensuration of solids, and a few general theorems relating to similar polyhedrons, these being occasionally required for college examinations. There is also added a brief sketch of the history of geometry, which all students are advised to read as a matter of general information, and a few of those recreations of geometry that add a peculiar interest to the subject.

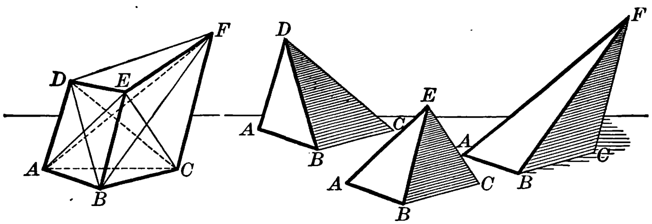
710. Similar Polyhedrons. Polyhedrons that have the same number of faces, respectively similar and similarly placed, and their corresponding polyhedral angles equal, are called *similar polyhedrons.*

It will be seen that this is analogous to the definition of similar polygons in plane geometry.

Proposition I. Theorem

711. *A truncated triangular prism is equivalent to the sum of three pyramids whose common base is the base of the prism and whose vertices are the three vertices of the inclined section.*





Given a truncated triangular prism *ABC-DEF* whose base is *ABC* and inclined section *DEF*, the truncated prism being divided into the three pyramids *E-ABC*, *E-ACD*, and *E-CFD*.

To prove ABC-DEF equivalent to the sum of the three pyramids E-ABC, D-ABC, and F-ABC.

Proof. *E-ABC* has the base *ABC* and the vertex *E*.

Now pyramid *E-ACD* = pyramid *B-ACD*. § 558

(*For they have the same base, ACD, and the same altitude, since their vertices E and B are in the line EB* ∥ *to the plane ACD.*)

But the pyramid *B-ACD* may be regarded as having the base *ABC* and the vertex *D*; that is, as pyramid *D-ABC*.

Then since $\triangle$ CFD and ACF have the common base CF and equal altitudes, their vertices lying in the line AD which is parallel to CF, they are equivalent. § 326

Furthermore, pyramids $E\text{-}CFD$ and $B\text{-}ACF$ not only have equivalent bases, the $\triangle$ CFD and ACF, but they have the same altitude, since their vertices E and B are in the line EB which is parallel to the plane of their bases.

$\therefore$ pyramid $E\text{-}CFD =$ pyramid $B\text{-}ACF$. § 558

But the pyramid $B\text{-}ACF$ may be regarded as having the base ABC and the vertex F; that is, as pyramid $F\text{-}ABC$.

Therefore the truncated triangular prism $ABC\text{-}DEF$ is equivalent to the sum of the three pyramids $E\text{-}ABC$, $D\text{-}ABC$, and $F\text{-}ABC$. **Q. E. D.**

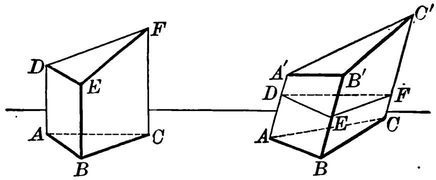


FIG. 1 FIG. 2

712. COROLLARY 1. *The volume of a truncated right triangular prism is equal to the product of its base by one third the sum of its lateral edges.*

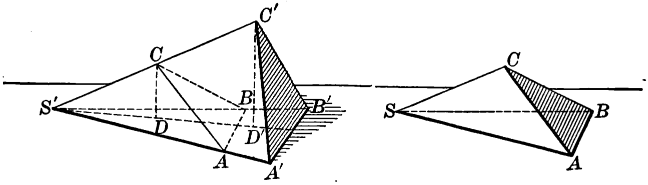
For the lateral edges DA, EB, FC (Fig. 1), being perpendicular to the base ABC, are the altitudes of the three pyramids whose sum is equivalent to the truncated prism. It is interesting to consider the special case in which $\triangle DEF$ is parallel to $\triangle ABC$.

713. COROLLARY 2. *The volume of any truncated triangular prism is equal to the product of its right section by one third the sum of its lateral edges.*

For the right section DEF (Fig. 2) divides the truncated prism into two truncated right prisms.

PROPOSITION II. THEOREM

714. *The volumes of two tetrahedrons that have a trihedral angle of the one equal to a trihedral angle of the other are to each other as the products of the three edges of these trihedral angles.*



Given the two tetrahedrons $S\text{-}ABC$ and $S'\text{-}A'B'C'$, having the trihedral angles S and S' equal, v and v' denoting the volumes.

To prove that $$\frac{v}{v'} = \frac{SA \times SB \times SC}{S'A' \times S'B' \times S'C'}.$$

Proof. Place the tetrahedron $S\text{-}ABC$ upon $S'\text{-}A'B'C'$ so that the trihedral $\angle S$ shall coincide with the equal trihedral $\angle S'$.

Draw CD and $C'D' \perp$ to the plane $S'A'B'$,

and let their plane intersect $S'A'B'$ in $S'DD'$.

The faces $S'AB$ and $S'A'B'$ may be taken as the bases, and CD, $C'D'$ as the altitudes, of the triangular pyramids $C\text{-}S'AB$ and $C'\text{-}S'A'B'$ respectively.

Then $$\frac{v}{v'} = \frac{S'AB \times CD}{S'A'B' \times C'D'} = \frac{S'AB}{S'A'B'} \times \frac{CD}{C'D'}.$$ § 562

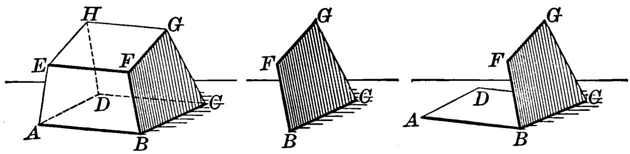
But $$\frac{S'AB}{S'A'B'} = \frac{S'A \times S'B}{S'A' \times S'B'},$$ § 332

and $$\frac{CD}{C'D'} = \frac{S'C}{S'C'}.$$ § 282

$$\therefore \frac{v}{v'} = \frac{S'A \times S'B \times S'C}{S'A' \times S'B' \times S'C'} = \frac{SA \times SB \times SC}{S'A' \times S'B' \times S'C'},$$ by Ax. 9. Q.E.D.

Proposition III. Theorem

715. *In any polyhedron the number of edges increased by two is equal to the number of vertices increased by the number of faces.*



Given the polyhedron AG, e denoting the number of edges, v the number of vertices, and f the number of faces.

To prove that $e + 2 = v + f.$

Proof. Beginning with one face $BCGF$, we have $e = v$.

Annex a second face $ABCD$ by applying one of its edges to a corresponding edge of the first face, and there is formed a surface of two faces having *one* edge BC and *two* vertices B and C common to the two faces.

Therefore for two faces $e = v + 1$.

Annex a third face $ABFE$, adjoining each of the first two faces. This face will have *two* edges AB, BF and *three* vertices A, B, F in common with the surface already formed.

Therefore for three faces $e = v + 2$.

In like manner, for four faces, $e = v + 3$, and so on.

Therefore for $(f - 1)$ faces $e = v + (f - 2)$.

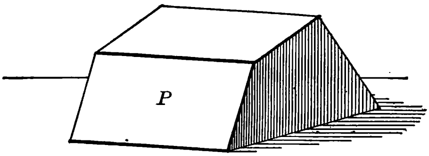
But $f - 1$ is the number of faces of the polyhedron when only one face is lacking, and the addition of this face will not increase the number of edges or vertices. Hence for f faces

$$e = v + f - 2, \text{ or } e + 2 = v + f. \qquad \text{Q.E.D.}$$

This theorem is due to the great Swiss mathematician, Euler.

Proposition IV. Theorem

716. *The sum of the face angles of any polyhedron is equal to four right angles taken as many times, less two, as the polyhedron has vertices.*



Given the polyhedron P, e denoting the number of edges, v the number of vertices, f the number of faces, and s the sum of the face angles.

To prove that $s = (v - 2)\, 4$ *rt.* $\angle$s.

Proof. Since e denotes the number of edges, $2e$ will denote the number of sides of the faces, considered as independent polygons, for each edge is common to two polygons.

If an exterior angle is formed at each vertex of every polygon, the sum of the interior and exterior angles at each vertex is 2 rt. $\angle$s; and since there are $2e$ vertices, the sum of the interior and exterior angles of all the faces is

$$2e \times 2 \text{ rt. } \angle\text{s, or } e \times 4 \text{ rt. } \angle\text{s.}$$

But the sum of the ext. $\angle$s of each face is 4 rt. $\angle$s. § 146

Therefore the sum of all the ext. $\angle$s of f faces is

$$f \times 4 \text{ rt. } \angle\text{s.}$$

Therefore $s = e \times 4 \text{ rt. } \angle\text{s} - f \times 4 \text{ rt. } \angle\text{s}$

$= (e - f)\, 4 \text{ rt. } \angle\text{s.}$

But $e + 2 = v + f$; § 715

that is, $e - f = v - 2$. Ax. 2

Therefore $s = (v - 2)\, 4 \text{ rt. } \angle\text{s.}$ Q. E. D.

EXERCISE 115

Find the volumes of truncated triangular prisms, given the bases b, and the distances of the three vertices p, q, r from the planes of the bases, as follows:

1. $b = 8$ sq. in., $p = 3$ in., $q = 4$ in., $r = 5$ in.

2. $b = 9$ sq. in., $p = 6$ in., $q = 3$ in., $r = 4\frac{1}{2}$ in.

3. $b = 15$ sq. in., $p = 7$ in., $q = 9$ in., $r = 8.1$ in.

4. $b = 32$ sq. in., $p = 9$ in., $q = 12$ in., $r = 9.3$ in.

5. $b = 48$ sq. in., $p = 16$ in., $q = 15$ in., $r = 18$ in.

6. A triangular rod of iron is cut square off (i.e. in right section) at one end, and slanting at the other end. The right section is an equilateral triangle $1\frac{1}{2}$ in. on a side. The edges of the rod are 3 ft. 2 in., 3 ft. 3 in., and 3 ft. 3 in. Find the weight of the rod, allowing 0.28 lb. per cubic inch.

7. Two triangular pyramids with a trihedral angle of the one equal to a trihedral angle of the other have the edges of these angles 3 in., 4 in., $3\frac{1}{2}$ in., and 5 in., $5\frac{1}{2}$ in., 6 in. respectively. Find the ratio of the volumes.

8. Make a table giving the number of edges, vertices, and faces of each of the five regular polyhedrons, showing that in every case the number conforms to Euler's theorem (§ 715).

9. Make a table similar to that of Ex. 8, giving the sum of the face angles in each of the five regular polyhedrons, showing that in every case $s = (v - 2)$ 4 rt. $\angle s$ (§ 716).

10. There can be no seven-edged polyhedron.

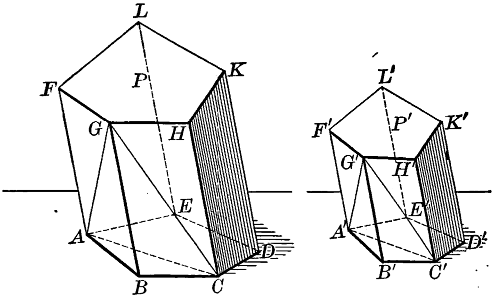
11. Can there be a nine-edged polyhedron?

12. What is the sum of the face angles of a six-edged polyhedron?

13. What is the sum of the face angles of a polyhedron with five vertices? with four vertices? Consider the possibility of a polyhedron with three vertices.

PROPOSITION V. THEOREM

717. *Two similar polyhedrons can be separated into the same number of tetrahedrons similar each to each and similarly placed.*



Given two similar polyhedrons P and P'.

To prove that P and P' can be separated into the same number of tetrahedrons similar each to each and similarly placed.

Proof. Let G and G' be corresponding vertices.

Divide all the faces of P and P', except those which include the angles G and G', into corresponding triangles by drawing corresponding diagonals.

Pass a plane through G and each diagonal of the faces of P; also pass a plane through G' and each corresponding diagonal of P'.

Any two corresponding tetrahedrons $G\text{-}ABC$ and $G'\text{-}A'B'C'$ have the faces ABC, GAB, GBC similar respectively to the faces $A'B'C'$, $G'A'B'$, $G'B'C'$. § 292

Since $$\frac{AG}{A'G'} = \frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'} = \frac{GC}{G'C'},$$ § 282

$\therefore$ the face GAC is similar to the face $G'A'C'$. § 289

They also have the corresponding trihedral ∠s equal. § 498

∴ the tetrahedron *G-ABC* is similar to *G'-A'B'C'*. § 710

If *G-ABC* and *G'-A'B'C'* are removed, the polyhedrons remaining continue similar; for the new faces *GAC* and *G'A'C'* have just been proved similar, and the modified faces *AGF* and *A'G'F'*, *GCH* and *G'C'H'*, are similar (§ 292); also the modified polyhedral ∠s *G* and *G'*, *A* and *A'*, *C* and *C'* remain equal each to each, since the corresponding parts taken from these angles are equal.

The process of removing similar tetrahedrons can be carried on until the polyhedrons are separated into the same number of tetrahedrons similar each to each and similarly placed. Q.E.D.

718. Corollary 1. *The corresponding edges of similar polyhedrons are proportional.*

For the corresponding faces are similar. Therefore their corresponding sides are proportional (§ 282).

719. Corollary 2. *Any two corresponding lines in two similar polyhedrons have the same ratio as any two corresponding edges.*

For these lines may be shown to be sides of similar polygons, and hence § 282 applies.

720. Corollary 3. *Two corresponding faces of similar polyhedrons are proportional to the squares on any two corresponding edges.*

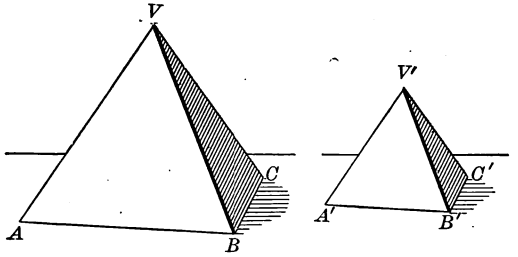
For they are similar polyhedrons, and hence they are to each other as the squares on any two corresponding sides (§ 334).

721. Corollary 4. *The entire surfaces of two similar polyhedrons are proportional to the squares on any two corresponding edges.*

For the corresponding faces are proportional to the squares on any two corresponding edges (§ 720), and hence their sum has the same proportion, by § 269.

Proposition VI. Theorem

722. *The volumes of two similar tetrahedrons are to each other as the cubes on any two corresponding edges.*



Given two similar tetrahedrons *V-ABC* and *V′-A′B′C′*, with volumes *v* and *v′*, *VB* and *V′B′* being two corresponding edges.

To prove that $$\frac{v}{v'} = \frac{\overline{VB}^3}{\overline{V'B'}^3}.$$

Proof. Since the two polyhedrons are similar, Given

$\therefore$ the corresponding polyhedral angles are equal, § 710

and, in particular, the trihedral angles V and V' are equal.

$$\therefore \frac{v}{v'} = \frac{VB \times VC \times VA}{V'B' \times V'C' \times V'A'} \qquad \S\ 714$$

$$= \frac{VB}{V'B'} \times \frac{VC}{V'C'} \times \frac{VA}{V'A'}.$$

Furthermore, since the tetrahedrons are similar, Given

$$\therefore \frac{VB}{V'B'} = \frac{VC}{V'C'} = \frac{VA}{V'A'}. \qquad \S\ 718$$

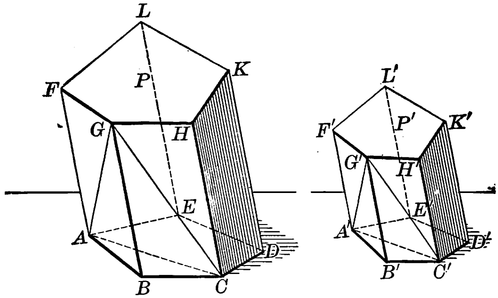
Substituting $\frac{VB}{V'B'}$ for its equals, we have

$$\frac{v}{v'} = \frac{VB}{V'B'} \times \frac{VB}{V'B'} \times \frac{VB}{V'B'}, \qquad \text{Ax. 9}$$

or $$\frac{v}{v'} = \frac{\overline{VB}^3}{\overline{V'B'}^3}. \qquad \text{Q.E.D.}$$

Proposition VII. Theorem

723. *The volumes of two similar polyhedrons are to each other as the cubes of any two corresponding edges.*



Given two similar polyhedrons P and P', with volumes v and v', GB and $G'B'$ being any two corresponding edges.

To prove that $v : v' = \overline{GB}^3 : \overline{G'B'}^3$.

Proof. Separate P and P' into tetrahedrons similar each to each and similarly placed (§ 717), denoting their respective volumes by $v_1, v_2, v_3, \cdots, v_1', v_2', v_3', \cdots$.

Then since $v_1 : v_1' = \overline{GB}^3 : \overline{G'B'}^3$,

$v_2 : v_2' = \overline{GB}^3 : \overline{G'B'}^3$, and so on. § 722

$\therefore v_1 + v_2 + v_3 + \cdots : v_1' + v_2' + v_3' + \cdots = \overline{GB}^3 : \overline{G'B'}^3$. § 269

But $v_1 + v_2 + v_3 + \cdots = v$, and $v_1' + v_2' + v_3' + \cdots = v'$.

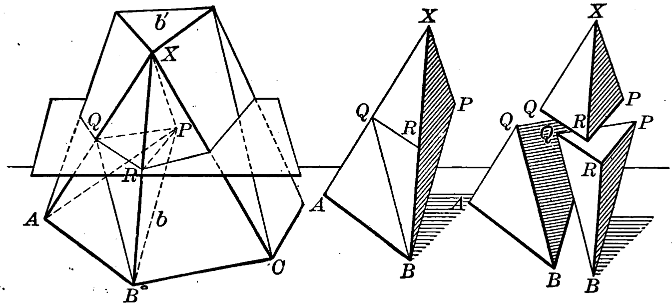
$\therefore v : v' = \overline{GB}^3 : \overline{G'B'}^3$, by Ax. 9. Q.E.D.

724. Prismatoid. A polyhedron having for bases two polygons in parallel planes, and for lateral faces triangles or trapezoids with one side common with one base, and the opposite vertex or side common with the other base, is called a *prismatoid*.

The *altitude* is the distance between the planes of the bases. The *mid-section* is the section made by a plane parallel to the bases and bisecting the altitude.

Proposition VIII. Theorem

725. *The volume of a prismatoid is equal to the product of one sixth of its altitude into the sum of its bases and four times its mid-section.*



Given a prismatoid of volume v, bases b and b', mid-section m, and altitude a.

To prove that $v = \frac{1}{6} a (b + b' + 4m)$.

Proof. If any lateral face is a trapezoid, divide it into two triangles by a diagonal.

Take any point P in the mid-section and join P to the vertices of the polyhedron and of the mid-section.

Separate the prismatoid into pyramids which have their vertices at P, and for their respective bases the lower base b, the upper base b', and the lateral faces of the prismatoid.

The pyramid $P\text{-}XAB$, which we may call a lateral pyramid, is composed of the three pyramids $P\text{-}XQR$, $P\text{-}QBR$, and $P\text{-}QAB$.

Now $P\text{-}XQR$ may be regarded as having vertex X and base PQR, and $P\text{-}QBR$ as having vertex B and base PQR.

Hence the volume of $P\text{-}XQR$ is equal to $\frac{1}{6} a \cdot PQR$,

and the volume of $P\text{-}QBR$ is equal to $\frac{1}{6} a \cdot PQR$. § 559

The pyramids $P\text{-}QAB$ and $P\text{-}QBR$ have the same vertex P. The base QAB is twice the base QBR (§ 327), since the $\triangle\ QAB$ has its base AB twice the base QR of the $\triangle\ QBR$ (§ 136), and these triangles have the same altitude (§ 724).

Hence the pyramid $P\text{-}QAB$ is equivalent to twice the pyramid $P\text{-}QBR$. § 563

Hence the volume of $P\text{-}QAB$ is equal to $\frac{2}{6}\, a \cdot PQR$.

Therefore the volume of $P\text{-}XAB$, which is composed of $P\text{-}XQR$, $P\text{-}QBR$, and $P\text{-}QAB$, is equal to $\frac{4}{6}\, a \cdot PQR$.

In like manner, the volume of each lateral pyramid is equal to $\frac{4}{6}\, a \times$ the area of that part of the mid-section which is included within it; and therefore the total volume of all these lateral pyramids is equal to $\frac{4}{6}\, am$.

The volume of the pyramid with base b is $\frac{1}{6}\, ab$,
and the volume of the pyramid with base b' is $\frac{1}{6}\, ab'$. § 559

Therefore $v = \frac{1}{6}\, a\,(b + b' + 4\,m)$. Q.E.D.

EXERCISE 116

Deduce from the formula for the volume of a prismatoid, $v = \frac{1}{6}\, a\,(b + b' + 4\,m)$, *the following formulas:*

1. Cube, $v = a^3$.

2. Prism, $v = ba$.

3. Pyramid, $v = \frac{1}{3}\, ba$.

4. Parallelepiped, $v = ba$.

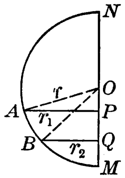
5. Frustum of a pyramid, $v = \frac{1}{3}\, a\left(b + b' + \sqrt{bb'}\right)$.

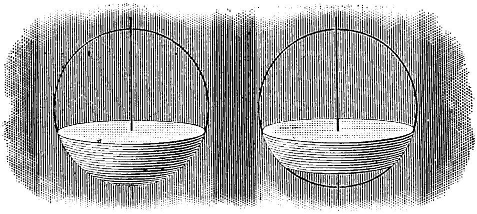
6. A prismatoid has an upper base 3 sq. in., a lower base 7 sq. in., an altitude 3 in., and a mid-section 4 sq. in. What is the volume?

7. A wedge has for its base a rectangle l in. long and w in. wide. The cutting edge is e in. long, and is parallel to the base. The distance from e to the base is d in. Deduce a formula for the volume of the wedge. Apply this formula to the case in which $l = 6$, $w = 1$, $e = 5$, $d = 3$.

PROPOSITION IX. THEOREM

726. *The volume of a spherical segment is equal to the product of one half the sum of its bases by its altitude, increased by the volume of a sphere having that altitude for its diameter.*





Given a spherical segment of volume *v*, generated by the revolution of *ABQP* about *MN* as an axis, *r* being the radius of the sphere, *AP* being represented by r_1, *BQ* by r_2, and *PQ* by *a*.

To prove that $v = \frac{1}{2}a(\pi r_1^2 + \pi r_2^2) + \frac{1}{6}\pi a^3.$

Proof. We shall first find the volume of the spherical segment with one base, generated by *AMP*.

$$\text{Area of zone } AM = 2\pi r \cdot PM. \qquad \S\ 691$$

$$\therefore \text{volume of sector generated by } OAM = \tfrac{1}{3}r \times 2\pi r \cdot PM. \qquad \S\ 708$$

$$\text{But the cone generated by } OAP = \tfrac{1}{3}\pi r_1^2(r - PM). \qquad \S\ 611$$

$$\therefore \text{volume } AMP = \tfrac{1}{3}r \times 2\pi r \cdot PM \cdot - \tfrac{1}{3}\pi r_1^2(r - PM). \qquad \text{Ax. 2}$$

$$\text{But } r_1^2 = PM \times NP = PM(2r - PM). \qquad \S\ 297$$

$$\therefore \text{volume } AMP = \tfrac{1}{3}r \times 2\pi r \cdot PM$$

$$- \tfrac{1}{3}\pi \cdot PM(2r - PM)(r - PM) \qquad \text{Ax. 9}$$

$$= \pi \cdot \overline{PM}^2(r - \tfrac{1}{3}PM).$$

In the same way, volume $BMQ = \pi \cdot \overline{QM}^2(r - \frac{1}{3}QM)$.

$$\therefore v = \text{volume } AMP - \text{volume } BMQ$$

$$= \pi \cdot \overline{PM}^2 \cdot r - \tfrac{1}{3}\pi \cdot \overline{PM}^3 - \pi \cdot \overline{QM}^2 \cdot r + \tfrac{1}{3}\pi \cdot \overline{QM}^3$$

$$= \pi r(\overline{PM}^2 - \overline{QM}^2) - \tfrac{1}{3}\pi(\overline{PM}^3 - \overline{QM}^3).$$

But $PM - QM = a$. Given

$\therefore v = \pi ra(PM + QM) - \frac{1}{3}\pi a(\overline{PM}^2 + PM \cdot QM + \overline{QM}^2)$. Ax. 9

But $a^2 = \overline{PM}^2 - 2\,PM \cdot QM + \overline{QM}^2$. Ax. 5

$\therefore a^2 + 3\,PM \cdot QM = \overline{PM}^2 + PM \cdot QM + \overline{QM}^2$. Ax. 1

$\therefore v = \pi ra(PM + QM) - \frac{1}{3}\pi a(a^2 + 3\,PM \cdot QM)$. Ax. 9

Furthermore $(2r - PM)PM = r_1^2$,

and $(2r - QM)QM = r_2^2$. § 297

$\therefore 2r \cdot PM + 2r \cdot QM - \overline{PM}^2 - \overline{QM}^2 = r_1^2 + r_2^2$. Ax. 1

$\therefore r \cdot PM + r \cdot QM = \frac{r_1^2 + r_2^2}{2} + \frac{\overline{PM}^2 + \overline{QM}^2}{2}$. Axs. 1, 4

$$\therefore v = \pi a\left(\frac{r_1^2 + r_2^2}{2} + \frac{\overline{PM}^2 + \overline{QM}^2}{2} - \frac{a^2}{3} - PM \cdot QM\right)$$

$$= \pi a\left(\frac{r_1^2 + r_2^2}{2} + \frac{a^2}{2} + PM \cdot QM - \frac{a^2}{3} - PM \cdot QM\right)$$

$$= \tfrac{1}{2}a(\pi r_1^2 + \pi r_2^2) + \tfrac{1}{6}\pi a^3.$$ Q.E.D.

EXERCISE 117

Find the volumes of spherical segments having bases b and b', and altitudes a, as follows:

1. $b = 4$, $b' = 5$, $a = 1$.
2. $b = 4$, $b' = 6$, $a = 1\frac{1}{4}$.
3. $b = 5$, $b' = 7$, $a = 2\frac{1}{8}$.
4. $b = 6$, $b' = 8$, $a = 1\frac{1}{2}$.
5. $b = 8$, $b' = 12$, $a = 2$.
6. $b = 12$, $b' = 15$, $a = 3\frac{1}{2}$.
7. $b = 27$ sq. in., $b' = 32$ sq. in., $a = 2.33$ in.

Find the volumes of spherical segments having radii of bases r_1 and r_2, and altitudes a, as follows:

8. $r_1 = 3$, $r_2 = 4$, $a = 2$.
9. $r_1 = 4$, $r_2 = 7$, $a = 3$.
10. $r_1 = 8$, $r_2 = 5$, $a = 4\frac{1}{2}$.
11. $r_1 = 5$, $r_2 = 3$, $a = 1\frac{1}{2}$.
12. $r_1 = 6$, $r_2 = 5$, $a = 1\frac{1}{4}$.
13. $r_1 = 9$, $r_2 = 10$, $a = 2\frac{3}{4}$.
14. $r_1 = 9$ in., $r_2 = 7$ in., $a = 4.75$ in.

EXERCISE 118

Examination Questions

1. A pyramid 6 ft. high is cut by a plane parallel to the base, the area of the section being $\frac{1}{9}$ that of the base. How far from the vertex is the cutting plane?

2. Find the area of a spherical triangle whose angles are 100°, 120°, and 140°, the diameter of the sphere being 16 in.

3. Two angles of a spherical triangle are 80° and 120°. Find the limits of the third angle, and prove that the greatest possible area of the triangle is four times the least possible area, the sphere on which it is drawn being given.

4. An irregular portion, less than half, of a material sphere is given. Show how the radius can be found, compasses and ruler being allowed.

5. Find the volume of a cone of revolution, the area of the total surface of which is 200π sq. ft., and the altitude of which is 16 ft.

6. The volumes of two similar polyhedrons are 64 cu. ft. and 216 cu. ft. respectively. If the area of the surface of the first polyhedron is 112 sq. ft., find the area of the surface of the second polyhedron.

7. A solid sphere of metal of radius 12 in. is recast into a hollow sphere. If the cavity is spherical, of the same radius as the original sphere, find the thickness of the shell.

8. The stone spire of a church is a regular pyramid 50 ft. high on a hexagonal base each side of which is 10 ft. There is a hollow part which is also a regular pyramid 45 ft. high, on a hexagonal base of which each side is 9 ft. Find the number of cubic feet of stone in the spire.

9. The volumes of a hemisphere, right circular cone, and right circular cylinder are equal. Their bases are also equal, each being a circle of radius 10 in. Find the altitude of each.

10. A sphere of radius 5 ft. and a right circular cone also of radius 5 ft. stand on a plane. If the height of the cone is equal to a diameter of the sphere, find the position of the plane that cuts the two solids in equal circular sections.

11. The vertices of one regular tetrahedron are at the centers of the faces of another regular tetrahedron. Find the ratio of the volumes.

12. Find the area of a spherical triangle, if the perimeter of its polar triangle is 297° and the radius of the sphere is 10 centimeters.

13. The radii of two spheres are 13 in. and 15 in. respectively, and the distance between the centers is 14 in. Find the volume of the solid common to both spheres, — a spherical lens.

14. The radius of the base of a right circular cylinder is r and the altitude of the cylinder is a. Find the radius and the volume of a sphere whose surface is equivalent to the lateral surface of the cylinder.

15. If the polyhedral angle at the vertex of a triangular pyramid is trirectangular, and the areas of the lateral faces are a, b, and c respectively, and the area of the base is d, then $a^2 + b^2 + c^2 = d^2$.

16. If the earth is a sphere with a diameter of 8000 mi., find the area of the zone bounded by the parallels 30° north latitude and 30° south latitude. Show that this zone and the planes of the circles include $\frac{11}{16}$ of the volume of the earth.

17. The altitude of a cone of revolution is 12 centimeters and the radius of its base is 5 centimeters. Compute the radius of the sector of paper which, when rolled up, will just cover the convex surface of the cone, and compute the size of the central angle of this sector in degrees, minutes, and seconds.

18. The volume of any regular pyramid is equal to one third of its lateral area multiplied by the perpendicular distance from the center of its base to any lateral face.

19. If the area of a zone of one base is n times the area of the circle which forms its base, the altitude of the zone is $\frac{1}{n}(n-1)$ times the diameter of the sphere. Discuss the special case when $n=1$.

20. If the four sides of a spherical quadrilateral are equal, its diagonals are perpendicular to each other.

21. Find the volume of a pyramid whose base contains 30 square centimeters if one lateral edge is 5 centimeters and the angle formed by this edge and the plane of the base is 45°.

22. On the base of a right circular cone a hemisphere is constructed outside the cone. The surface of the hemisphere equals the surface of the cone. If r is the radius of the hemisphere, find the slant height of the cone, the inclination of the slant height to the base, and the volume of the entire solid.

23. Find the total surface and the volume of a regular tetrahedron whose edge equals 8 centimeters.

24. If a spherical quadrilateral is inscribed in a small circle, the sum of two opposite angles is equal to the sum of the other two angles.

25. By what number must the dimensions of a cylinder of revolution be multiplied to obtain a similar cylinder of revolution with surface n times that of the first? with volume n times that of the first?

26. A pyramid is cut by a plane parallel to the base midway between the vertex and the plane of the base. Compare the volumes of the entire pyramid and the pyramid cut off.

27. The height of a regular hexagonal pyramid is 36 ft. and one side of the base is 6 ft. What are the dimensions of a similar pyramid whose volume is $\frac{1}{20}$ that of the first?

28. One of the lateral edges of a pyramid is 4 meters. How far from the vertex will this edge be cut by a plane parallel to the base, which divides the pyramid into two equivalent parts?

727. Recreations of Geometry. The following simple puzzles and recreations of geometry may serve the double purpose of adding interest to the study of the subject and of leading the student to exercise greater care in his demonstrations. They have long been used for this purpose and are among the best known puzzles of geometry.

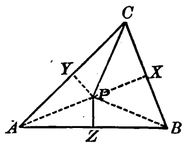
EXERCISE 119

1. To prove that every triangle is isosceles.

Let ABC be a $\triangle$ that is not isosceles.

Take CP the bisector of $\angle ACB$, and ZP the $\perp$ bisector of AB.

These lines must meet, as at P, for otherwise they would be $\parallel$, which would require CP to be $\perp$ to AB, and this could happen only if $\triangle ABC$ were isosceles, which is not the case by hypothesis.



From P draw $PX \perp$ to BC and $PY \perp$ to CA, and draw PA and PB.

Then since ZP is the $\perp$ bisector of AB, $\therefore PA = PB$.

And since CP is the bisector of $\angle ACB$, $\therefore PX = PY$.

$\therefore$ the rt. $\triangle$s PBX and PAY are congruent, and $BX = AY$.

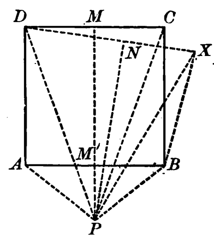
But the rt. $\triangle$s PXC and PYC are also congruent, and $\therefore XC = YC$.

Adding, we have $BX + XC = AY + YC$, or $BC = AC$.

$\therefore \triangle ABC$ is isosceles, even though constructed as not isosceles.

2. To prove that part of an angle equals the whole angle.

Take a square $ABCD$, and draw $MM'P$, the $\perp$ bisector of CD. Then $MM'P$ is also the $\perp$ bisector of AB.



From B draw any line BX equal to AB.

Draw DX and bisect it by the $\perp$ NP.

Since DX intersects CD, $\perp$s to these lines cannot be parallel, but must meet as at P.

Draw PA, PD, PC, PX, and PB.

Since MP is the $\perp$ bisector of CD, $PD = PC$.

Similarly $PA = PB$, and $PD = PX$.

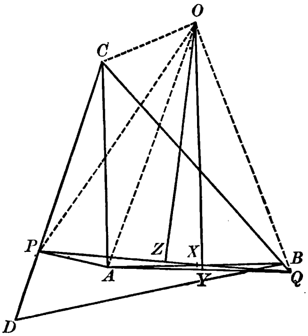
$\therefore PX = PD = PC$.

But $BX = BC$ by construction, and PB is common to $\triangle$s PBX and PBC.

$\therefore \triangle PBX$ is congruent to $\triangle PBC$, and $\angle XBP = \angle CBP$.

$\therefore$ **the whole** $\angle XBP$ equals its part, the $\angle CBP$.

3. To prove that part of an angle equals the whole angle.



Take a right triangle ABC and construct upon the hypotenuse BC an equilateral triangle BCD, as shown.

On CD lay off CP equal to CA.

Through X, the mid-point of AB, draw PX to meet CB produced at Q. Draw QA.

Draw the $\perp$ bisectors of QA and QP, as YO and ZO. These must meet at some point O because they are $\perp$ to two intersecting lines.

Draw OQ, OA, OP, and OC.

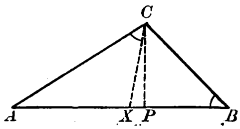
Since O is on the $\perp$ bisector of QA, $\therefore OQ = OA$.

Similarly $OQ = OP$, and $\therefore OA = OP$.

But $CA = CP$, by construction, and $CO = CO$.

$\therefore \triangle AOC$ is congruent to $\triangle POC$, and $\angle ACO = \angle PCO$.

4. To prove that part of a line equals the whole line.



Take a triangle ABC and draw $CP \perp$ to AB.
From C draw CX, making $\angle ACX = \angle B$.
Then $\triangle s$ ABC and ACX are similar.

$$\therefore \triangle ABC : \triangle ACX = \overline{BC}^2 : \overline{CX}^2.$$

Furthermore $\triangle ABC : \triangle ACX = AB : AX.$

$$\therefore \overline{BC}^2 : \overline{CX}^2 = AB : AX,$$

or

$$\overline{BC}^2 : AB = \overline{CX}^2 : AX.$$

But

$$\overline{BC}^2 = \overline{AC}^2 + \overline{AB}^2 - 2\,AB \cdot AP,$$

and

$$\overline{CX}^2 = \overline{AC}^2 + \overline{AX}^2 - 2\,AX \cdot AP.$$

$$\therefore \frac{\overline{AC}^2 + \overline{AB}^2 - 2\,AB \cdot AP}{AB} = \frac{\overline{AC}^2 + \overline{AX}^2 - 2\,AX \cdot AP}{AX},$$

or

$$\frac{\overline{AC}^2}{AB} + AB - 2\,AP = \frac{\overline{AC}^2}{AX} + AX - 2\,AP.$$

$$\therefore \frac{\overline{AC}^2}{AB} - AX = \frac{\overline{AC}^2}{AX} - AB,$$

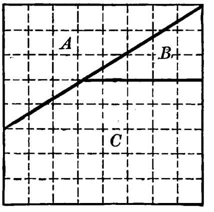
or

$$\frac{\overline{AC}^2 - AB \cdot AX}{AB} = \frac{\overline{AC}^2 - AB \cdot AX}{AX}.$$

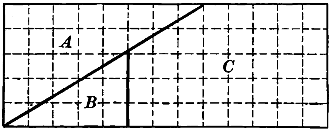
$$\therefore AB = AX.$$

5. To show geometrically that $1 = 0$.

Take a square that is 8 units on a side, and cut it into three parts, A, B, C, as shown in the right-hand figure. Fit these parts together as in the left-hand figure.



Now the square is 8 units on a side, and therefore contains 8×8, or 64, small squares, while the rectangle is 13 units long and 5 units high, and therefore contains 5×13, or 65, small squares.



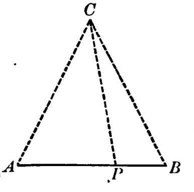
But the two figures are each made up of $A + B + C$ (Ax.11), and therefore are equal (Ax.8).

$\therefore$ $65 = 64$, and by subtracting 64 we have $1 = 0$ (Ax. 2).

6. To prove that any point on a line bisects it.

Take any point P on AB.

On AB construct an isosceles $\triangle ABC$, having $AC = BC$; and draw PC.



Then in $\triangle$s APC and PBC, we have

	$\angle A = \angle B$,	§ 74
	$AC = BC$,	Const.
and	$PC = PC$.	Iden.

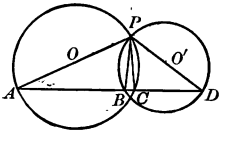
Three independent parts (that is, not merely the three angles) of one triangle are respectively equal to three parts of the other, and the triangles are congruent; therefore $AP = BP$ (§ 67).

7. To prove that it is possible to let fall two perpendiculars to a line from an external point.

Take two intersecting ⓢ with centers O and O'.

Let one point of intersection be P, and draw the diameters PA and PD.

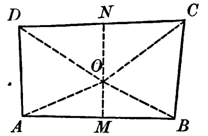
Draw AD cutting the circumferences at B and C. Then draw PB and PC.



Since $\angle PCA$ is inscribed in a semicircle, it is a right angle. In the same way, since $\angle DBP$ is inscribed in a semicircle, it also is a right angle.

$\therefore$ PB and PC are both $\perp$ to AD.

8. To prove that if two opposite sides of a quadrilateral are equal the figure is an isosceles trapezoid.



Given the quadrilateral $ABCD$, with $BC = DA$.

To prove that AB is $\parallel$ to DC.

Draw MO and NO, the $\perp$ bisectors of AB and CD, to meet at O.

If AB and DC are parallel, the proposition is already proved.

If AB and DC are not parallel, then MO and NO will meet at O, either inside or outside the figure. Let O be supposed to be inside the figure.

Draw OA, OB, OC, OD.

Then since OM is the $\perp$ bisector of AB, $\therefore OA = OB$.

Similarly $OD = OC$.

But DA is given equal to BC.

$\therefore \triangle AOD$ is congruent to $\triangle BOC$,

and $\angle DOA = \angle BOC$.

Also, rt. $\triangle$s OCN and ODN are congruent,

and $\angle NOD = \angle CON$.

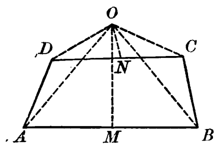
Similarly rt. $\triangle$s AMO and BMO are congruent,

and $\angle AOM = \angle MOB$.

$\therefore \angle NOD + \angle DOA + \angle AOM = \angle CON + \angle BOC + \angle MOB$,

or $\angle NOM = \angle MON =$ a st. $\angle$.

Therefore the line MON is a straight line, and hence AB is $\parallel$ to DC.



If the point O is outside the quadrilateral, as in the second figure, the proof is substantially the same.

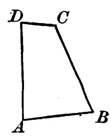
For it can be easily shown that

$$\angle DON - \angle DOA - \angle AOM$$
$$= \angle NOC - \angle BOC - \angle MOB,$$

which is possible only if

$$\angle DON = \angle DOM,$$

or if ON lies along OM.



But that the proposition is not true is evident from the third figure, in which $BC = DA$, but AB is not $\parallel$ to DC.

728. History of Geometry. The geometry of very ancient peoples was largely the mensuration of simple areas and volumes such as is taught to children in elementary arithmetic to-day. They learned how to find the area of a rectangle, and in the oldest mathematical records that we have there is some discussion of triangles and of the volumes of solids.

The earliest documents that we have, relating to geometry, come to us from Babylon and Egypt. Those from Babylon were written about 2000 B.C. on small clay tablets, some of them about the size of the hand, these tablets afterwards having been baked in the sun. They show that the Babylonians of that period knew something of land measures, and perhaps had advanced far enough to compute the area of a trapezoid. For the mensuration of the circle they later used, as did the early Hebrews, the value $\pi = 3$.

The first definite knowledge that we have of Egyptian mathematics comes to us from a manuscript copied on papyrus, a kind of paper used about the Mediterranean in early times. This copy was made by one Aah-mesu (The Moon-born), commonly called Ahmes, who probably flourished about 1700 B.C. The original from which he copied, written about 2300 B.C., has been lost, but the papyrus of Ahmes, written nearly four thousand years ago, is still preserved and is now in the British Museum. In this manuscript, which is devoted chiefly to fractions and to a crude algebra, is found some work on mensuration. Among the curious rules are the incorrect ones that the area of an isosceles triangle equals half the product of the base and one of the equal sides; and that the area of a trapezoid having bases b, b', and nonparallel sides each equal to a, is $\frac{1}{2}\,a(b + b')$. One noteworthy advance appears however. Ahmes gives a rule for finding the area of a circle, substantially as follows: Multiply the square on the radius by $(\frac{16}{9})^2$, which is equivalent to taking for π the value 3.1605. Long before the time of Ahmes, however, Egypt had a good working knowledge of practical geometry, as witness the building of the pyramids, the laying out of temples, and the digging of irrigation canals.

From Egypt and possibly from Babylon geometry passed to the shores of Asia Minor and Greece. The scientific study of the subject begins with Thales, one of the Seven Wise Men of the Grecian civilization. Born at Miletus about 640 B.C., he died there in 548 B.C. He spent his early manhood as a merchant, accumulating the wealth that enabled him to spend his later years in study. He visited Egypt and is said to have learned such elements of geometry as were known there. He founded a school of mathematics and philosophy at Miletus, known as the Ionic School. How elementary the knowledge of geometry then was, may be understood from the fact that tradition attributes only about four propositions to Thales, substantially those given in §§ 60, 72, 74, and 215 of this book.

The greatest pupil of Thales, and one of the most remarkable men of antiquity, was Pythagoras. Born probably on the island of Samos, just off the coast of Asia Minor, about the year 580 B.C., Pythagoras set forth as a young man to travel. He went to Miletus and studied under Thales, probably spent several years in study in Egypt, very likely went to Babylon, and possibly went even to India, since tradition asserts this and the nature of his work in mathematics confirms it. In later life he went to southern Italy, and there, at Crotona, in the southeastern part of the peninsula, he founded a school and established a secret society to propagate his doctrines. In geometry he is said to have been the first to demonstrate the proposition that the square on the hypotenuse of a right triangle is equivalent to the sum of the squares on the other two sides (§ 337). The proposition was known before his time, at any rate for special cases, but he seems to have been the first to prove it. To him or to his school seems also to have been due the construction of the regular pentagon (§§ 397, 398) and of the five regular polyhedrons. The construction of the regular pentagon requires the dividing of a line in extreme and mean ratio (§ 311), and this problem is commonly assigned to the Pythagoreans, although it played an important part in Plato's school. Pythagoras is also said to have known that six equilateral triangles, three regular hexagons, or four squares, can be placed about a point so as just to fill the 360°, but that no other regular polygons can be so placed. To his school is also due the proof that the sum of the angles of a triangle equals two right angles (§ 107), and the construction of at least one star-polygon, the star-pentagon, which became the badge of his fraternity.

For two centuries after Pythagoras geometry passed through a period of discovery of propositions. The state of the science may be seen from the fact that Œnopides of Chios, who flourished about 465 B.C., showed how to let fall a perpendicular to a line (§ 227), and how to construct an angle equal to a given angle (§ 232). A few years later, about 440 B.C., Hippocrates of Chios wrote the first Greek textbook on mathematics. He knew that the areas of circles are proportional to the squares on their radii, but was ignorant of the fact that equal central angles or equal inscribed angles intercept equal arcs.

About 430 B.C. Antiphon and Bryson, two Greek teachers, worked on the mensuration of the circle. The former attempted to find the area by doubling the number of sides of a regular inscribed polygon, and the latter by doing the same for both inscribed and circumscribed polygons. They thus substantially exhausted the area between the circle and the polygon, and hence this method was known as the Method of Exhaustions.

During this period the great philosophic school of Plato (429–348 B.C.) flourished at Athens, and to this school is due the first systematic attempt to create exact definitions, axioms, and postulates, and to distinguish between elementary and higher geometry. At this time elementary geometry became limited to the use of the compasses and the unmarked straightedge, which took from this domain the possibility of constructing a square equivalent to a given circle ("squaring the circle"), of trisecting any given angle, and of constructing a cube with twice the volume of a given cube ("duplicating the cube"), these being the three most famous problems of antiquity. Plato and his school were interested in the so-called Pythagorean numbers, numbers that represent the three sides of a right triangle. Pythagoras had already given a rule to the effect that $\frac{1}{4}(m^2+1)^2 = m^2 + \frac{1}{4}(m^2-1)^2$. The school of Plato found that $[(\frac{1}{2}m)^2+1]^2 = m^2 + [(\frac{1}{2}m)^2-1]^2$. By giving various values to m, different numbers will be found such that the sum of the squares of two of them is equal to the square of the third.

The first great textbook on geometry, and the most famous one that has ever appeared, was written by Euclid, who taught mathematics in the great university at Alexandria, Egypt, about 300 B.C. Alexandria was then practically a Greek city, having been named in honor of Alexander the Great, and being ruled by the Greeks.

Euclid's work is known as the "Elements," and, as was the case with all ancient works, the leading divisions were called books, as is seen in the Bible and in such Latin writers as Cæsar and Vergil. This is why we speak of the various books of geometry to-day. In this work Euclid placed all the leading propositions of plane geometry as then known, and arranged them in a logical order. Most subsequent geometries of any importance since his time have been based upon Euclid, improving the sequence, symbols, and wording as occasion demanded.

Euclid did not give much solid geometry because not much was known then. It was to Archimedes (287–212 B.C.), a famous mathematician of Syracuse, on the island of Sicily, that some of the most important propositions of solid geometry are due, particularly those relating to the sphere and cylinder.

He also showed how to find the approximate value of π by a method similar to the one we teach to-day (§ 404), proving that the real value lies between $3\frac{1}{7}$ and $3\frac{10}{71}$. Tradition says that the sphere and cylinder were engraved upon his tomb. The Greeks contributed little more to elementary geometry, although Apollonius of Perga, who taught at Alexandria between 250 and 200 B.C., wrote extensively on conic sections; and Heron of Alexandria, about the beginning of the Christian era, showed that the area of a triangle whose sides are a, b, c, equals $\sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$ (see p. 211).

The East did little for geometry, although contributing considerably to algebra. The first great Hindu writer was Aryabhatta, who was born in 476 A.D. He gave the very close approximation for π, expressed in modern notation as 3.1416. The Arabs, about the time of the Arabian Nights Tales (800 A.D.), did much for mathematics, translating the Greek authors into their own language and also bringing learning from India. Indeed, it is to them that modern Europe owes its first knowledge of Euclid. They contributed nothing of importance to geometry, however.

Euclid was translated from the Arabic into Latin in the twelfth century, Greek manuscripts not being then at hand, or being neglected because of ignorance of the language. The leading translators were Athelhard of Bath (1120), an English monk who had learned Arabic in Spain or in Egypt; Gerhard of Cremona, an Italian monk; and Johannes Campanus, chaplain to Pope Urban IV.

In the Middle Ages in Europe nothing worthy of note was added to the geometry of the Greeks. The first edition of Euclid was printed in Latin in 1482, the first one in English appearing in 1570. Our symbols are modern, $+$ and $-$ first appearing in a German work in 1489; $=$ in Recorde's "Whetstone of Witte" in 1557; $>$ and $<$ in the works of Harriot (1560–1621); and $\times$ in a publication by Oughtred (1574–1660).

729. Areas of Solid Figures. The following are the more important areas of solid figures:

Prism,	$l = ep$ (§ 512).
Regular pyramid,	$l = \frac{1}{2} sp$ (§ 553).
Frustum of regular pyramid,	$l = \frac{1}{2}(p + p')s$ (§ 554).
Cylinder of revolution,	$l = ac = 2\pi ra$ (§ 588).
Cone of revolution,	$l = \frac{1}{2} sc = \pi rs$ (§ 609).
Frustum of cone of revolution,	$l = \frac{1}{2}(c + c')s$ (§ 615).
Sphere,	$s = 4\pi r^2$ (§ 689).
Zone,	$s = 2\pi ra$ (§ 691).
Lune,	$s = \frac{\angle A}{360} \cdot 4\pi r^2 = \frac{\angle A}{90} \cdot \pi r^2$ (§ 694).

730. Volumes. The following are the more important volumes:

Rectangular parallelepiped,	lwa (§ 534).
Prism or cylinder,	ba (§§ 539, 589).
Pyramid or cone,	$\frac{1}{3} ba$ (§§ 561, 611).
Frustum of pyramid or cone,	$\frac{1}{3} a(b + b' + \sqrt{bb'})$ (§§ 565, 617).
Right-circular cylinder,	$\pi r^2 a$ (§ 590).
Cone of revolution,	$\frac{1}{3}\pi r^2 a$ (§ 612).
Frustum of cone of revolution,	$\frac{1}{3}\pi a(r^2 + r'^2 + rr')$ (§ 618).
Prismatoid,	$\frac{1}{6} a(b + b' + 4m)$ (§ 725).
Sphere,	$\frac{4}{3}\pi r^3 = \frac{1}{6}\pi d^3$ (§ 706).
Spherical pyramid,	$\frac{1}{3} br$.
Spherical sector,	$\frac{1}{3} zr$ (§ 708).
Spherical segment,	$\frac{1}{2} a(\pi r_1^2 + \pi r_2^2) + \frac{1}{6}\pi a^3$ (§ 726).

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